ORIGINAL PAPER

ON RULED SURFACE GENERATED BY DARBOUX VECTOR OF NATURAL LIFT CURVE IN MINKOWSKI SPACE

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Abstract. *In this study, Darboux vector $\overline{W}$ of the natural lift curve $\bar{\alpha}$ for the given curve α is calculated in terms of the types of curves in Minkowski space. Then striction lines, distribution parameters, Gaussian and mean curvatures of ruled surface pairs generated by the Darboux vector of the natural lift curve have been examined. Furthermore, some examples are given to support the main results.*

Keywords: *natural lift curve; ruled surface; Darboux vector.*

1. INTRODUCTION

In differential geometry, the theories of curves and surfaces have important research areas to understand the geometric structure of them in detail [1]. Moreover, considering the Frenet operators between the main curve and its pair, many significiant results have been obtained. One of the well-known curve mentioned the previous sentence is natural lift curves. The natural lift curve has been defined in J.A. Thorpe's book, see [2]. According to the definition, the natural lift curve is defined as a curve obtained by the endpoints of the unit tangent vectors of the main curve. The properties of the natural lift curve and the relation for Frenet vectors between the main curve and its natural lift curve are investigated in [3, 4].

Beside the theory for curves, the surface theory has extensively been studied by several mathematicians. One of the significiant surface is ruled surfaces. The ruled surface was firstly been examined by G. Monge in literature. Generally, the ruled surface, which consist of the base curve and the rulling, has been defined by the surface obtained as moving a line along a curve.

The concepts and fundamental theorems of the ruled surfaces in $\mathbb{R}^3_1$ are given in [5, 6]. The idea of ruled surfaces generated by a curve and its natural lift curve in Minkowski space has been highlighted in [7]. In that study, the striction curve and distribution parameter of these ruled surfaces are calculated. Furthermore, the isomorphism among unit dual sphere, the subset of the unit 2-sphere and the ruled surfaces generated by the natural lift curves are studied in [8]. In the same study, the developability condition of these ruled surfaces have been extensively denoted. In the light of this study, all characterizations have been modified by the isomorphism among pseudo-sphere, the subsets of the pseudo-spheres and the ruled surfaces generated by the natural lift curves, see [9]. Additionally, the natural lift curve and the concepts of geodesic spray are defined in $\mathbb{R}^3_1$, see [10, 11]. In [12], the distribution parameter of a ruled surface, which is formed by a straight line and moves along two different

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spacelike curves with the same parameter, has been represented by using Frenet trihedron. However, there is a little research about the ruled surfaces generated by Darboux vector of the natural lift curves. Hence, we examine the striction lines, distribution parameters, Gaussian and mean curvatures of these ruled surfaces according to the types of curves in $\mathbb{R}_1^3$. Moreover, some examples have been given to verify the obtained results.

This study is organized as follows: In Section 2, some basic definitions and theorems are given about the natural lift curve and the Frenet operators in $\mathbb{R}_1^3$. In Section 3, the ruled surfaces generated by the natural lift curves are denoted and some examples are given to support the results. Finally, in Section 4, the obtained results are discussed.

2. PRELIMINARIES

In this section, some elementary definitions and theorems about the Frenet operators of the curve in Minkowski space, the properties of the natural lift curve and the concepts of the ruled surfaces are denoted, respectively.

Definition 2.1. Let $\mathbb{R}_1^3$ equipped with the Lorentzian inner product h given by

$$h(Y,Y) = -y_1^2 + y_2^2 + y_3^2,$$

where $Y = (y_1, y_2, y_3)$ is the vector in $\mathbb{R}^3$. A vector $Y = (y_1, y_2, y_3)$ is called timelike if $h(Y,Y) < 0$, spacelike if $h(Y,Y) > 0$ and lightlike (or null) if $h(Y,Y) = 0$. Likewise, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{R}^3$ where s is a pseudo-arclength parameter, can locally be timelike, spacelike or ligtlike (or null) if all of its velocity vectors $\alpha'(s)$ are timelike, spacelike or lightlike (or null) for all $s \in I \subset R$, see [3].

Definition 2.2. The norm of a vector Y is defined by

$$\|Y\|_{\mathbb{R}_1^3} = \sqrt{|h(Y,Y)|}.$$

Definition 2.3. Let $\alpha : I \to \mathbb{R}_1^3$ be a parametrized curve. We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve α. In this trihedron, T, N, B represent the tangent vector, the principal normal vector and the binormal vector, respectively. The following assertions are satisfied, see [12].

Let α be a unit speed timelike space curve with curvature κ and torsion. Let Frenet vector fields of α be $\{T, N, B\}$. In this trihedron, T is timelike vector field, N and B are spacelike vector fields. Then Frenet formulas are given by

$$\begin{cases} T' = \kappa N\,, \\ N' = \kappa T + \tau B, \\ B' = -\tau N\,. \end{cases}$$

Let α be a unit speed spacelike space curve with a spacelike binormal. In this trihedron, we assume that T and B are spacelike vector fields and N is a timelike vector field. Then, Frenet formulas are given by

$$\begin{cases} T' = \kappa N\,, \\ N' = \kappa T + \tau B, \\ B' = \tau N\,. \end{cases}$$

Let α be a unit speed spacelike space curve with a timelike binormal. In this trihedron, we assume that T and N are spacelike vector fields and B is a timelike vector field. Then, Frenet formulas are given by

$$\begin{cases} T' = \kappa N\,, \\ N' = -\kappa T + \tau B, \\ B' = -\tau N\,. \end{cases}$$

Definition 2.4. Let $V = (v_1, v_2, v_3)\,, W = (w_1, w_2, w_3) \in \mathbb{R}_1^3$. The Lorentzian vector product of V and W are given by [4]

$$\mathrm{V} \times \mathrm{W} = \det \begin{bmatrix} e_1 & -e_2 & -e_3 \\ \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \\ \mathrm{w}_1 & \mathrm{w}_2 & \mathrm{w}_3 \end{bmatrix} = (\mathrm{v}_2\mathrm{w}_3 - \mathrm{v}_3\mathrm{w}_2, \mathrm{v}_1\mathrm{w}_3 - \mathrm{v}_3\mathrm{w}_1, \mathrm{v}_2\mathrm{w}_1 - \mathrm{v}_1\mathrm{w}_2)\,.$$

Next, we give some definitions and properties of the natural lift curve in Minkowski space as follows:

Definition 2.5. Let M be a hypersurface in $\mathbb{R}_1^3$ and $\alpha : I \to M$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{ds}\big(\alpha(s)\big) = X\big(\alpha(s)\big), \textit{(for all } s \in I),$$

where X is a smooth tangent vector field on M . Hence, we have

$$TM = \bigcup_{P \in M} T_p M$$

where TpM is the tangent space of M at p [2].

Definition 2.6. For any parametrized curve : $I \to M$, $\bar{\alpha} : I \to TM$ given by

$$\bar{\alpha}(s) = \big(\alpha(s), \alpha'(s)\big) = \alpha'(s)|_{\alpha(s)}$$

is called the natural lift of a α on TM in [2]. Thus, we can write

$$\frac{d\bar{\alpha}}{ds} = \frac{d}{ds}\big(\alpha'(s)|_{\alpha(s)}\big) = D_{\alpha'(s)}\alpha'(s)$$

where D is the Levi-Civita connection on $\mathbb{R}_1^3$, see [2].

Definition 2.7. We denote by $\{T(s), N(s), B(s), \kappa(s), \tau(s)\}$ the Frenet operators of the natural lift curve α.

i) Let α be a unit speed timelike space curve and $\bar{\alpha}$ be the natural lift curve of α. Then, we get

$$\bar{T}(s) = N(s),$$

$$\bar{N}(s) = -\frac{\kappa(s)}{\|W\|}T(s) - \frac{\tau(s)}{\|W\|}B(s),$$

$$\bar{B}(s) = -\frac{\tau(s)}{\|W\|}T(s) - \frac{\kappa(s)}{\|W\|}B(s),$$

$$\bar{\kappa}(s) = \frac{\kappa^2(s) - \tau^2(s)}{\|W\|},$$

$$\bar{\tau}(s) = \frac{-\kappa'(s)\tau(s) + \kappa(s)\tau'(s)}{\|W\|^2}.$$

ii) Let α be a unit speed spacelike space curve with a spacelike binormal and $\overline{\alpha}$ be the natural lift curve of α. Then, we get,

$$\bar{T}(s) = N(s),$$

$$\bar{N}(s) = \frac{\kappa(s)}{\|W\|}T(s) + \frac{\tau(s)}{\|W\|}B(s),$$

$$\bar{B}(s) = \frac{\tau(s)}{\|W\|}T(s) - \frac{\kappa(s)}{\|W\|}B(s),$$

$$\bar{\kappa}(s) = \frac{\kappa^2(s) + \tau^2(s)}{\|W\|},$$

$$\bar{\tau}(s) = \frac{\kappa'(s)\tau(s) + \kappa(s)\tau'(s)}{\|W\|^2}.$$

iii) Let α be a unit speed spacelike space curve with a timelike binormal and $\overline{\alpha}$ be the natural lift curve of α Then, we get,

$$\bar{T}(s) = N(s),$$

$$\bar{N}(s) = -\frac{\kappa(s)}{\|W\|}T(s) - \frac{\tau(s)}{\|W\|}B(s),$$

$$\bar{B}(s) = \frac{\tau(s)}{\|W\|}T(s) + \frac{\kappa(s)}{\|W\|}B(s),$$

$$\bar{\kappa}(s) = \frac{\kappa^2(s) + \tau^2(s)}{\|W\|},$$

$$\bar{\tau}(s) = \frac{-\kappa'(s)\tau(s) + \kappa(s)\tau'(s)}{\|W\|^2},$$

see for details in [2].

Proposition 2.1. Let α be the natural lift curve in $\mathbb{R}_1^3$. Then, there is a $\{T(s), N(s), B(s), \kappa(s), \tau(s)\}$ Frenet frame at each point of the curve. In the coordinate neighborhood, Darboux vectors are classified as follows:

i) Assume that α is the smooth curve with a timelike tangent vector and ψ is the spacelike binormal $-B$ between the Lorentzian timelike angle and the Frenet instantaneous rotation vector that is formulated by $\overline{W} = \bar{\tau}\overline{T} + \bar{\kappa}\overline{B}$.

a) Provided that $|\tau| < |\kappa|$, then W is a spacelike vector. So we write,

$$\kappa = \|W\| cosh\psi \, , \tau = \|W\| sinh\psi$$

which satisfy $\|W\|^2 = h(W, W) = \kappa^2 - \tau^2$.

b) Provided that $|\tau| < |\kappa|$, then W is a timelike vector. So we write,

$$\kappa = \|W\| sinh\psi \, , \tau = \|W\| cosh\psi$$

which satisfy $\|W\|^2 = -h(W, W) = -(\kappa^2 - \tau^2)$.

ii) Assume that α is the smooth curve with a spacelike curve with a spacelike binormal and ψ is the spacelike binormal $-B$ between the Lorentzian timelike angle and the Frenet instantaneous rotation vector that is formulated by $\overline{W} = \bar{\tau}\overline{T} - \bar{\kappa}\overline{B}$. Moreover, $h(W, W) > 0$ and W is a spacelike vector. So we write,

$$\kappa = \|W\| cos\psi \, , \tau = \|W\| sin\psi$$

which satisfy $\|W\| = \sqrt{|\kappa^2 + \tau^2|} = \|W\|^2 = h(W, W) = \kappa^2 + \tau^2$.

iii) Assume that α is the smooth curve with spacelike curve with a timelike binormal and ψ is the spacelike binormal $-B$ between the Lorentzian timelike angle and the Frenet instantaneous rotation vector that is formulated by $\overline{W} = -\bar{\tau}\overline{T} + \bar{\kappa}\overline{B}$.

a) Provided that $|\tau| < |\kappa|$, then W is a spacelike vector. So we write,

$$\kappa = \|W\| sinh\psi \, , \tau = \|W\| cosh\psi$$

which satisfy $\|W\|^2 = h(W, W) = \tau^2 - \kappa^2$.

b) Provided that $|\tau| < |\kappa|$, then W is a timelike vector. So we write,

$$\kappa = \|W\| cosh\psi \, , \tau = \|W\| sinh\psi$$

which satisfy $\|W\|^2 = -h(W, W) = -(\tau^2 - \kappa^2)$, see for details in [9].

Next, we will mention the concepts of the ruled surface. Additionally, the concepts of the striction curve, distribution parameter, Gaussian and mean curvatures;

Definition 2.8. Given a one-parameter family of straight lines,

$$\chi(s, v) = \alpha(s) + ve(s)$$

is called the ruled surface, where $\alpha(s)$ represents the base curve and e is a unit vector of a straight line, see [3]. In [9], the striction curve is defined as

$$\beta(s) = \alpha(s) - \frac{h(\alpha', e')}{h(e', e')} e(s).$$

Also, the distribution parameter on ruled surface χ is given by [11].

$$P_e = \frac{\det(\alpha', e, e')}{\|e'\|^2}.$$

Theorem 2.1. The ruled surface is developable if and only if $Pe = 0$ [3].

Definition 2.9. The Gaussian curvature of $\chi(s, v)$ is

$$K(s, v) = -\frac{\left(det(\alpha', e, e')\right)^2}{(EG - F^2)^2},$$

which consists of

$$E = E(s, v) = \|\alpha'(s) + ve'(s)\|^2, F = F(s, v) = \alpha'(s).e(s), G = G(s, v) = 1.$$

The mean curvature of $\chi(s, v)$ is,

$$H(s, v) = \frac{A}{2(EG - F^2)^{3/2}},$$

which consists of

$$E = E(s, v) = \|\alpha'(s) + ve'(s)\|^2, F = F(s, v) = \alpha'(s).e(s), G = G(s, v) = 1.$$

Here,

$$A = -2\big(\alpha(s).e(s)\big)det\big(\alpha'(s), e(s), e'(s)\big) + det\big(\alpha''(s) + ve''(s), \alpha'(s) + ve'(s), e(s)\big),$$

for more details, see [5].

3. ON RULED SURFACE GENERATED BY DARBOUX VECTOR OF NATURAL LIFT CURVE IN MINKOWSKI SPACE

In this section, Darboux vectors are calculated according to the types of α and $\overline{\alpha}$. Then, the ruled surfaces generated by Darboux vectors are defined and some characterizations related to these surfaces are investigated. Furthermore, some examples are given to support the main results. Now, we will examine the ruled surface pairs generated by the main curve and its natural lift curve according to Darboux vector in Minkowski space:

i) Assume that α is a unit speed timelike space curve and $\chi(s, v)$ is the ruled surface which is parametrized by

$$\chi(s, v) = \alpha(s) + vW(s).$$

After some basic calculations, we get the following properties:
The striction curve of χ is

$$\lambda = \frac{\tau'}{(\tau')^2 - (\kappa')^2}.$$

The distribution parameter is calculated as

$$P_W = 0.$$

The Gaussian and mean curvatures are given as

$$K(s,v) = 0$$

and

$$H(s,v) = \frac{-\big(\kappa + v(\tau'\kappa - \kappa'\tau)\big)(\kappa(1+v\tau') - v\kappa'\tau)}{2(-(1+v\tau')^2 + (v\kappa')^2 - (-\tau)^2)^{3/2}}.$$

Assume that $\bar{\alpha}$ is a unit speed timelike space curve of α given above. The ruled surface $\bar{\chi}(s,v)$ generated by the natural lift curve is

$$\bar{\chi}(s,v) = \bar{\alpha}(s) + v\bar{W}(s).$$

The spacelike Darboux vector of $\bar{\alpha}$ is

$$\bar{W} = \bar{\tau}\bar{T} + \bar{\kappa}\bar{B} = \left(\frac{-\kappa'\tau + \kappa\tau'}{\|W\|^2}\right) N + \frac{\kappa^2 - \tau^2}{\|W\|}\left(-\frac{\tau}{\|W\|}T - \frac{\kappa}{\|W\|}B\right).$$

Here, we get

$$\bar{W}' = A'N + (\kappa A - \tau')T + (\tau A - \kappa')B$$

where

$$A = \left(\frac{-\kappa'\tau + \kappa\tau'}{\|W\|^2}\right)$$

and

$$\bar{W}' = \left(\frac{-\kappa'\tau + \kappa\tau'}{\|W\|^2}\right)' N + \left(\kappa\left(\frac{-\kappa'\tau + \kappa\tau'}{\|W\|^2}\right) - \tau'\right)T + \left(\tau\left(\frac{-\kappa'\tau + \kappa\tau'}{\|W\|^2}\right) - \kappa'\right)B.$$

Also, the striction curve is obtained as

$$\mu = \frac{\kappa A'}{(A')^2 - A^2(\kappa^2 - \tau^2) + 2A(\kappa\tau' - \tau\kappa') + (\kappa')^2 - (\tau')^2}.$$

The distribution parameter is calculated by

$$\bar{P}_{\bar{W}} = 0.$$

Gaussian and the mean curvatures are

$$\bar{K}(s,v) = 0.$$

and

$$\bar{H}(s,v)=\frac{(\Psi)}{2(-v^2(\kappa A-\tau')^2+(\tau A-\kappa')^2+2v\kappa A'+A^2(v^2-\kappa^2)+\tau^2)^{3/2}}$$

where

$$\begin{aligned}\Psi &= -\left(\kappa^2+v(\kappa A'+(\kappa A-\tau')')\right)(\kappa(\kappa+vA')+v(\tau A-\kappa')A)\\ &+\left(\kappa'+v\left(A''+\kappa(\kappa A-\tau')-(\tau A-\kappa')\right)\right)\left(-\kappa v(\kappa A-\tau')+2v(\tau A-\kappa')\right)\\ &+\left(\kappa\tau+v(\tau A'+(\tau A-\kappa')')\right)\left(-Av(\kappa A-\tau')-\tau(\kappa+vA')\right)\end{aligned}$$

From the above calculations, we deduce that the ruled surfaces χ and $\bar{\chi}$ are developable.

ii) Assume that α is a unit speed spacelike space curve with a spacelike binormal. The ruled surface $\chi(s,v)$ is parametrized by

$$\chi(s,v) = \alpha(s) + vW(s).$$

After some basic calculations, we get the following properties:
The striction curve is

$$\lambda=\frac{\tau'}{(\tau')^2+(\kappa')^2}.$$

The distribution parameter is

$$P_W = 0.$$

Hence, the ruled surface is developable. Gaussian and mean curvatures are given as follows:

$$K(s,v) = 0$$

and

$$H(s,v)=\frac{\left(\kappa+v(\tau'\kappa-\kappa'\tau)\right)(-\kappa(1+v\tau')+v\kappa'\tau)}{2((1+v\tau')^2+(v\kappa')^2-(\tau)^2)^{3/2}}.$$

Assume that $\bar{\alpha}$ is a unit speed spacelike curve with a spacelike binormal of α. The ruled surface $\bar{\chi}(s,v)$ is parametrized by

$$\bar{\chi}(s,v)=\bar{\alpha}(s)+v\bar{W}(s)$$

The Darboux vector is

$$\bar{W}=\bar{\tau}\bar{T}-\bar{\kappa}\bar{B} =\left(\frac{\kappa'\tau-\kappa\tau'}{\|W\|^2}\right)N-\frac{\kappa^2+\tau^2}{\|W\|}\left(\frac{\tau}{\|W\|}T-\frac{\kappa}{\|W\|}B\right).$$

So, we calculate

$$\bar{W}'=A'N+(\kappa A-\tau')T+(\tau A+\kappa')B$$

where

$$A=\left(\frac{\kappa'\tau-\kappa\tau'}{\|W\|^2}\right)$$

and

$$A' = \frac{(-\kappa''\tau + \kappa\tau'')\|W\|^2 - 2\kappa\tau((\kappa')^2 - (\tau')^2) - 2\kappa'\tau'(\tau^2 - \kappa^2)}{(\|W\|^2)^2}.$$

The striction curve is

$$\mu = \frac{-\kappa A'}{-(A')^2 + A^2(\kappa^2 + \tau^2) - 2A(\kappa\tau' - \tau\kappa') + (\kappa')^2 + (\tau')^2}.$$

The distribution parameter is

$$\bar{P}_{\bar{W}} = \frac{-2\kappa(-\kappa'\tau + \kappa\tau')}{-(A')^2 + A^2(\kappa^2 + \tau^2) - 2A(\kappa\tau' - \tau\kappa') + (\kappa')^2 + (\tau')^2}.$$

The Gaussian curvature is

$$v^2A^2(\kappa^2 + \tau^2) + 2Av^2(\tau\kappa' + \kappa\tau') + v^2((\kappa')^2 + (\tau')^2) - \kappa^2 - 2\kappa vA' - v^2(A')^2 + (\kappa^2A^2) \ldots (\varrho).$$

Then we have

$$\bar{K}(s,v) = \frac{-4\kappa^2(-\kappa'\tau + \kappa\tau')^2}{(\varrho)^2}.$$

The mean curvature is calculated by

$$\bar{H}(s,v) = \frac{(\Theta)}{2(\Omega)^{3/2}}$$

where

$$\begin{aligned}\Theta = {} & -2\big((\kappa A)N - \tau T + \kappa B\big)(-\kappa A') \\ & -\big(\kappa^2 + v(\kappa A' + (\kappa A - \tau')')\big)(-v\kappa(\kappa + vA') + v(\tau A + \kappa')A) \\ & + \Big(\kappa' + v\big(A'' + \kappa(\kappa A - \tau') + (\tau A + \kappa')\big)\big(\kappa v(\kappa A - \tau') + v\tau(\tau A + \kappa')\big)\Big) \\ & + \big(\kappa\tau + v(\tau A' + (\tau A + \kappa')')\big)\big(-vA(\kappa A - \tau') + \tau(\kappa + vA')\big)\end{aligned}$$

and

$$\Omega = v^2A^2(\kappa^2 + \tau^2) + 2Av^2(\tau\kappa' + \kappa\tau') + v^2((\kappa')^2 + (\tau')^2) - \kappa^2 - 2\kappa vA' - v^2(A')^2).$$

iii) Assume that α is a unit speed spacelike curve with a timelike binormal. The ruled surface χ(s,v) is parametrized by

$$\chi(s,v) = \alpha(s) + vW(s).$$

The striction curve is calculated as

$$\lambda = \frac{\tau'}{(\kappa')^2 - (\tau')^2}.$$

The distribution parameter is obtained by

$$P_W = 0.$$

Hence, we deduce that the ruled surface is developable. The Gaussian and mean curvatures are obtained as

$$K(s,v) = 0$$

and

$$H(s,v) = \frac{\left(\kappa + v(-\tau'\kappa + \kappa'\tau)\right)(\kappa(1 - v\tau') + v\kappa'\tau)}{2((1 - v\tau')^2 - (v\kappa')^2 - (-\tau)^2)^{3/2}}.$$

Assume that $\bar{\alpha}$ is a unit speed spacelike curve with a timelike binormal. The ruled surface $\bar{\chi}(s,v)$ is given by

$$\bar{\chi}(s,v) = \bar{\alpha}(s) + v\bar{W}(s)$$

Here, Darboux vector is given as

$$\bar{W} = -\bar{\tau}\bar{T} + \bar{\kappa}\bar{B} = -\left(\frac{-\kappa'\tau + \kappa\tau'}{\|W\|^2}\right) N + \frac{\kappa^2 + \tau^2}{\|W\|}\left(\frac{\tau}{\|W\|}T + \frac{\kappa}{\|W\|}B\right).$$

Then, we obtain

$$\bar{W}' = (A' + 2\tau\kappa C)N + (-\kappa A + \tau C')T + (\tau A + \kappa C')B$$

where

$$A = \left(\frac{-\kappa'\tau + \kappa\tau'}{\|W\|^2}\right), C = \left(\frac{\kappa^2 + \tau^2}{\|W\|^2}\right)$$

and

$$A' = \frac{(-\kappa''\tau + \kappa\tau'')\|W\|^2 - 2\kappa'\tau'(\tau^2 + \kappa^2) + 2\kappa\tau\left((\kappa')^2 + (\tau')^2\right)}{(\|W\|^2)^2},$$

$$C' = \frac{4\kappa\tau(\kappa'\tau - \kappa\tau')}{(\tau^2 - \kappa^2)^2}.$$

After some calculations, we obtain the some elementary results as follows:
The striction curve is

$$\mu = \frac{\kappa(A' + 2\tau\kappa C)}{(A' + 2\tau\kappa C) + (\kappa^2 - \tau^2)(A^2 - (C')^2)}.$$

The distribution parameter is denoted as

$$\bar{P}_{\bar{W}} = \frac{-\kappa CA(\tau^2 + \kappa^2)}{(A' + 2\tau\kappa C) + (\kappa^2 - \tau^2)(A^2 - (C')^2)}.$$

The Gaussian and the mean curvatures are

$$\bar{K}(s,v) = \frac{-\kappa^2 C^2 A^2 (\tau^2 + \kappa^2)^2}{(\varsigma)^2}$$

where

$$\varsigma = \kappa^2 + 2\kappa v(A' + 2\tau\kappa C)v^2\left((A')^2 + 2A'\tau\kappa C + 4\tau^2\kappa^2 C^2 + A^2(\kappa^2 - \tau^2) - 4\kappa A\tau C' + (C')^2(\kappa^2 + \tau^2)\right) - \kappa^2 A^2.$$

$$\overline{H}(s,v)=\frac{(\mathfrak{J})}{2(\mathfrak{S})^{3/2}}.$$

Here

$$\mathfrak{J}=-2\big((\kappa A)N+2CT+\kappa CB\big)\big(-\kappa CA(\kappa^2+\tau^2)\big)-$$
$$\Big(\big(\kappa^2+v(-\kappa(A'+2\tau\kappa C)\big)+(-\kappa A+\tau C')'\left(-\big(\kappa+v(A'+2\kappa\tau C)\big)\right)\kappa Cv(\tau A+\kappa C')A'\Big)$$
$$+\Big((\kappa'+v(A'+2\tau\kappa C)')+(-\kappa a+\tau C')+(\tau A+\kappa C')\big(v\kappa C(-\kappa A+\tau C')\big)$$
$$-v\big((\tau A+\kappa C')\big)\tau C\Big)$$
$$+(\kappa\tau^2(A'+2\tau\kappa C)+(\tau A+\kappa C')')\big(-vA'(-\kappa A+\tau C')+\kappa+v\tau C(A'+2\kappa\tau C)\big),$$

$$\mathfrak{S}=\kappa^2+2\kappa v(A'+2\tau\kappa C)+v^2((A')^2+2A'\tau\kappa C+4\tau^2\kappa^2C^2+A^2(\kappa^2-\tau^2)-4\kappa A\tau C')+(C')^2(\kappa^2+\tau^2)-\big(\kappa^2A^2+C^2(\tau^2-\kappa^2)\big).$$

Now, we will consider some examples to verify the obtained results given above:

Example 3.1. Assume that $\alpha(s)=\left(\frac{1}{\sqrt{3}}coss,1-sins,\frac{-2}{\sqrt{3}}coss\right)$ is spacelike curve with timelike binormal. Frenet operators of α are calculated as

$$T(s)=\left(-\frac{1}{\sqrt{3}}sins,-coss,\frac{2}{\sqrt{3}}sins\right),$$

$$N(s)=\left(-\frac{1}{\sqrt{3}}coss,sins,\frac{2}{\sqrt{3}}coss\right),$$

$$B(s)=\left(\frac{-2}{\sqrt{3}},0,\frac{1}{\sqrt{3}}\right),$$

$$\kappa(s)=1,$$

$$\tau(s)=0.$$

Hence, Darboux vector is $W=-\tau T+\kappa B=\left(\frac{-2}{\sqrt{3}},0,\frac{1}{\sqrt{3}}\right)$. The ruled surface generated by α (Fig. 1) is

$$\chi(s,v)=\left(\frac{1}{\sqrt{3}}coss,1-sins,\frac{-2}{\sqrt{3}}coss\right)+v\left(\frac{-2}{\sqrt{3}},0,\frac{1}{\sqrt{3}}\right)$$

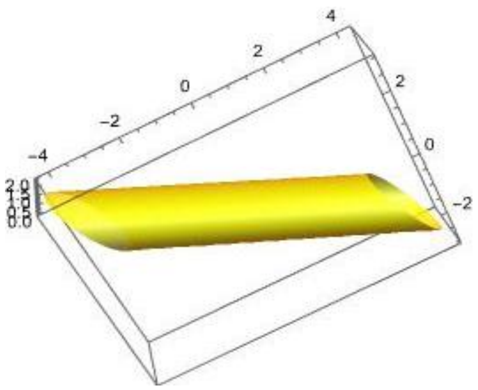

Figure 1. Timelike ruled surface generated by *α*.

The normal vector of χ is

$$N=\frac{\chi_s\times\chi_v}{\|\chi_s\times\chi_v\|}=\left(\frac{-1}{\sqrt{3}}coss,sins,\frac{2}{\sqrt{3}}coss\right)$$

The shape operator, Gaussian and mean curvatures, first and second fundamental forms are given as follows:

$$S(\chi)=\begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix},$$

$$K(s)=0,$$

$$H(s)=-\frac{1}{2},$$

$$I(U,V)=\langle U,V\rangle,$$

$$II(U,V)=\langle S(U),V\rangle=-\tfrac{1}{2}\langle U,V\rangle.$$

The natural lift curve of α is

$$\bar{\alpha}(s)=\left(-\frac{1}{\sqrt{3}}sins,-coss,\frac{2}{\sqrt{3}}sins\right)$$

Frenet operators of $\overline{\alpha}$ are given as

$$\bar{T}(s)=\left(-\frac{1}{\sqrt{3}}coss,sins,\frac{2}{\sqrt{3}}coss\right),$$

$$\bar{N}(s)=\left(\frac{1}{\sqrt{3}}sins,coss,-\frac{2}{\sqrt{3}}sinss\right),$$

$$\bar{B}(s)=\left(\frac{-2}{\sqrt{3}},0,\frac{1}{\sqrt{3}}\right),$$

$$\bar{\kappa}(s)=1,$$

$$\bar{\tau}(s)=0.$$

Thus, Darboux vector is $\overline{W}=-\overline{\tau}\overline{T}+\overline{\kappa}\overline{B}=\left(\frac{-2}{\sqrt{3}},0,\frac{1}{\sqrt{3}}\right)$. The ruled surface generated by $\overline{\alpha}$ (Fig. 2) is

$$\bar{\chi}(s,v)=\left(-\frac{1}{\sqrt{3}}sins,-coss,\frac{2}{\sqrt{3}}sinss\right)+v\left(\frac{-2}{\sqrt{3}},0,\frac{1}{\sqrt{3}}\right)$$

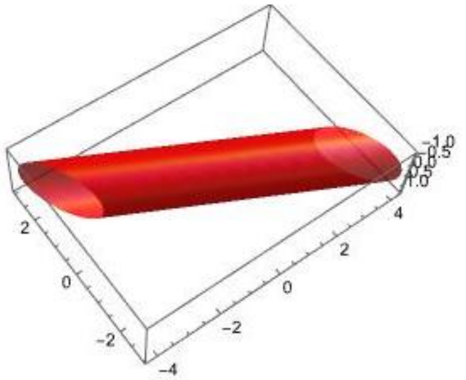

Figure 2. Timelike ruled surface generated by $\overline{\alpha}$.

The normal vector of χ is

$$\overline{N} = \frac{\overline{\chi}_s \times \overline{\chi}_v}{\|\overline{\chi}_s \times \overline{\chi}_v\|} = \left(\frac{1}{\sqrt{3}} sins, coss, -\frac{2}{\sqrt{3}} sins\right).$$

The shape operator, Gaussian and mean curvatures, first and second fundamental forms are calculated as follows:

$$S(\chi) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$K(s) = 0,$$

$$H(s) = -\frac{1}{2},$$

$$I(U, V) = \langle U, V \rangle,$$

$$II(U, V) = \langle S(U), V \rangle = -\frac{1}{2} \langle U, V \rangle.$$

Example 3.2. Assume that $\alpha(s) = \left(\frac{1}{\sqrt{5}} sins, 1 + coss, \frac{\sqrt{6}}{\sqrt{5}} sins\right)$ is spacelike curve with timelike binormal. Frenet operators are given as

$$T(s) = \left(\frac{1}{\sqrt{5}} coss, -sins, \frac{\sqrt{6}}{\sqrt{5}} coss\right),$$

$$N(s) = \left(-\frac{1}{\sqrt{5}} sins, -coss, -\frac{\sqrt{6}}{\sqrt{5}} sins\right),$$

$$B(s) = \left(\frac{\sqrt{6}}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right),$$

$$\kappa(s) = 1,$$

$$\tau(s) = 0.$$

Hence, Darboux vector is $W = -\tau T + \kappa B = \left(\frac{\sqrt{6}}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)$. The ruled surface generated by α (Fig. 3) is

$$\chi(s,v) = \left(\frac{1}{\sqrt{5}}sins, 1 + coss, \frac{\sqrt{6}}{\sqrt{5}}sins\right) + v\left(\frac{\sqrt{6}}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)$$

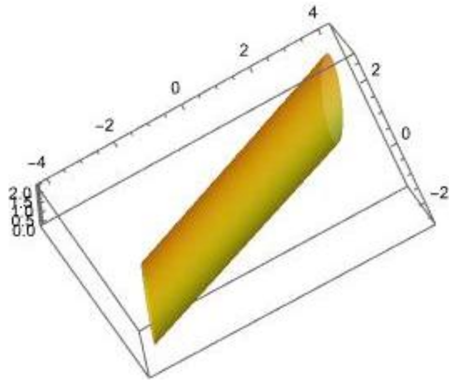

Figure 3. Timelike ruled surface generated by α.

The normal vector of χ is

$$N = \frac{\chi_s \times \chi_v}{\|\chi_s \times \chi_v\|} = \left(\frac{-1}{\sqrt{5}}sins, -coss, -\frac{\sqrt{6}}{\sqrt{5}}sins\right)$$

The shape operator, Gaussian and mean curvatures, first and second fundamental forms are given as follows:

$$S(\chi) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$K(s) = 0,$$

$$H(s) = -\frac{1}{2},$$

$$I(U,V) = \langle U, V\rangle,$$

$$II(U,V) = \langle S(U), V\rangle = -\tfrac{1}{2}\langle U, V\rangle.$$

The natural lift curve of α is

$$\bar{\alpha}(s) = \left(\frac{1}{\sqrt{5}}coss, -sins, \frac{\sqrt{6}}{\sqrt{5}}coss\right)$$

Similarly, Frenet operators are

$$\bar{T}(s) = \left(-\frac{1}{\sqrt{5}}sins, -coss, -\frac{\sqrt{6}}{\sqrt{5}}coss\right),$$

$$\bar{N}(s) = \left(-\frac{1}{\sqrt{5}}coss, sins, -\frac{\sqrt{6}}{\sqrt{5}}coss\right),$$

$$\bar{B}(s) = \left(\frac{\sqrt{6}}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right),$$

$$\bar{\kappa}(s) = 1,$$

$$\bar{\tau}(s) = 0.$$

Thus, Darboux vector is $\bar{W} = -\bar{\tau}\bar{T} + \bar{\kappa}\bar{B} = \left(\frac{\sqrt{6}}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)$. The ruled surface generated by $\bar{\alpha}$ (Fig. 4) is

$$\bar{\chi}(s,v) = \left(\frac{1}{\sqrt{5}}coss, -sins, \frac{\sqrt{6}}{\sqrt{5}}coss\right) + v\left(\frac{\sqrt{6}}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right).$$

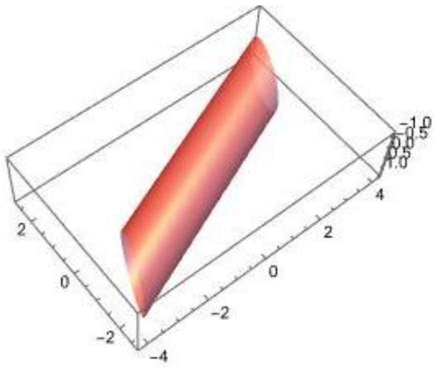

Figure 4. Timelike ruled surface generated by $\bar{\alpha}$.

The normal vector of $\bar{\chi}$ is

$$\bar{N} = \frac{\bar{\chi}_s \times \bar{\chi}_v}{\|\bar{\chi}_s \times \bar{\chi}_v\|} = \left(-\frac{1}{\sqrt{5}}coss, sins, -\frac{\sqrt{6}}{\sqrt{5}}coss\right).$$

The shape operator, Gaussian and mean curvatures, first and second fundamental forms are calculated as

$$S(\chi) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$K(s) = 0,$$

$$H(s) = -\frac{1}{2},$$

$$I(U,V) = \langle U,V \rangle,$$

$$II(U,V) = \langle S(U),V \rangle = -\frac{1}{2}\langle U,V \rangle.$$

4. CONCLUSION

Study mapping plays an important role to construct correspondence between Euclidean space and dual space. In this paper, the isomorphism between the subset of unit 2-sphere, $T\bar{M}$ and unit dual sphere, DS^2 is constructed. Then we obtain ruled surfaces generated by the natural lift curves in $\mathbb{R}^3$. Using these results, it is possible to model motions by using the natural lift curves on the subset of unit 2-sphere instead of the unit dual sphere. Natural lift curves on $T\bar{M}$ can also be used to modeling motions in $\mathbb{R}^3$.

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