ORIGINAL PAPER

SOME NEW EXACT SOLUTIONS OF A THREE-DIMENSIONAL KUDRYASHOV–SINELSHCHIKOV EQUATION IN THE BUBBLY LIQUID

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Abstract. *Fractional calculus is of vital importance and its significance is increased a lot since last many years. In this article, fractional derivatives in Caputo's logic is used to construct exact solutions for three-dimensional Kudryashov–Sinelshchikov (KS) equation of fractional order, which describes the convey of nonlinear waves in a bubbly liquid. A comprehensive fractional complex transform is appropriately used to renovate this equation to ordinary differential equation which afterward resulted into number of exact solutions via Exp function method. The efficiency of under study method is checked by computational work and graphical representation. We develop the corresponding solutions containing the periodic, solitary wave solutions for* $\chi = 0$*, and kink wave solutions for* $\chi \neq 0$*. Effects of* χ *on the dilation factor, steepness and velocity of the kink wave solutions are discussed. With the increasing magnitude of* χ*, the dilation factor and steepness of the kink wave solutions increase, while the velocity of the kink wave solutions first decreases and then increases.*

Keywords: *Bubbly liquid, Kudryashov–Sinelshchikov equation, fractional calculus, solitons.*

1. INTRODUCTION

The class of fractional calculus is one of the most expedient classes of fractional differential equation which viewed as generalized differential equations [1]. In the intellect that, much of the theory and, hence, applications of differential equation can be extensive smoothly to fractional differential equations with the same taste and character of the empire of differential equation. The seeds of fractional calculus (that is, the theory of integrals and derivatives of any arbitrary real or complex order) were planted over 300 years ago. Since then, many researchers have contributed to this field. Freshly, it has twisted out those differential equations involving derivatives of non-integer [2]. For example, the nonlinear oscillation of earthquakes can be modeled with fractional derivatives [3]. There has been some crack to solve linear problems with multiple fractional derivatives (the so -called multi-term equations) [3, 4]. Not much work has been done on nonlinear problems and only a few numerical schemes have been proposed for solving nonlinear fractional differential equations. More recently, applications have included classes of nonlinear equation with multi-order fractional derivatives. We apply a

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generalized fractional complex transform [5-9] to convert fractional order differential equation to ordinary differential equation. Lastly, we obtain exact solutions for it by using a narrative practice [10, 11] called exp-function method, to obtain generalized solitary solutions and periodic solutions. Mohyud-Din [12-15] extended the same for nonlinear physical problems plus higher-order BVPs; Oziz [16] tried this novel approach for Fisher's equation; Wu et. al. [17, 18] for the extension of solitary, periodic and compacton-like solutions; Yusufoglu [19] for MBBN equations, Zhang [20] for high-dimensional nonlinear evolutions; Zhu [21, 22] for the Hybrid-Lattice system and discrete m KdV lattice; Kudryashov [23] for exact soliton solutions of the generalized evolution equation of wave dynamics; Momani [24] for an explicit and numerical solutions of the fractional KdV equation; It is to be highlighted that Ebaid [25] proved that $c = d$ and $p = q$ are the only relations that can be obtained by applying exp-function method to any nonlinear ordinary differential equation. Mainly scientific problems and phenomena in different fields of sciences and engineering occur nonlinearly. This method has been effectively and accurately shown to solve a large class of nonlinear problems. The solution procedure of this method, with the aid of Maple, is of sheer simplicity and this method can easily extended to other kinds of nonlinear evolution equations. In this research, we use the Exp-function method next to with generalized fractional complex transform to obtain new solitary wave solutions for the three-dimensional KS equation [26]. KS equation has been measured to illustrate the physical characteristics of nonlinear waves in a bubbly liquid[27,28], where χ represent the density of the bubbly liquid, the scalar quantity χ depends on the kinematic viscosity of the bubbly liquid *x*, *y* and *z* are the scaled space coordinates, *t* is the scaled time coordinate and the subscripts denote the partial derivatives.

2. PRELIMINARIES AND NOTATION

In this segment, we give some fundamental definitions and properties of the fractional calculus theory which will be used additional in this work. For the finite derivative in $[a,b]$ we define the following fractional integral and derivatives.

Definition 1. A real function $f(x), x > 0,$ is said to be in the space $C\mu, \mu \in R,$ If there exists a real number $(p > \mu)$ such that $f(x) = x^p f_1(x),$ where $f_1(x) = C(0,\infty)$ and it is said to be in the space $C_\mu^m \mu$ if $f^m \in C\mu, m \in N.$

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C\mu, \mu \geq -1,$ is defined as

$$J^\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \alpha > 0, x > 0, J^0(x) = f(x). \tag{1}$$

Some properties of the operator J^a are discussed in the following

For $f \in C\mu, \mu \geq -1,, \alpha, \beta \geq 0$ and $\gamma \geq -1$

$$J^{\alpha} J^{\beta} f(x) = J^{\alpha+B} f(x),$$
$$J^{\alpha} J^{B} f(x) = J^{B} J^{\alpha} f(x), \tag{2}$$
$$J^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

The Riemann--Liouville derivative has convinced disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator suggested by M. Caputo in his struggle on the theory of viscoelasticity [2].

Definition 3. For m to be the smallest integer that exceeds, α the Caputo time fractional derivative operator of order $\alpha > 0$ and defined as

$$D_t^{\alpha} f(x) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(m-a)} \int_0^x (x-t)^{m-\alpha-1} f(t)dt, -1 << m, m \in N \\ \dfrac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}, \alpha = m \end{cases} \tag{3}$$

3. CHAIN RULE FOR FRACTIONAL CALCULUS AND FRACTIONAL COMPLEX TRANSFORM

In [3-6], the authors used the following chain rule $\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial u}{\partial s}\frac{\partial^{\alpha} s}{\partial t^{\alpha}}$ to convert a fractional differential equation with Jumarie's modification of Riemann-Liouville derivative into its classical differential partner. In [8], the authors showed that this chain rule is invalid and show following relation [8].

$$D_t^{a} u = \sigma_t^{'} \frac{du}{d\eta} D_t^{a} \eta \text{ and } D_x^{a} u = \sigma_x^{'} \frac{du}{d\eta} D_x^{a} \eta,$$

To determine σ_s we consider a special case as follows

$$s = t^{\alpha} \text{ and } u = s^{m}$$

and we have

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\Gamma(1+m\alpha) t^{m\alpha-\alpha}}{\Gamma(1+m\alpha-\alpha)} = \sigma.\frac{\partial u}{\partial s} = \sigma m t^{m\alpha-\alpha}.$$

Thus we can calculate σ_s as

$$\sigma_s = \frac{\Gamma(1+m\alpha)}{\Gamma(1+m\alpha-\alpha)}.$$

Other fractional indexes $\left(\sigma_x^{'},\sigma_y^{'},\sigma_z^{'}\right)$ can settle on in similar way. Li and He [2-8] proposed following fractional complex transform for converting fractional differential equations into ordinary differential equations, so that all analytical methods for advanced calculus can be easily applied to fractional calculus.

$$u(x,t)=u(\eta),\eta=\frac{kx^{\beta}}{\Gamma(1+\beta)}+\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}+\frac{Mx^{\gamma}}{\Gamma(1+\gamma)}, \tag{4}$$

Where k,ω and M are constants.

4. EXP-FUNCTION TECHNIQUE [29-32]

We consider the general nonlinear fractional partial differential equation of the type

$$P\left(u,u_t,u_x,u_{xx},\ldots,D_t^{\alpha}u,D_x^{\alpha}u,D_{xx}^{\alpha}u,\ldots\right)=0, \qquad 0<\alpha\leq 1. \tag{5}$$

Where $D_t^{\alpha}u,D_x^{\alpha}u,D_{xx}^{\alpha}u$ are the modified Riemann-Liouville derivative of u w.r.t. t,x,xx correspondingly.

Using a transformation [4] nonlinear ordinary differential equation in the general form is

$$F\left(\phi,\phi',\phi'',\phi''',\ldots\right)=0 \tag{6}$$

Where prime represents differentiation w.r.t η

Permitting to Exp-function method, we take up that the solitary wave solutions can be articulated in the subsequent procedure

$$\phi(\eta)=\frac{\sum_{i=-c}^{d}a_i e^{i\eta}}{\sum_{j=-p}^{q}b_j e^{j\eta}} \tag{7}$$

In last equation c,d and p,q are the positive integers and need to be calculated, a_i and b_j are constants. Equation (10) can be expressed in the subsequent corresponding way

$$\phi(\eta)=\frac{a_c e^{c\eta}+\cdots+a_{-d} e^{-d\eta}}{b_p e^{p\eta}+\cdots+a_{-q} e^{-q\eta}} \tag{8}$$

The outcome of equivalent formulation is an imperative and vital analytic solutions of the governing differential equation. Calculating values by using [25], finally results in

$$p=c,q=d \tag{9}$$

5. SOLUTION PROCEDURE

Consider the following Kudryashov–Sinelshchikov (KS) equation of fraction order.

$$\left(D_t^{\alpha}u + uu_x + u_{xxx} - \chi u_{xx}\right)_x + \frac{1}{2}\left(u_{yy} + u_{zz}\right) = 0 \tag{10}$$

Using (4) equation (10) can be converted to an ordinary differential equation

$$cu'' + \frac{1}{2}\left(u^2\right)'' + u'''' - \chi u''' + u'' = 0, \tag{11}$$

Where the prime denotes the differentiation with respect to η. The solution of the equation (11) can be expressed in the form, equation (8). To determine the value of c, d and p, q, by using [25], we have

$$p = c, q = d. \tag{12}$$

Case I. We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set
$p = c = 1$ and $q = d = 1$

$$u(\eta) = \frac{a_1 e^{\eta} + a_0 + a_{-1} e^{-\eta}}{b_1 e^{\eta} + b_0 + b_{-1} e^{-\eta}} \tag{13}$$

Substituting equation (13) into equation (11), we have

$$\frac{1}{A}\left[c_4 e^{4\eta} + c_3 e^{3\eta} + c_2 e^{2\eta} + c_1 e^{\eta} + c_0 + c_{-1} e^{-\eta} + c_{-2} e^{-2\eta} + c_{-3} e^{-3\eta} + c_{-4} e^{-4\eta}\right] = 0 \tag{14}$$

Where $A = \left(b_1 e^{\eta} + b_0 + b_{-1} e^{-\eta}\right)^4$ and c_i are the constants acquired by Maple 17. Associating the coefficients of $e^{n\eta}$ equal to zero, we gain

$$\left[c_4 = 0, c_3 = 0, c_2 = 0, c_1 = 0, c_0 = 0, c_{-1} = 0, c_{-2} = 0, c_{-3} = 0, c_{-4} = 0\right] \tag{15}$$

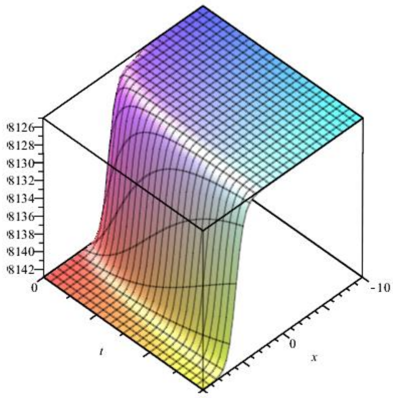
The solution set placate the given equation (10) are given below

1st Solution set

$$\left\{a_{-1} = 4b_{-1}\chi - 2(\sigma.\omega)b_{-1} - 10b_{-1}, a_0 = 0, a_1 = 4b_1\chi - 2(\sigma.\omega)b_1 - 10b_1, b_{-1} = b_{-1}, b_0 = 0, b_1 = b_1\right\}$$

We, therefore, find the subsequent generalized solitary solution $u(x,t)$ of equation (10)

$$u(x,t)=\frac{\left(4b_{-1}\chi-2(\sigma.\omega)-10b_{-1}\right)e^{-x+\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}}+\left(4b_{1}\chi-2(\sigma.\omega)b_{1}-10b_{1}\right)e^{x-\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}}}{b_{-1}e^{-x+\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}}+b_{1}e^{x-\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}}} \tag{16}$$



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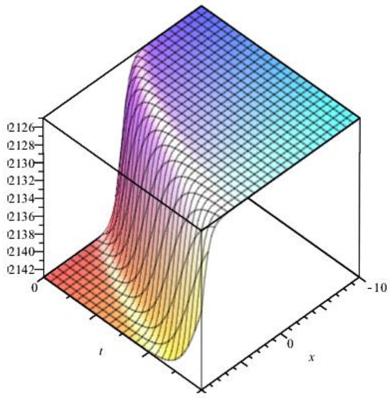

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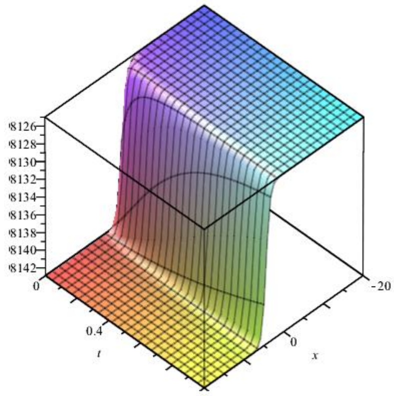

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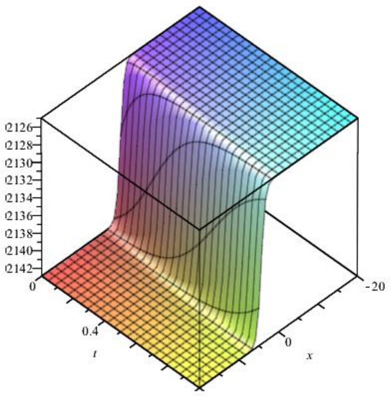

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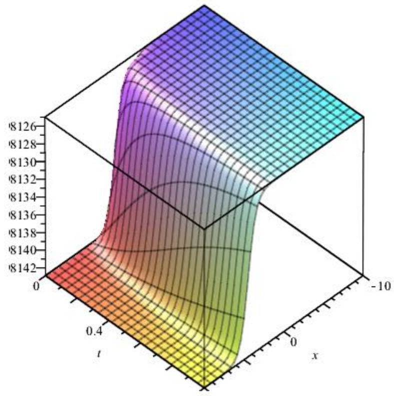

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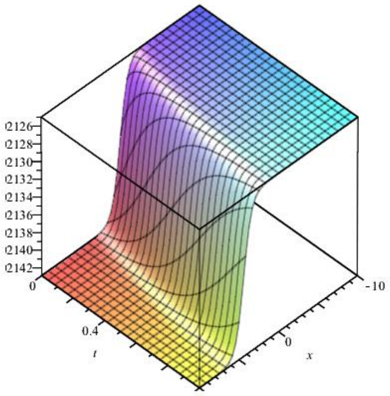

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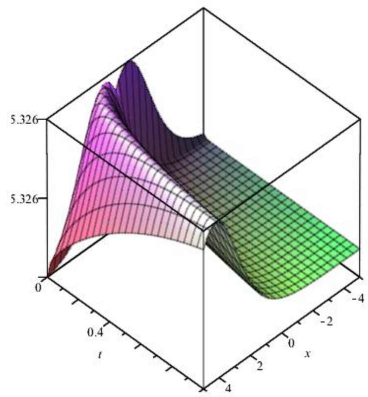
2nd Solution set

$$\left\{a_{-1}=b_{-1}\chi-\sigma.\omega b_{-1}-2b_{-1},a_{0}=b_{01}\chi-\sigma.\omega b_{0}-2b_{0},a_{1}=(\chi-\sigma.\omega-2)b_{1},b_{-1}=b_{-1},b_{0}=b_{0},b_{1}=b_{1}\right\}$$

We, thus, observed the following generalized solitary solution $u(x,t)$ of equation (10)

$$u(x,t)=\frac{\left(b_{-1}\chi-\sigma.\omega b_{-1}-2b_{-1}\right)e^{-x-y-z+\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}}+b_{0}\chi-\sigma.\omega b_{0}-2b_{0}+\left(\chi-\sigma.\omega-2\right)b_{1}e^{x+y+z-\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}}}{b_{-1}e^{-x-y-z+\frac{\omega t^{\alpha}}{\Gamma(1+a)}}+b_{0}+b_{1}e^{x+y+z-\frac{\omega t^{\alpha}}{\Gamma(1+a)}}} \tag{17}$$



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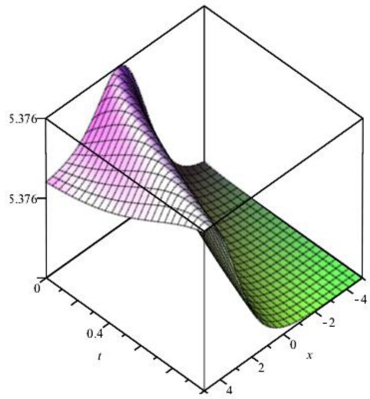

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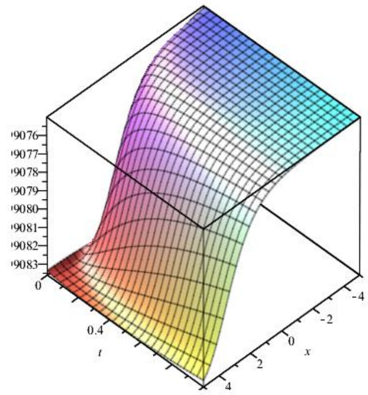

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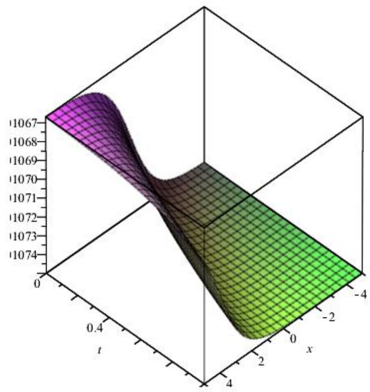

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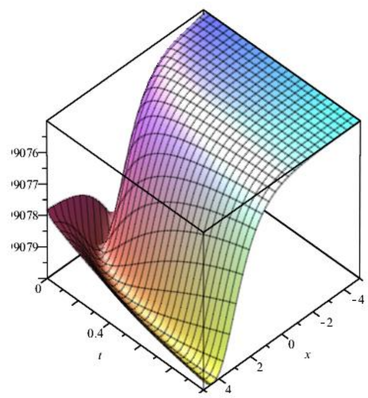

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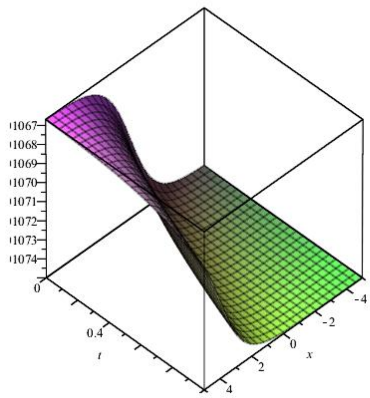


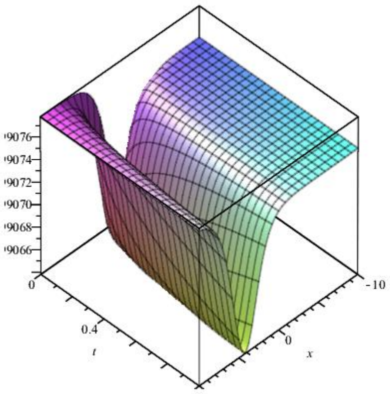
$\chi\angle 0,\alpha=.75$

3rd Solution set

$$\left\{a_{-1}=(\chi+\sigma.\omega+2)b_{-1}, a_0=b_0\chi+\sigma.\omega b_0+2b_0, a_1=b_1\chi+\sigma.\omega b_1, b_{-1}=b_{-1}, b_0=b_0, b_1=b_1\right\}$$

We, hence, get the generalized solitary solution $u(x,t)$ of equation (10)

$$u(x,t)=\frac{(\chi+\sigma.\omega+2)b_{-1}e^{-x-y-z+\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}}+b_0\chi+\sigma.\omega b_0+2b_0+(b_1\chi+\sigma.\omega b_1+2b_1)e^{x+y+z-\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}}}{b_{-1}e^{-x-y-z+\frac{\omega t^{\alpha}}{\Gamma(1+a)}}+b_0+b_1e^{x+y+-\frac{\omega t^{\alpha}}{\Gamma(1+a)}}} \quad (18)$$



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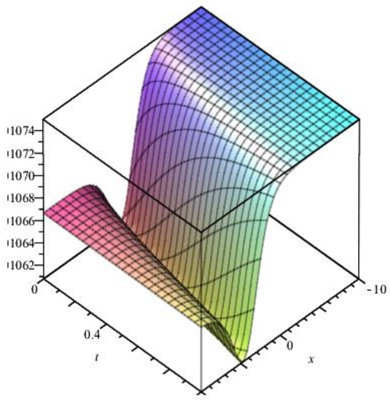

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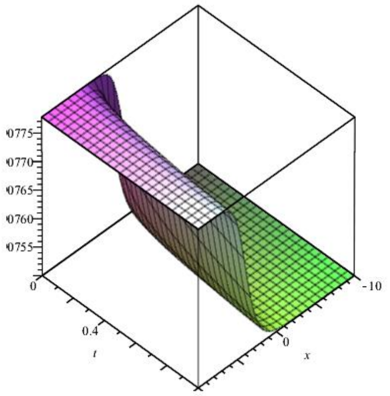

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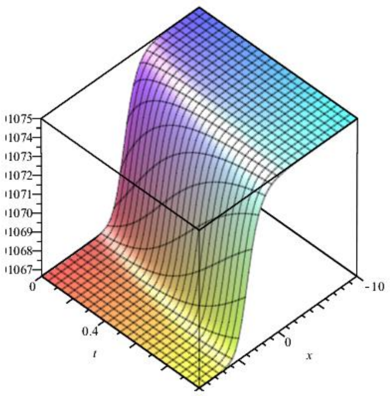

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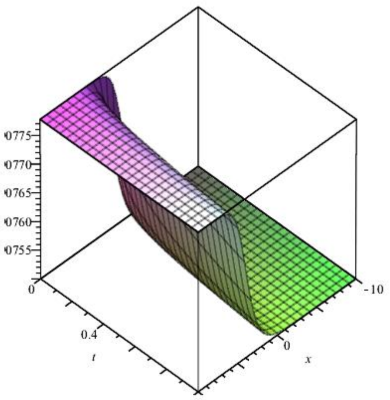

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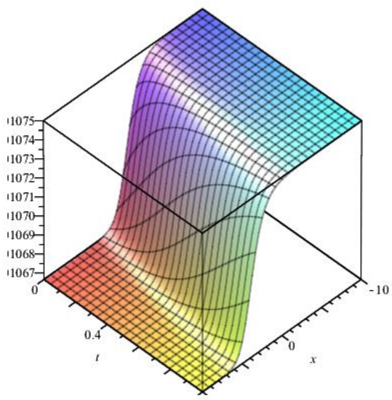

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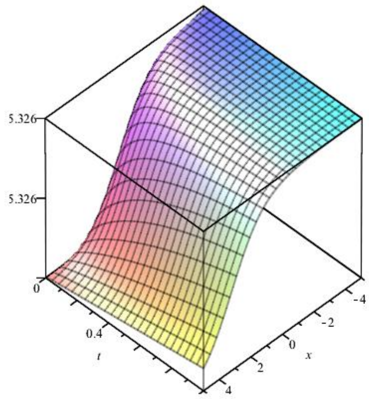
4^{th} Solution set

$$\left\{a_{-1}=0, a_0=\left(b_0\chi-\sigma.\omega b_0-2b_0\right), a_1=b_1\chi-\sigma.\omega b_1-2b_1, b_{-1}=0, b_0=b_0, b_1=b_1\right\}$$

Constructing the following soliton type solution $u(x,t)$ of equation (10)

$$u(x,t)=\frac{\left(b_0\chi-\sigma.\omega b_0-2b_0\right)+\left(b_1\chi-\sigma.\omega b_1-2b_1\right)e^{x+y+z-\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}}}{b_0+b_1e^{x+y+z-\frac{\omega t^{\alpha}}{\Gamma(1+a)}}} \tag{19}$$



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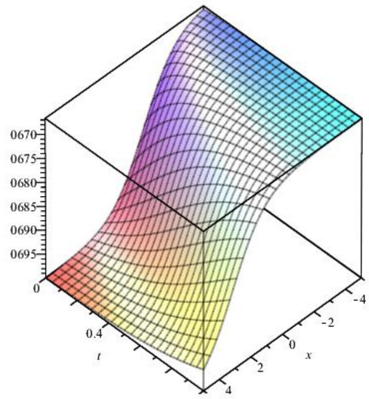

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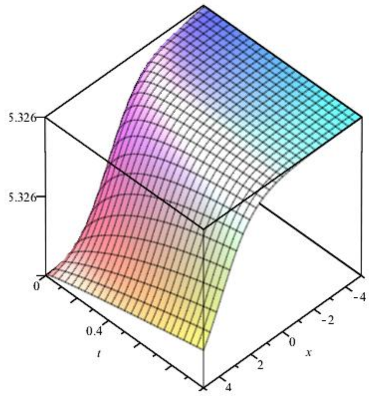

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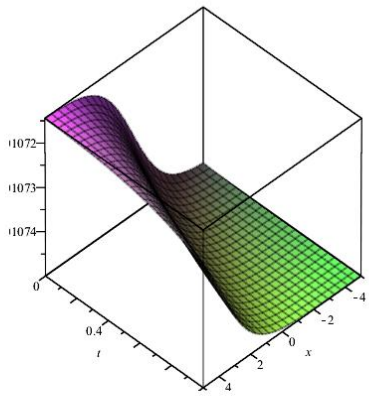

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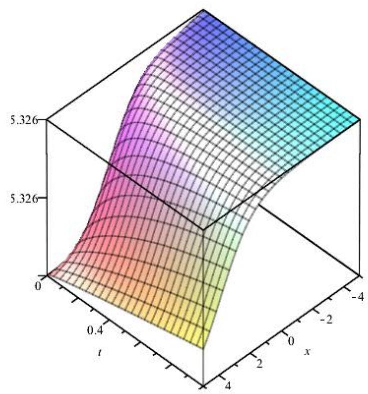

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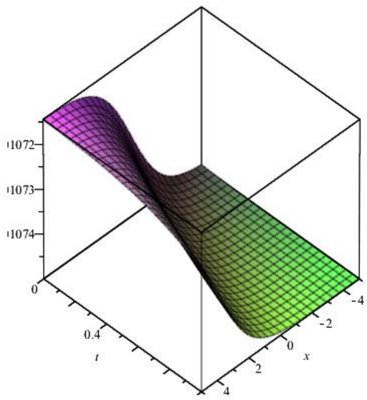


$\chi\angle 0, \alpha=.75$

Case II. By taking $p=c=2$ and $q=d=1$ then the solution, equation (10) becomes

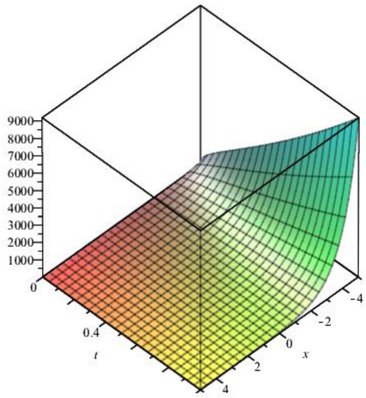
$$u(\eta)=\frac{a_2 e^{2\eta}+a_1 e^{\eta}+a_0+a_{-1} e^{-\eta}}{b_2 e^{2\eta}+b_1 e^{\eta}+b_0+b_{-1} e^{-\eta}} \tag{20}$$

Solving in previous manner, we attain

$$\{\omega=\omega, a_{-1}=0, a_0=0, a_1=a_1, a_2=\sigma.\omega b_2+\chi b_2+2b_2, b_{-1}=0, b_0=0, b_1=0, b_2=b_2\},$$

Hence we acquire the general solitary wave solution of equation (10)

$$u(x,t)=\left\{\frac{a_1 e^{x+y+z-\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}}+(\sigma.\omega b_2+\chi b_2+2b_2)e^{2x+2y+2z-\frac{2\omega t^{\alpha}}{\Gamma(1+\alpha)}}}{b_0 e^{2x+2y+2z-\frac{2\omega t^{\alpha}}{\Gamma(1+a)}}}\right\} \tag{21}$$



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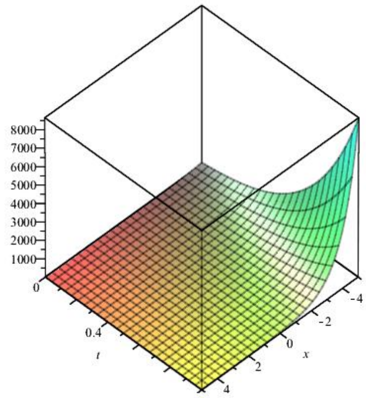

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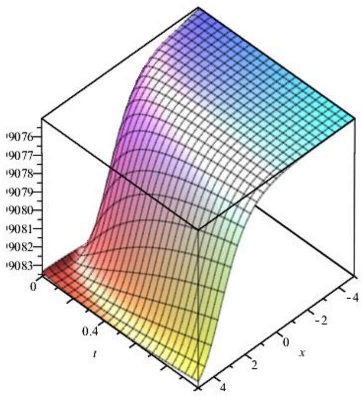

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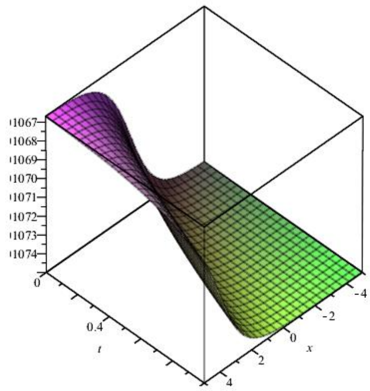

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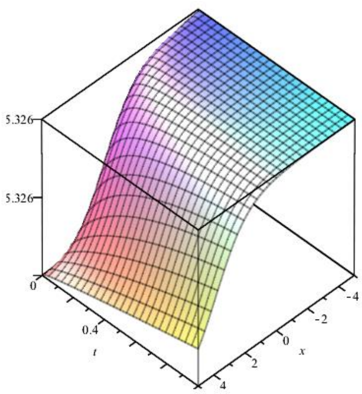

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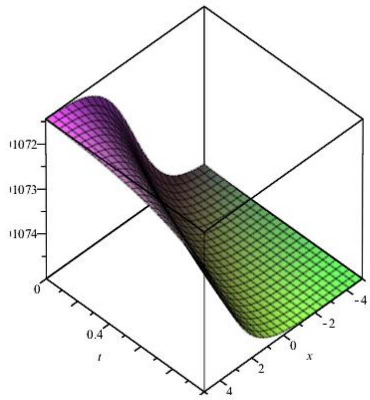

$\chi\angle 0, \alpha=.75$

In both cases, for different choices of c, p, d and q we get the same soliton solutions which clearly illustrate that final solution does not strongly depends upon these parameters.

6. CONCLUSION

In this work periodic and kink wave solutions of nonlinear Kudryashov–Sinelshchikov (KS) equation of fractional order is obtained successfully by making use of Exp-function method. The steadfastness of understudy method is wholly sustained by the computation work, the subsequent results and graphical representations. It is perceived that Exp-function technique is much handy to affect and is very suitable for obtaining novel type solutions of many physical problems [33] of fractional orders. One important finding is that by using Exp-fucntion method, we can conveniently obtain travelling wave solutions of different nonlinear problems whose solutions cannot be obtained by other classical techniques.

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