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HOW ASYMPTOTIC SERIES HELP US TO FIND BOUNDS FOR SOME EXPRESSIONS

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Abstract. *The aim of this article is to show how asymptotic series theory can be used to establish bounds for some expressions. For sake of explicitness, we discuss two problems posed by N. Schaumberger and M. S. Klamkin.*

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1. INTRODUCTION

Norman Schaumberger [1] proposed the following double inequality:

$$\frac{n^2+n}{n^2+n+1}<\left(\frac{n}{n+1}\right)^{1/(n+1)}<\frac{n^2+2n}{n^2+2n+1} \quad (n=1,2,3,\dots) \tag{1}$$

Several solutions were submitted, some using the monotonicity of the sequences $(1+1/n)^n$ and $(1+1/n)^{n+1}$ (which are increasing, respective decreasing). A complete proof presented by Donald Batman, M.I.T. Lincoln Laboratory, Lexington, MA can be found in [2].

We ask a natural question, namely which of the approximations

$$\left(\frac{n}{n+1}\right)^{1/(n+1)} \sim \frac{n^2+n}{n^2+n+1} \tag{2}$$

or

$$\left(\frac{n}{n+1}\right)^{1/(n+1)} \sim \frac{n^2+2n}{n^2+2n+1} \tag{3}$$

is better, as n approaches infinity?

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Moroever, in order to find the best approximation of the form $\left(\frac{n}{n+1}\right)^{1/(n+1)}$ we launch the idea of using the quotient of two polynomials of 3rd degree, where *a, b, c, d, e, f* are any real numbers.

$$\left(\frac{n}{n+1}\right)^{1/(n+1)} \sim \frac{n^3+an^2+bn+c}{n^3+dn^2+en+f} \quad (n\to\infty) \tag{4}$$

Using a software for symbolic computation such as Maple we get the following:

$$a=d=\frac{58}{15},\ b=\frac{52}{15},\ c=-\frac{43}{180},\ e=\frac{67}{15},\ f=\frac{383}{180}$$

This means that

$$\left(\frac{n}{n+1}\right)^{1/(n+1)} \sim \frac{n^3+\frac{58}{15}n^2+\frac{52}{15}n-\frac{43}{180}}{n^3+\frac{58}{15}n^2+\frac{67}{15}n+\frac{383}{180}} \quad (n\to\infty) \tag{5}$$

is the best the approximation of the form (4) and consequently much better than (2)-(3).

2. THE BEST APPROXIMATION

We talked about the best approximation in the previous section, but what is it? How to compare the accuracy of two given approximations?

Firstly, let us consider an approximation of the form

$$f(n)\sim g(n) \quad (n\to\infty)$$

in the sense that $f(n)-g(n)$ converges to zero, as n approaches infinity. In this case, we consider such an approximation better as the convergence rate of the sequence $f(n)-g(n)$ is higher.

This method has been widely applied in a series of papers by V.G. Cristea [3], S. Dumitrescu [4] and C. Mortici et al. [5-20] to improve, or to obtain some new results involving the gamma function and related functions.

In order to apply these remarks to our problem, we rewrite (4) in the form

$$\frac{1}{n+1}\ln\frac{n}{n+1}\sim \ln\frac{n^3+an^2+bn+c}{n^3+dn^2+en+f}$$

As we have explained, we are interested in finding when the difference

$$d(n)=\frac{1}{n+1}\ln\frac{n}{n+1}-\ln\frac{n^3+an^2+bn+c}{n^3+dn^2+en+f} \tag{6}$$

converges to zero with the highest possible rate of convergence. By looking carefully at the expression (6), we realize that $d(n)$ can be written as n^{-1} expansion:

$$d(n)=\frac{\alpha}{n}+\frac{\beta}{n^2}+\frac{\gamma}{n^3}+\frac{\delta}{n^4}+\frac{\varepsilon}{n^5}+\frac{\zeta}{n^6}+\frac{\eta}{n^7}+O\left(\frac{1}{n^8}\right) \tag{7}$$

where

$$\alpha=d-a$$

$$\beta=\frac{a^2}{2}-b-\frac{d^2}{3}+e-1$$

$$\gamma=-\frac{a^3}{3}+ab-c+\frac{d^3}{3}-ed+f+\frac{3}{2}$$

$$\delta=\frac{a^4}{4}-a^2b+ac+\frac{b^2}{2}-\frac{d^4}{4}+ed^2-df-\frac{e^2}{2}-\frac{11}{6}$$

$$\varepsilon=-\frac{a^5}{5}+a^3b-a^2c-ab^2+bc+\frac{d^5}{5}-ed^3+d^2f+e^2d-ef+\frac{25}{12}$$

$$\zeta=\frac{a^6}{6}-a^4b+a^3c+\frac{3a^2b^2}{2}-2abc-\frac{b^3}{3}+\frac{c^2}{2}-\frac{d^6}{6}+ed^4-d^3f$$

$$-\frac{3e^2d^2}{2}$$

$$+2edf-\frac{f^2}{2}+\frac{e^3}{3}-\frac{137}{60}$$

$$\eta=-\frac{a^7}{7}+a^5b-a^4c-2a^3b^2+3a^2bc+ab^3-ac^2-b^2c+\frac{d^7}{7}$$

$$-d^5e+d^4f+2d^3e^2-3d^2ef-de^3+df^2+e^2f+\frac{49}{20}$$

Theoretically, the expansion (7) can be obtained by using the standard Maclaurin series of $\ln(1+x)$, but some difficulties appear. The salvation can be given by Maple, which gives the answer in less than a second.

The representation (7) is of main role in our study. More exactly, if one of the coefficients $\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta$ is not zero, then the rate of convergence of the sequence $d(n)$ can be $n^{-1},n^{-2},n^{-3},\ldots$ possibly n^{-7}. The highest rate of convergence of $d(n)$ is obtained when

$$\alpha=\beta=\gamma=\delta=\varepsilon=\zeta=0$$

that is for

$$a=d=\frac{58}{15},\ b=\frac{52}{15},\ c=-\frac{43}{180},\ e=\frac{67}{15},\ f=\frac{383}{180}$$

With these values, the difference (6) converges to zero as n^{-7}, since

$$\begin{aligned} d(n)=&\frac{68}{675n^7}-\frac{790733}{1134000n^8}+\frac{99064243}{34020000n^9}-\frac{4902009439}{510300000n^{10}}\\ &+\frac{425225113169}{15309000000n^{11}}-\frac{187496249135557}{2525985000000n^{12}}+O\left(\frac{1}{n^{13}}\right) \end{aligned} \tag{8}$$

via Maple software. The justification of the best approximation (5) is now completed.

3. BOUNDS VIA ASYMPTOTIC SERIES

It is of general knowledge that by truncation of an asymptotic series, in- creasingly accurate approximations can be obtained. We use (8) to present the following bounds:

Theorem 1. The following double inequality is valid, for every real number $x\geq 1$ in the left-hand side and $x\geq 2$ in the right-hand side:

$$\frac{x^3+\frac{58}{15}x^2+\frac{52}{15}x-\frac{43}{180}}{x^3+\frac{58}{15}x^2+\frac{67}{15}x+\frac{383}{180}}<\left(\frac{x}{x+1}\right)^{1/(x+1)}<\frac{x^3+\frac{58}{15}x^2+\frac{52}{15}x-\frac{43}{180}}{x^3+\frac{58}{15}x^2+\frac{67}{15}x+\frac{383}{180}}\exp\left\{\frac{68}{675x^7}\right\} \tag{9}$$

The lower bound in (9) is better than that presented in (1), since

$$\frac{x^3+\frac{58}{15}x^2+\frac{52}{15}x-\frac{43}{180}}{x^3+\frac{58}{15}x^2+\frac{67}{15}x+\frac{383}{180}}-\frac{x^2+x}{x^2+x+1}=\frac{90x^2+198x-43}{\left(x^2+x+x\right)\left(180x^3+696x^2+804x+383\right)}>0$$

The comparison of the upper bounds in (1) and (9) reduces to $f>0$, where

$$f(x)=\ln\left(\frac{x^2+2x}{x^2+2x+1}\cdot\frac{x^3+\frac{58}{15}x^2+\frac{67}{15}x+\frac{383}{180}}{x^3+\frac{58}{15}x^2+\frac{52}{15}x-\frac{43}{180}}\right)-\frac{68}{675x^7}$$

This inequality is true for every real number $x\geq 2$, as the function f is strictly decreasing on $[2,\infty)$, with $\lim_{x\to\infty} f(x)=0$. The monotonicity of f follows by

$$f'(x) = -\frac{2}{675x^8}\frac{P(x-2)}{(x+1)(x+2)(180x^3+696x^2+804x+383)(180x^3+696x^2+624x-43)}$$

where

$$\begin{aligned}P(x) = {} & 16402500x^{12}+496813500x^{11}+6820025850x^{10}+56148823350x^9\\ & +308999148300x^8+1198238656095x^7+3358700766x^6\\ & +6856779596076x^5+10110031374984x^4+10474636123800x^3\\ & +7199459013230x^2+2914154297018x+512711919240.\end{aligned}$$

The Proof of Theorem 1. Inequality (9) reduces to $u<0$ and $v>0$, where

$$u(x) = (x+1)\ln\frac{x^3+\frac{58}{15}x^2+\frac{52}{15}x-\frac{43}{180}}{x^3+\frac{58}{15}x^2+\frac{67}{15}x+\frac{383}{180}} - \ln\frac{x}{x+1}$$

$$v(x) = (x+1)\ln\frac{x^3+\frac{58}{15}x^2+\frac{52}{15}x-\frac{43}{180}}{x^3+\frac{58}{15}x^2+\frac{67}{15}x+\frac{383}{180}} - \ln\frac{x}{x+1} + \frac{68(x+1)}{675x^7}$$

The function u is strictly concave on $[1,\infty)$, while v is strictly convex on $[2,\infty)$, since

$$u''(x) = -\frac{A(x-1)}{x^2(x+1)^2(180x^3+696x^2+804x+383)^2(180x^3+696x^2+624x-43)^2} < 0$$

$$v''(x) = \frac{B(x-2)}{675x^9(x+1)^2(180x^3+696x^2+804x+383)^2(180x^3+696x^2+624x-43)^2} > 0$$

where

$$\begin{aligned}A(x) = {} & 4441651200x^8+78044653440x^7+587546481600x^6+2471990616864x^5\\ & +6349156539912x^4+10181214978000x^3+9940682863656x^2\\ & +5395084123798x+1244235607773\end{aligned}$$

$$\begin{aligned}B(x) = {} & 27669329136000x^{14}+1145044616846400x^{13}+21957633737701920x^{12}\\ & +258672002117477256x^{11}+2091833910240483168x^{10}\\ & +12286132343996502888x^9+54054516475076856750x^8\\ & +180989464892503530411x^7+463459400541038965050x^6\\ & +903211507451646120348x^5+1318637416090425155592x^4\\ & +1398345675714504322296x^3+1018058125888481260480x^2\\ & +455402672470658776200x+94411982935981882800\end{aligned}$$

are polynomials of 8th, respective 14th degree, with all coefficients positive. Moreover, $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} v(x) = 0$, so $u < 0$ on $[1,\infty)$ and $v > 0$ on $[2,\infty)$ and the proof is completed. ■

Note that Robert E. Schafer, Berkeley, CA made some efforts to improve the lower bound in (1) by exploiting some inequalities related to $\frac{1}{x+1}\ln\frac{x}{x+1}$. More precisely, he proved

$$\left(\frac{x}{x+1}\right)^{\frac{1}{x+1}} > \rho(x) := \frac{(x+1)\sqrt[3]{x\left(x+\frac{1}{2}\right)(x+1)} - \frac{1}{2}}{(x+1)\sqrt[3]{x\left(x+\frac{1}{2}\right)(x+1)} + \frac{1}{2}} \tag{10}$$

Our lower bound in (9) was obtained by truncation the series (8), but more accurate results can be presented when more terms are considered. As in the proof of Theorem 1, the following inequality can be stated:

$$\left(\frac{x}{x+1}\right)^{\frac{1}{x+1}} > \mu(x) := \frac{x^3 + \frac{58}{15}x^2 + \frac{52}{15}x - \frac{43}{180}}{x^3 + \frac{58}{15}x^2 + \frac{67}{15}x + \frac{383}{180}}$$

Numerical computations show that this inequality is better than (10), as we can see from the following table:

x	$\mu(x) - \rho(x)$
10	1.016786965823581943 x 10^{-2}
50	1.04597261811131268 x 10^{-3}
100	3.5942335559134891 x 10^{-4}
500	2.767503415802049 x 10^{-5}
1000	8.98813100554576 x 10^{-6}

4. CONCLUSIONS

We presented above our method applied on a concrete case, but we invite the reader to keep in mind the method used. According to the personal experience of the author, this method using the asymptotic series theory is very useful in establishing and proving a wide types of inequalities.

Randomly, or not, the following inequality due to M. S. Klamkin was listed

$$2 \ge \frac{(x-1)^{x-1}x^x}{\left(x-\frac{1}{2}\right)^{2x-1}} > 1 \quad (x \in \mathbb{R};\ x > 1) \tag{11}$$

just after Schaumberger's problem in The College Mathematics Journal [21].

Pleasing for us, we can show once again the great applicability of the method proposed in the previous section. As

$$\ln \frac{(x-1)^{x-1} x^x}{\left(x-\frac{1}{2}\right)^{2x-1}} = \frac{1}{4x} + \frac{1}{8x^2} + \frac{7}{96x^3} + O\left(\frac{1}{x^4}\right)$$

we can improve in a first stage the lower bound in (11) as follows:

$$\frac{(x-1)^{x-1} x^x}{\left(x-\frac{1}{2}\right)^{2x-1}} > \exp\left\{\frac{1}{4x} + \frac{1}{8x^2} + \frac{7}{96x^3}\right\} \quad (x \in \mathbb{R};\ x > 1) \tag{12}$$

This follows from the fact that $w > 0$, where

$$w(x) = (x-1)\ln(x-1) + x\ln x - (2x-1)\ln\left(x-\frac{1}{2}\right) - \frac{1}{4x} - \frac{1}{8x^2} - \frac{7}{96x^3}.$$

Indeed, *w* is strictly convex on $[1, \infty)$, since

$$w''(x) = \frac{15x-7}{8x^5(x-1)(2x-1)} > 0 \quad (x \in \mathbb{R};\ x > 1)$$

and $\lim_{x\to\infty} w(x) = 0$. Hence $w > 0$ and (12) is completely justified.

It is true that methods using means inequalities, convexity, monotonicity of some sequences or functions provide estimates of nice form, but of limited accuracy. If someone is interested to obtain estimates which become better as the variable approaches infinity, then arguments from the theory of asymptotic series should be considered.

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