ORIGINAL PAPER

THE CHARACTERIZATION OF SOME LINEAR MAPS USING THE RANK

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Abstract. *For a linear map $T: V \to V$ where V is a vector space, there are two special subspaces: the kernel $(ker\, T)$ and the image $(Im\, T = T(V))$ with dimensions $d(T)$ (the defect of T), respectively $r(T)$ (the rank of T). In this paper we characterize some special linear maps (projections, symmetries and tripotent maps) using only these subspaces and their dimensions.*

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MSC 2010*: 15A03, 15A04, 15A24.*

1. INTRODUCTION

In general, for a linear map $T: V \to W$ between two vector spaces, the subspaces $ker\, T$ and $Im\, T$ give too little enough information to completely characterize the mapping's type. In this paper we study some linear maps for which this information is sufficient for a complete characterization. The starting points for this paper where [1, Theorem 4.3] which concerns the characterization of idempotent matrices and [2, Remark 26] which concerns the involutory matrices.

2. THE CHARACTERIZATION OF THE PROJECTIONS

Let V be a vector space over the field K.

Definition 2.1 *A linear map $P: V \to V$ is called a projection iff $P \circ P = P$.*

We will denote by $Im\, P = P(V)$ the image of P and by $ker\, P = \{x \in V | P(x) = 0\}$ the kernel of V. If these subspaces are finite dimensional, we will denote by $r(P) = dim(Im\, P)$ the rank of P and by $d(P) = dim(ker\, P)$ the defect of the map P.

The theorem of the dimension for linear maps (see [3]) in the finite dimensional case states that:

$$dim\, V = r(P) + d(P).$$

A first characterization of the projections is given by:

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Theorem 2.1 *The map $P: V \to V$ is a projection if and only if V is the direct sum of the subspaces $ker\ P$ and $ker\ (I-P)$ $\big(V = ker\ P \oplus ker\ (I-P)\big)$, where $I: V \to V$, $I(x) = x, \forall x \in V$ is the identity map of V.*

Proof: If $P \circ P = P$ then every vector $x \in V$ has the expression $x = x_1 + x_2$ with $x_1 = x - P(x) \in kerP$ and $x_2 = P(x) \in ker(I-P)$ because $P(x_1) = P(x) - P\big(P(x)\big) = 0$ and $(I-P)(x_2) = P(x) - P\big(P(x)\big) = 0$, so that $V = kerP + ker(I-P)$. On the other side, $kerP \cap ker(I-P) = \{0\}$, which implies the sum is a direct sum: $V = kerP \oplus ker(I-P)$. If $V = ker\ P \oplus ker\ (I-P)$, then every $x \in V$ can be written uniquely as $x = x_1 + x_2$ with $x_1 \in kerP$ and $x_2 \in ker(I-P)$. We have the following equalities: $P(x) = P(x_1) + P(x_2) = 0 + P(x_2) = x_2 + (I-P)(x_2) = x_2$ and then $P \circ P(x) = P(x_2) = x_2 = P(x)$ for all $x \in V$.■

Another characterization that uses the rank is:

Theorem 2.2 *If V is finite-dimensional then the linear map $P: V \to V$ is a projection iff:*

$$dimV = r(P) + r(I-P).$$

Proof: If P is a projection, then from Theorem 2.1 it follows that $V = kerP \oplus ker(I-P)$, consequently $dimV = d(P) + d(I-P) = dimV - r(P) + dimV - r(I-P)$, hence $dimV = r(P) + r(I-P)$. Conversely, from the given equality we obtain in the same way that $dimV = d(P) + d(I-P)$ and because $kerP \cap ker(I-P) = \{0\}$ it follows that $V = kerP \oplus ker(I-P)$. According to Theorem 2.1, it follows that $P \circ P = P$. ■

From Theorem 2.2 we obtain a characterization of the idempotent matrices using the rank (see [1]):

Corollary 2.1. T*he matrix $A \in \mathcal{M}_n(K)$ is idempotent $(A^2 = A)$ iff the following equality is satisfied:*

$$rank\ A + rank\ (I_n - A) = n.$$

Proof: If $T: V \to V$ is a linear map of the vector space V over the field K having the matrix A in some basis, then $A^2 = A$ iff $T^2 = T$ so that T is a projection and we can apply Theorem 2.2 and the equalities $rank\ T = rank\ A$ and $rank\ (I_n - A) = rank(I - T)$.■

3. CHARACTERIZATION OF THE SYMMETRIES

Let V be a vector space over the field K with characteristic not equal with 2.

Definition 3.1 *A linear map $S: V \to V$ is called a symmetry if $S \circ S = I$.*

Theorem 3.1 *The linear map $S: V \to V$ is a symmetry if and only if:*

$$V = ker(I-S) \oplus ker(I+S).$$

Proof: If S is a symmetry, we will prove that $ker(I-S) \cap ker(I+S) = \{0\}$. For $x \in ker(I-S)$ we have $S(x) = x$ and for $x \in ker(I+S)$ we have $S(x) = -x$ so that $x = -x$.

From this equality we obtain that $x = 0$. In addition, every vector $x \in V$ can be writted as: $x = x_1 + x_2$, where $x_1 = \frac{1}{2}(x+S(x))$, $x_2 = \frac{1}{2}(x-S(x))$ for that we have:

$$(I - S)(x_1) = \frac{1}{2}\left(x + S(x) - S(x) - S^2(x)\right) = 0$$
$$(I + S)(x_2) = \frac{1}{2}\left(x - S(x) + S(x) - S^2(x)\right) = 0,$$

then $x_1 \in ker(I - S)$ and $x_2 \in ker(I + S)$, so that $V = ker(I - S) \oplus ker(I + S)$.

Conversely, every vector $x \in V$ can be written uniquely as $x = x_1 + x_2$, with $x_1 \in ker(I - S)$ and $x_2 \in ker(I + S)$, so that we have $S(x) = S(x_1) + S(x_2) = x_1 - x_2$. It follows that $S^2(x) = S(x_1 - x_2) = x_1 + x_2 = x$. We obtain that for every $x \in V$: $S^2(x) = x$ hence S is a symmetry.■

Theorem 3.2 *If V is a finite-dimensional vector space, then the linear map $S: V \to V$ is a symmetry if and only if $dimV = r(I - S) + r(I + S)$.*

Proof: If S is a symmetry, then from Theorem 3.1 it follows that:

$$dimV = d(I - S) + d(I + S) = dimV - r(I - S) + dimV - r(I + S)$$

hence $dimV = r(I - S) + r(I + S)$. Conversely, from the given equality it follows that $dimV = d(I - S) + d(I + S)$. But $ker(I - S) \cap ker(I + S) = \{0\}$, hence

$$V = ker(I - S) \oplus ker(I + S)$$

and according to Theorem 3.1, it follows that $S^2 = I$. ■

Corollary 3.1. *The matrix $A \in \mathcal{M}_n(K)$ is an involutory matrix ($A^2 = I_n$) iff the following equality is satisfied*:

$$\text{rank}\,(I_n + \text{A}) + \text{rank}\,(\text{I}_\text{n} - \text{A}) = \text{n}$$

Proof: Let $\text{T}: \text{V} \to \text{V}$ be a linear map of a vector space V over the field K and having the matrix A in some basis. Then it is easy to prove that $T^2 = I$ iff $A^2 = I_n$, and we can use Theorem 3.2.■

4. CHARACTERIZATION OF THE TRIPOTENT MAPS

Let V be a vector space over the field $K$with characteristic not equal with 2.

Definition 4.1 *A linear map $T: V \to V$ is called tripotent if $T^3 = T$.*

Theorem 4.1 *The linear map $T: V \to V$ satisfies the equality $T^3 = T$ iff*

$$V = kerT \oplus ker(I - T) \oplus ker(I + T).$$

Proof: If $T^3 = T$, then we will show that every vector $x \in V$ has the form

$$x = x_1 + x_2 + x_3 \tag{1}$$

where: $T(x_1)=0$, $T(x_2)=x_2$ and $T(x_3)=-x_3$. We have from the above equality that $T(x)=x_2-x_3$ and $T^2(x)=x_2+x_3$. In (1) we choose

$$x_2=\frac{1}{2}\left(T(x)+T^2(x)\right), x_3=\frac{1}{2}\left(T^2(x)-T(x)\right) \; and \; x_1=x-T^2(x).$$

Conversely, if every vector $x\in V$ has the form (1) (which is unique) with $T(x_1)=0$, $T(x_2)=x_2$ and $T(x_3)=-x_3$ we obtain: $T(x)=x_2-x_3$, $T^2(x)=x_2+x_3$ and $T^3(x)=x_2-x_3=T(x)$, so that $T^3=T$. ∎

Theorem 4.2 *If V is a vector space with finite dimension then the linear map $T: V\to V$ is tripotent iff the following equality holds:*

$$r(T)+r(I-T)+r(I+T)=2dim\,V.$$

Proof: If $T^3=T$, then from Theorem 4.1 follows that:

$$dimV=d(T)+d(I-T)+d(I+T)=$$
$$=dimV-r(T)+dimV-r(I-T)+dimV-r(I+T)$$

so that $r(T)+r(I-T)+r(I+T)=2dim\,V$. Conversely, from the given equality it follows that $d(T)+d(I-T)+d(I+T)=dim\,V$ and because $ker(T)\cap ker(I-T)\cap ker(I+T)=\{0\}$, $ker(I-T)\cap\left(ker(T)+ker(I+T)\right)=\{0\}$ and $ker(I+T)\cap\left(ker(T)+ker(I-T)\right)=\{0\}$, it follows that $V=kerT\oplus ker(I-T)\oplus ker(I+T)$.

Applying Theorem 4.1 it follows that $T^3=T$. ∎

From Theorem 4.2 we obtain the following result:

Corollary 4.1. *The matrix $A\in\mathcal{M}_n(K)$ satisfies the equality $A^3=A$ iff the following equality holds:*

$$rankA+rank(I_n-A)+rank(I_n+A)=2n.$$

Remark 4.1 Using the same ideas we can characterize some linear maps that satisfy equalities$T^k=T^p$, for example:

<u>Problem 1.</u> The linear map $T: V\to V$ satisfies the equality $T^3=T^2$ iff:

$$r(T^2)+r(I-T)=dimV.$$

<u>Problem 2.</u> For a linear map $T: V\to V$ the following equalities are equivalent:

a) $T^3=T^5$.
b) $r(T^3)+r(I-T^2)=dimV$.
c) $r(T^3)+r(I-T)+r(I+T)=2dimV$.

REFERENCES

[1] Zhang, F., *Matrix Theory - Basic Results and Techniques*, Springer-Verlag, 1999.
[2] Prasolov, V.V. *Problems and Theorems in Linear Algebra*, Vol. 134. American Mathematical Soc., 1994.
[3] Pop, V., Rasa, I., *Linear Algebra*, Ed. Mediamira, Cluj-Napoca, 2005.
[4] Nichita, F.F., Pop, V., *Symmetries, projections and Yang-Baxter Equations*, Automat. Comput. Appl. Math., **21**(1), 101, 2012.