ORIGINAL PAPER

SAIGO FRACTIONAL INTEGRAL OPERATOR OF THE GAUSS HYPERGEOMETRIC FUNCTIONS

DAYA LAL SUTHAR[1], BINIYAM SHIMELIS[1]

Manuscript received: 26.06.2016; Accepted paper: 03.08.2016;
Published online: 30.09.2016

Abstract. *In this paper, using different kind of representations for Gauss hypergeometric function ${}_2F_1(.)$, we establish seven theorems for the Saigo operators of fractional integration. The theorems established in this paper are of general character.*
Keywords: *Saigo operator of fractional integration, Gauss hypergeometric function, hypergeometric function of two variables.*

1. INTRODUCTION

Let α, β and η be complex numbers and $x \in R_+ = (0, \infty)$. The fractional integral $(\Re(\alpha) > 0)$ and the fractional derivative $(\Re(\alpha) < 0)$ of the second kind of a function $f(x)$ on R_+ are given by (cf. [3]):

$$I_{x,\infty}^{\alpha,\beta,\eta} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1[\alpha+\beta, -\eta; \alpha; 1-x/t] f(t) dt, \qquad (\Re(\alpha) > 0). \tag{1.1}$$

$$= (-1)^n \frac{d^n}{dx^n} I_{x,\infty}^{\alpha+n,\beta-n,\eta-n} f(x), \qquad (\Re(\alpha) \le 0; 0 < \Re(\alpha)+n \le 1; n \in N_0). \tag{1.2}$$

The ${}_2F_1(.)$ function occurring in the right-hand side of (1.1) is the familiar Gaussian hypergeometric function defined by

$${}_2F_1[a, b; c; x] \equiv {}_2F_1\left[\begin{matrix} a,\ b; \\ \\ c; \end{matrix}\ x\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \tag{1.3}$$

Where c is neither zero nor a negative integer, for convergence $|x| < 1$; $x = 1$ and $\Re(c-a-b) > 0$, $x = -1$ and $\Re(c-a-b) > -1$; and $(\alpha)_n$ is the Pochhammer symbol defined by

$$(\alpha)_0 = 1, (\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, (\alpha)_{-n} = \frac{(-1)^n}{(1-\alpha)_n} \qquad (\alpha \in C; n \in N_o). \tag{1.4}$$

[1] Wollo University, Department of Mathematics, 1145 Dessie, Ethiopia. E-mail: dlsuthar@gmail.com; shimelisbiniyam@gmail.com.

Following Saigo [3], for ρ being real, the function $x^{\rho-1}$ has integral formula

$$I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\rho-1}\right\}=\frac{\Gamma(\beta-\rho+1)\Gamma(\eta-\rho+1)}{\Gamma(-\rho+1)\Gamma(\alpha+\beta+\eta-\rho+1)}\,x^{\rho-\beta-1},\quad (\rho>0;\rho<min.(\Re(\beta),\Re(\eta))+1). \tag{1.5}$$

The hypergeometric function of two variable due to Srivastava and Karlson [6] is defined as:

$$F_{l:\,m;\,n}^{p:q;k}\left[\begin{matrix}(a_p):(b_q);(c_k);\\(\alpha_l):(\beta_m);(\gamma_n);\end{matrix}\; x,y\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{1.6}$$

where, for convergence

(i) $p+q<l+m+1,\ p+k<l+n+1,\ |x|<\infty$ and $|y|<\infty$ or

(ii) $p+q=l+m+1,\ p+k=l+n+1$

$$|x|^{\frac{1}{(p-1)}}+|y|^{\frac{1}{(q-1)}}<1,\quad if\quad p>1,$$

$$max.\{|x|,|y|\}<1,\quad if\quad p\le 1.$$

We also have

$$(1-x)^{-a}=\sum_{n=0}^{\infty}\frac{(a)_n}{n!}x^n, \tag{1.7}$$

and the beta integral is defined by

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{1.8}$$

where $\Re(p)>0$ and $\Re(q)>0$.

Recently Singh and Singh [5], established certain theorems for the Saigo operator of fractional integration of first kind. In the present paper, we propose to add one more dimension to this study by introducing certain theorems for the Saigo operator of fractional integration of second kind. The theorems established in this paper are believed to be a new contribution in the theory of fractional calculus.

2. MAIN RESULTS

In this section, using different kind of representations for Gauss hypergeometric function ${}_2F_1(\cdot)$, we shall evaluate the following generalized fractional integrals. The main theorems are as under:

Theorem-1 *If* $|x|<1$ *and the familiar Gaussian hypergeometric function defined by equation (1.3), then we have*

$$I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\ {}_2F_1\left[\begin{matrix} a,\ b\,; \\ \\ c\,; \end{matrix}\ x\right]\right\} = x^{\sigma-\beta-1}\ \frac{\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\eta+\alpha+\beta-\sigma+1)}\ {}_4F_3\left[\begin{matrix} a,\ b,\ \sigma,\ \sigma-\alpha-\beta-\eta; \\ \\ c,\ \sigma-\beta,\ \sigma-\eta; \end{matrix}\ x\right] \tag{2.1}$$

where $\sigma>0$ and $\sigma<min.(\Re(\beta),\Re(\eta))+1$.

Proof: On multiplying equation (1.3) by $x^{\sigma-1}$, and operating the generalized fractional integral operator $I_{x,\infty}^{\alpha,\beta,\eta}(.)$ to both the sides, the left-hand side, say L of equation (2.1) yields to

$$L = I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\ {}_2F_1\left[\begin{matrix} a,\ b\,; \\ \\ c\,; \end{matrix}\ x\right]\right\} = I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}\right\} \tag{2.2}$$

On interchanging the order of integration and summation, which is valid under conditions given with (1.1)-(1.3), we obtain

$$L = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\,n\,!}\,I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma+n-1}\right\} \tag{2.3}$$

On using the formula (1.5), the above equation leads to

$$L = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\,n\,!}\,\frac{\Gamma(\beta-\sigma-n+1)\Gamma(\eta-\sigma-n+1)}{\Gamma(-\sigma-n+1)\Gamma(\alpha+\beta+\eta-\sigma-n+1)}\,x^{\sigma+n-\beta-1}, \tag{2.4}$$

which on further simplifications reduces to the right-hand side of the theorem-1, given by (2.1).

Theorem-2. *If* $|x|<1$, $\Re(c)>\Re(b)>0$ *and ([2, p.47, Theorem 16])*

$${}_2F_1\left[\begin{matrix} a,\ b; \\ \\ c; \end{matrix}\ x\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt, \tag{2.5}$$

then the following result holds

$$I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\ {}_2F_1\left[\begin{matrix} a,\ b\,; \\ \\ c\,; \end{matrix}\ x\right]\right\} = x^{\sigma-\beta-1}\ \frac{\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\eta+\alpha+\beta-\sigma+1)}\ {}_4F_3\left[\begin{matrix} a,b,\ \sigma,\sigma-\alpha-\beta-\eta; \\ \\ c,\ \sigma-\beta,\ \sigma-\eta; \end{matrix}\ x\right], \tag{2.6}$$

where c is neither zero nor negative integer, $\sigma>0$ and $\sigma<min.(\Re(\beta),\Re(\eta))+1$.

Proof: On making use of relation (2.5), the left-hand side, say L of equation (2.6) yields to

$$L = I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt\right\} \tag{2.7}$$

On using the binomial series (1.7) and than interchanging the order of integrations and summation, which is valid under conditions given with (2.6), we obtain

$$L = \sum_{m=0}^{\infty}\frac{(a)_m}{m\,!}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b+m-1}(1-t)^{c-b-1}dt\, I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma+m-1}\right\} \tag{2.8}$$

On making use of (1.8) and fractional integral formula (1.5), the above equation reduces to the right-hand side of the theorem-2, which is same as theorem-1.

Theorem-3. *If* $|x|<1$ *and we have [1, p.278, eqn.(8.13)]*

$$\frac{d^n}{dx^n}\,{}_2F_1\left[\begin{matrix}a,\,b;\\ \\ c;\end{matrix}\;x\right] = \frac{(a)_n(b)_n}{(c)_n}\,{}_2F_1\left[\begin{matrix}a+n,\,b+n;\\ \\ c+n;\end{matrix}\;x\right], \tag{2.9}$$

then the following result holds

$$I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\frac{d^n}{dx^n}\,{}_2F_1\left[\begin{matrix}a,\,b;\\ \\ c;\end{matrix}\;x\right]\right\} = \frac{\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\eta+\alpha+\beta-\sigma+1)}$$

$$\times\, x^{\sigma-\beta-1}\frac{(a)_n(b)_n}{(c)_n}\,{}_4F_3\left[\begin{matrix}a+n,b+n,\sigma,\sigma-\alpha-\beta-\eta;\\ \\ c+n,\sigma-\beta,\sigma-\eta;\end{matrix}\;x\right], \tag{2.10}$$

where c and $c+n$ is neither zero nor a negative integer, $n=1,2,\cdots$, $\sigma>0$ and $\sigma < min.(\Re(\beta),\Re(\eta))+1$.

Proof: Operating both sides of (2.9) by the fractional integral operator $I_{x,\infty}^{\alpha,\beta,\eta}x^{\sigma-1}$, then we obtain

$$I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\frac{d^n}{dx^n}\,{}_2F_1\left[\begin{matrix}a,\,b;\\ \\ c;\end{matrix}\;x\right]\right\} = \frac{(a)_n(b)_n}{(c)_n}\,I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\,{}_2F_1\left[\begin{matrix}a+n,\,b+n;\\ \\ c+n;\end{matrix}\;x\right]\right\} \tag{2.11}$$

On making use of theorem-1 in the right-hand side of above equation, we get the required result (2.10).

Theorem-4. *If* $|x|<1$ *and due to [2, p.60, eqn.(5)] we have*

$$ {}_2F_1\left[\begin{matrix} a,\ b\,; \\ \\ c\,; \end{matrix}\ x\right] = (1-x)^{c-a-b}\ {}_2F_1\left[\begin{matrix} c-a, c-b; \\ \\ c; \end{matrix}\ x\right], \tag{2.12} $$

then

$$ I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\ {}_2F_1\left[\begin{matrix} a,\ b\,; \\ \\ c\,; \end{matrix}\ x\right]\right\} = x^{\sigma-\beta-1}\,\frac{\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\eta+\alpha+\beta-\sigma+1)} $$

$$ \times F_{2:0;1}^{2:1;2}\left[\begin{matrix} \sigma, \sigma-\alpha-\beta-\eta: & a+b-c; & c-a, c-b; \\ & & \\ \sigma-\beta, \quad \sigma-\eta\ : & - \ ; & c \quad ; \end{matrix}\ x,\ x\right], \tag{2.13} $$

where $\sigma>0$ and $\sigma<min.(\Re(\beta),\Re(\eta))+1$.

Proof: In view of the transformation (2.12), the left-hand side of (2.13), say L yields to

$$ L = I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}(1-x)^{c-a-b}\,{}_2F_1\left[\begin{matrix} c-a, c-b\,; \\ \\ c\,; \end{matrix}\ x\right]\right\} \tag{2.14} $$

Using (1.7) and expansion formula for Gauss hypergeometric function in the right-hand side of the above equation, we obtain

$$ L = I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\sum_{m,n=0}^{\infty}\frac{(a+b-c)_m(c-a)_n(c-b)_n}{(c)_n}\frac{x^{m+n}}{m\,!n!}\right\} \tag{2.15} $$

On interchanging the order of integration and summations, which is valid under conditions given with (1.1)-(1.3) and (1.6), and than using the formula (1.5), the above equation leads to

$$ L = x^{\sigma-\beta-1}\sum_{m,n=0}^{\infty}\frac{(a+b-c)_m(c-a)_n(c-b)_n}{(c)_n}\frac{x^{m+n}}{m!n!}\ \frac{\Gamma(\beta-\sigma-m-n+1)\Gamma(\eta-\sigma-m-n+1)}{\Gamma(-\sigma-m-n+1)\Gamma(\alpha+\beta+\eta-\sigma-m-n+1)} \tag{2.16} $$

which, in view of the definition (1.6) further reduces to right-hand side of (2.13).

Theorem-5. *If* $|x|<1$, $\left|\frac{x}{1-x}\right|<1$ *and we have [2, p.60, eqn.(4)]*

$$ {}_2F_1\left[\begin{matrix} a,\, b; \\ \\ c; \end{matrix}\; x\right] = (1-x)^{-a}\; {}_2F_1\left[\begin{matrix} a, c-b\,; \\ \\ c; \end{matrix}\; \frac{-x}{1-x}\right] \tag{2.17} $$

then

$$ I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}{}_2F_1\left[\begin{matrix} a,\, b; \\ \\ c; \end{matrix}\; x\right]\right\} = x^{\sigma-\beta-1}\,\frac{\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\eta+\alpha+\beta-\sigma+1)}\; F_{2:0;1}^{3:0;1}\left[\begin{matrix} \sigma, \sigma-\alpha-\beta-\eta, a : -; c-b; \\ \\ \sigma-\beta, \sigma-\eta \quad : \; -; \quad c\;\; ; \end{matrix}\; x, -x\right], \tag{2.18} $$

where $\sigma > 0$ and $\sigma < min.(\Re(\beta), \Re(\eta)) + 1$.

Proof: In view of the transformation (2.17), the left-hand side of (2.18), say L yields to

$$ L = I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}(1-x)^{-a}\; {}_2F_1\left[\begin{matrix} a, c-b\;; \\ \\ c\;; \end{matrix}\; \frac{-x}{1-x}\right]\right\} \tag{2.19} $$

On making use of (1.7) and expansion formula for Gauss hypergeometric function in the right-hand side of the above equation, we obtain

$$ L = I_{x,\,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\sum_{m,n=0}^{\infty}\frac{(a+n)_m\,(a)_n\,(c-b)_n}{(c)_n}\,\frac{x^{m+n}}{m!\,n!}\right\} \tag{2.20} $$

Applying the formula $(a)_n(a+n)_m = (a)_{n+m}$ and changing the order of integration and summations, which is permissible under the conditions given with theorem, and than using the formula (1.5), the above equation leads to

$$ L = x^{\sigma-\beta-1}\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(c-b)_n}{(c)_n}\,\frac{x^{m+n}}{m!\,n!}\,\frac{\Gamma(\beta-\sigma-m-n+1)\Gamma(\eta-\sigma-m-n+1)}{\Gamma(-\sigma-m-n+1)\Gamma(\alpha+\beta+\eta-\sigma-m-n+1)} \tag{2.21} $$

which, on further simplifications arrive at right-hand side of (2.18).

Theorem-6. *If* $|x| < 1$, $\left|\frac{x}{1-x}\right| < 1$ *and we have*

$$ {}_2F_1\left[\begin{matrix} a,\, b; \\ \\ c; \end{matrix}\; x\right] = (1-x)^{-b}\; {}_2F_1\left[\begin{matrix} c-a, b\;; \\ \\ c; \end{matrix}\; \frac{-x}{1-x}\right], \tag{2.22} $$

then

$$I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\ {}_2F_1\left[\begin{matrix}a,\ b;\\ \\ c;\end{matrix}\ x\right]\right\}=x^{\sigma-\beta-1}\ \frac{\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\eta+\alpha+\beta-\sigma+1)}\quad F_{2:0;1}^{3:0;1}\left[\begin{matrix}\sigma,\sigma-\alpha-\beta-\eta,b:-;c-a;\\ \\ \sigma-\beta,\sigma-\eta\qquad :-;\ \ c\ \ ;\end{matrix}\ x,-x\right] \tag{2.23}$$

where $\sigma>0$ and $\sigma<min.(\Re(\beta),\Re(\eta))+1$.

Proof: On interchanging a and b in the theorem-5, one can easily prove the theorem-6. For sake of brevity we omite the proof of the theorem-6.

Theorem-7. *If* $|x|<1$ *and we have*

$$G\left[\begin{matrix}a,\ b;\\ \\ c;\end{matrix}\ x\right]=\frac{\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}\ {}_2F_1\left[\begin{matrix}a,\ b;\\ \\ c;\end{matrix}\ x\right]+\frac{\Gamma(c-1)}{\Gamma(a)\Gamma(b)}\ x^{1-c}\ {}_2F_1\left[\begin{matrix}a+1-1,b+1-c;\\ \\ 2-c;\end{matrix}\ x\right] \tag{2.24}$$

then

$$I_{x,\infty}^{\alpha,\beta,\eta}\left\{x^{\sigma-1}\ G\left[\begin{matrix}a,\ b;\\ \\ c;\end{matrix}\ x\right]\right\}=x^{\sigma-\beta-1}\ \frac{\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}$$

$$\times\frac{\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\eta+\alpha+\beta-\sigma+1)}\ {}_4F_3\left[\begin{matrix}a,b,\sigma,\sigma-\alpha-\beta-\eta;\\ \\ c,\sigma-\beta,\sigma-\eta;\end{matrix}\ x\right]$$

$$+x^{\sigma-c-\beta}\ \frac{\Gamma(\beta-\sigma+c)\Gamma(\eta-\sigma+c)}{\Gamma(c-\sigma)\Gamma(\eta+\alpha+\beta-\sigma+c)}\ {}_4F_3\left[\begin{matrix}a,b,\sigma-c+1,\sigma-c-\alpha-\beta-\eta-c+1;\\ \\ c,\sigma-\beta-c+1,\sigma-\eta-c+1;\end{matrix}\ x\right] \tag{2.25}$$

where equation (2.24) represents hypergeometric function of second kind [1, p. 285, eqn (28)], $\sigma>0$ and $\sigma<min.(\Re(\beta),\Re(\eta))+1$.

Proof: Operating both sides of (2.24) by the fractional integral operator $I_{x,\infty}^{\alpha,\beta,\eta}x^{\sigma-1}$, and then on making use of theorem-1, one can easily prove the theorem-7.

3. CONCLUDING REMARKS

The fractional integral operator with Gaussian hypergeometric function in the kernel defined by (1.1) includes important and useful fractional integral operators like Weyl fractional integral operator and Kober fractional integral operator. The results given by theorems 1 to 7 can be applied to yield the corresponding results for the aformentioned integral operator.

Thus if we put $\beta = 0$ in (1.1) and noting the relationship

$$I_{x,\infty}^{\alpha,0,\eta} f(x) = K_{x,\infty}^{\alpha,\eta} f(x),$$

where $K_{x,\infty}^{\alpha,\eta}(.)$ denotes the Kober fractional integral operator, we can easily deduce the corresponding results from theorems 1 to 7.

REFERENCES

[1] George, E., Andrews, L.C., *Special Function for Engineering and Applied Mathematics*. Macmillan, New York, 1985.

[2] Rainville, E.D., *Special Function*. Macmillan, New York, 1960.

[3] Saigo, M., *Rep. College General Ed., Kyushu Univ.* **11**, 135, 1978.

[4] Saigo, M., Raina, R.K., *Fukuoka, Univ. Sci. Rep.*, **18**(1), 15, 1988.

[5] Singh, L.S., Singh, D.K., *Acta Ciencia Indica,* **XXXII M**(1), 427, 2006.

[6] Srivastava, H.M., Karlson, P.W., *Multiple Gaussian hypergeometric series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1985.