ORIGINAL PAPER

ON THE NORMAL CURVATURES OF HYPERSURFACES UNDER THE CONFORMAL MAPS

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Abstract. *In this paper, the normal curvatures of hypersurfaces are investigated under conformal, homothety and isometry maps. At first, an equation is obtained between normal curvatures of hypersurfaces, if conformal map which defined between hypersurfaces in E^n is a homothety. In the last section, it is shown that first and second fundamental forms of hypersurfaces are invariant if conformal map is an isometry.*

Keywords: *Conformal Map, Homothety, Isometry, Normal Curvature, Hypersurface, q^{th}fundamental form.*

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1. INTRODUCTION

Properties of connection preserving and conformal maps in $n-$dimensional $C^{\infty}-$manifolds were given by N. J. Hicks in 1963 [4]. He proved that a conformal map f which defined between $C^{\infty}-$class differentiable manifolds M and M' is connection preserving if and only if f is a homothety [4]. Furthemore, N. J. Hicks studied the connection preserving spray maps [6]. He investigated the finding necessary and sufficient conditions for a spray map to be connection preserving. Then F. Erkekoğlu studied the differential geometry of the connection preserving maps [2]. C. Tezer showed that for $n \neq 3,7$ a conformal diffeo morphism of S^n into itself admits no invariant connection except the trivial case where it admits an invariant Riemannian metric [10]. Besides, S. T. Pamuk proved that the results in [10] without any restiriction on the dimension of spheres [8]. On the other hand, in 1989 A. Kılıç proved that for $(n-1)-$dimensional hypersurfaces M and M' in E^n if the conformal map $f: E^n \to E^n$ for $f(M) = M'$ is a homothety, then

$$f_*(S(X)) = \delta S'(f_*X) \tag{1}$$

where S and S' are Weingarten maps on M and M', respectively, [8].

Let M and M'be for $(n-1)-$dimensional hypersurfaces in $E^n$and let $f: E^n \to E^n$ be conformal map for $f(M) = M'$. In this paper, we investigate that if f is a homothety, the normal curvatures of M and M' are invariant or not. Then, we get some results. Later if f is an isometry, we prove first and second fundamental forms of hypersurfaces are invariant.Thus, the q^{th} $-$fundamental form is invariant.

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2. PRELIMINARIES

M is called a hypersurface in n −dimensional Euclidean space E^n if $M = f^{-1}(\{c\})$ for a smooth function $f : M \to IR$ and $c \in IR$, $grad\, f(q) \neq 0$ [1].

Let M be a hypersurface in $E^n$and N be a unit normal vector field of M, also ∇ be Riemannian connection, so we have

$$S(X) = \nabla_X N$$

where $X \in \chi(M)$, S is Weingarten map and $\chi(M)$ is the space of all vector fields on M [1].

Let M and $M\,'$ be $C^\infty -$ Riemannian manifolds, and $f : M \to M'$ be a C^∞ −map. For Jacobian map f_* of f , if there is a C^∞ real-valued function $G > 0$ on M for any P in M, then

$$\langle f_* X, f_* Y \rangle = G(P) \langle X, Y \rangle \tag{2}$$

for all $X, Y \in \chi(M)$; f is called conformal. Here if G is a constant function, then f is homothety. If $G = 1$, f is an isometry [4]. Let M and $M\,'$ be $C^\infty -$ manifolds with onnections ∇ and ∇', respectively. A C^∞ −map $f : M \to M'$ is connection preserving if

$$f_*(\nabla_X Y) = \nabla'_{f_* X} f_* Y \tag{3}$$

for all $X, Y \in \chi(M)$ [4].

Theorem 1. Let M and $M\,'$ be n-dimensional $C^\infty -$ Riemannian manifolds with M connected, and let f be a $C^\infty -$ conformal map of M into M' with G. Then f is connection preserving if and only if f is homothety [4].

Theorem 2. Let M and $M\,'$ be be $(n-1)-$ dimensional hypersurfaces in E^N for $f(M) = M'$ and $f : E^n \to E^n$ be a conformal map. If f is a homothety, then for all $X \in \chi(M)$

$$f_*\big(S(X)\big) = \delta\, S'(f_* X) \tag{4}$$

where S and $S'$are Weingarten maps on M and $M\,'$, respectively. Here $\delta = \frac{1}{\|f_* N\|}$ is a constant homothety ratio [7].

Corollary 1. Let M and $M\,'$ be $(n-1)-$ dimensional hypersurfaces in E^N for $f(M) = M'$ and $f : E^n \to E^n$ be a conformal map. If f is an isometry, then for all $X \in \chi(M)$ [7],

$$f_*\big(S(X)\big) = S'(f_* X) \tag{5}$$

For every $P \in M$, $T_M(P)$ is the tangent space to M at P. The function of $k_n : T_M(P) \to IR$ is defined at $X_P \in T_M(P)$ is given by

$$k_n(X_P) = \frac{\langle S(X_P), X_P \rangle}{\langle X_P, X_P \rangle} \tag{6}$$

where k_n is a normal curvature [3].

Let α be a regular $C^{\infty}-$ curve on $n-$dimensional Riemannian manifold $M.$ If $k_n = 0$ for all X_P tangent to α, then α is an asymptotic curve [9].

Let M be a hypersurface in E^n and let S be Weingarten map of M . For $1 \leq q \leq n$, the function $I^q : \chi(M) \times \chi(M) \rightarrow ! \ C^{\infty}(M;\ IR)$ is defined at

$$I^q(X,Y) = \langle S^{q-1}(X), Y\rangle \quad . \tag{7}$$

Then I^q is called the $q^{th}-$fundamental form of hypersurface M [1].

3. ON THE NORMAL CURVATURES OF HYPERSURFACES UNDER THE CONFORMAL MAPS

Throughout this section we will suppose that $f : E^n \rightarrow E^n$ for $f(M) = M'$ is conformal map also f: E^n→E^nare $(n-1)-$ dimensional hypersurfaces in E^n.

Theorem 3. Let $f : E^n \rightarrow E^n$ conformal map for $f(M) = M'$be a homothety. Assume that k_n and k_n' is the normal curvatures of M and M', respectively. Then for all $X_P \in T_M(P)$ we have

$$k_n'(f_*(X_P)) = \delta k_n(X_P). \tag{8}$$

Proof: From (6) we may write

$$k_n'(f_*(X_P)) = \frac{\langle S'(f_*(X_P)), f_*(X_P)\rangle}{\langle f_*(X_P), f_*(X_P)\rangle}. \tag{9}$$

and

$$k_n(X_P) = \frac{\langle S(X_P), X_P\rangle}{\langle X_P, X_P\rangle} \tag{10}$$

where S and S' is the Weingarten maps of M and M'.

Since f is a conformal map, we get

$$k_n'(f_*(X_P)) = \frac{\langle S(X_P), X_P\rangle}{\langle X_P, X_P\rangle}. \tag{11}$$

where G > 0.

On the other hand, since the conformal map f is a homothety, by theorem 2 we get

$$k_n'(f_*(X_P)) = \delta \frac{\langle S'(f_*(X_P)), f_*(X_P)\rangle}{\langle f_*(X_P), f_*(X_P)\rangle}. \tag{12}$$

From (9), (10) and (12) we obtain

$$k_n'(f_*(X_P)) = \delta k_n(X_P).$$

This is completed the proof.

Using Theorem 3 we can give the following corollaries.

Corollary 2. Let conformal map $f : E^n \rightarrow E^n$ be a homothety. If α is an asymptotic curve on M, then the curve $f \circ \alpha = \beta$ is an asymptotic curve on M'.

Proof: Assume that α is an asymptotic curve on M. Then we have $k_n = 0$. From (6) we know that

$$k'_n(f_*(X_P)) = \delta k_n(X_P).$$

Then, substituting k_n =0 into the last equation we get $k'_n = 0$. Then the curve $f \circ \alpha = \beta$ is an asymptotic curve, too.

Corollary 3. Let $f : E^n \to E^n$ be a conformal map for $f(M) = M'$ If f is an isometry, then we may write

$$k'_n(f_*(X_P)) = \delta k_n(X_P)$$

Proof: From (8), we know that

$$k'_n(f_*(X_P)) = \delta k_n(X_P))$$

where δ_ is a constant homothety ratio. Since f is an isometry, we get $\delta = 1$. Then, for all $X_P \in T_M(P)$ we obtain

$$k'_n(f_*(X_P)) = \delta k_n(X_P).$$

Theorem 4. Let $f : E^n \to E^n$be a conformal map. If f is an isometry, then first and second fundamental forms of hypersurfaces are invariant.

Proof: At first,we will show that the .first fundamental forms of hypersurfaces are invariant under a map f. If f is an isometry, then we can write

$$\langle f_*X, f_*Y\rangle = \langle X, Y\rangle \tag{13}$$

On the other hand, the first fundamental forms I and $I'$of hypersurfaces M and M', respectively, are

$$I(X, Y) = \langle X, Y\rangle \tag{14}$$

and

$$I'\ (f_*X, f_*Y) = \langle f_*X, f_*Y\rangle. \tag{15}$$

Since f is an isometry we obtain

$$I'(f_*X, f_*Y) = \langle X, Y\rangle = I(X, Y). \tag{16}$$

Now we will show that the second fundamental forms of hypersurfaces are invariant under a map f.

Similarly, for the second fundamental forms II and II' of hypersurfaces M and M', respectively, we have

$$II(X, Y) = \langle S(X), Y\rangle \tag{17}$$

and

$$II'(f_*X, f_*Y) = \langle S'(f_*X), f_*Y\rangle. \tag{18}$$

Using (5) we get

$$II'(f_*X, f_*Y) = \langle f_*(S(X)), Y\rangle. \tag{19}$$

Since f is a isometry we obtain

$$II'(f_*X, f_*Y) = \langle S(X), Y \rangle. \tag{20}$$

From (17) and (20) we have

$$II(X, Y) = II'(f_*X, f_*Y) = \langle S(X), Y \rangle \tag{21}$$

Proposition 1. Let $f : E^n \rightarrow E^m$ be a conformal map. If f is an isometry, then we have

$$I'^{(q)}(f_*X, f_*Y) = I^{(q)}(X, Y) \tag{22}$$

where $\forall X, Y \in \chi(M)$.

Proof: The proof will be done by the inductive method.

STEP 1: For $q = 1$, (22) is true, since

$$I'\,(f_*X, f_*Y) = I(X, Y) = \langle X, Y \rangle.$$

STEP 2: For $q = k$, Suppose (22) is true for some $q = k \geq 1$; that is,

$$I'^{(k)}(f_*X, f_*Y) = I^{(k)}(X, Y) \tag{23}$$

By using (7), we get

:

$$\langle S'^{(k-1)}(f_*X), f_*Y \rangle = \langle S^{(k-1)}(X), Y \rangle. \tag{24}$$

STEP 3: Prove that (22) is true for q = k + 1, that is

$$I'^{(k+1)}(f_*X, f_*Y) = I^{(k+1)}(X, Y) \tag{25}$$

By (7) we may write

$$\begin{aligned} I'^{(k+1)}(f_*X, f_*Y) &= \langle S'^{(k)}(f_*X), f_*Y \rangle \\ &= \langle S'^{(k-1)}(S'(f_*X)), f_*Y \rangle. \end{aligned}$$

From (5), we get

$$\begin{aligned} I'^{(k+1)}(f_*X, f_*Y) &= \langle S'^{(k-1)}(f_*(S(X))), f_*Y \rangle. \\ &\vdots \\ &= \langle S'^{(k-2)}(f_*(S(S(X)))), f_*Y \rangle \\ &\vdots \\ &= \langle f_*(S^{(k)}(X)), f_*Y \rangle. \end{aligned}$$

Since f is a isometry we have

$$I'^{(k+1)}(f_*X, f_*Y) = \langle S^{(k)}(X), Y \rangle = I^{(k+1)}(X, Y)$$

This is completed the proof .

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