ORIGINAL PAPER

BRANS - DICKE THEORY IN GÖDEL TYPE SPACE-TIME

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Abstract.*Considering cylindrically symmetric spacetimes and its a specific case, Gödel spacetime, we study Brans-Dicke theory and find the identities under which deceleration parameter and scalar field can be calculated. As an example we take a particular case of matter dominant universe. More specifically, we show that a constant curvature cylindrically symmetric spacetime must have a scalar field of exponential type in radial and time coordinates.*

Keywords: *Brans Dicke Theory, Gödel Space time, Hubble Constant, Deceleration parameter.*

1. INTRODUCTION

Brans Dicke Theory is a well known example of scalar tensor theories [1] in which Mach principle is successfully incorporated in General Relativity by considering Gravitational constant as a background of scalar field and Gravitational "constant" no longer be a constant but a bondage of time. Brans and Dicke [2] coupled this field with matter part of field equation in order to support weak equivalence principle. Scalar fields able to explain expanded universe and provide a tool for studying dark matter of the universe [3]. Many field considered as scalar field like Higgs field for short range [4], inflation in early universe for long range and finding a suitable time coordinate in quantum cosmology. Brans Dicke theory has connected with other alternative theories like $f(R)$ theory [5] and quantum gravity [6], however, equivalence between these alternative theories could not be established [7]. BDT has also been used to develop anisotropic dark energy cosmological model [8] by considering unified dark fluid as a source of energy. Anisotropy in expansion rates does not affect the scalar field, the self interacting potential, but it controls the non-evolving part of BD parameter. The amount of dark energy is related to the mass of this field when BDT is formulated in terms of Jordan scalar field ϕ. For late universe solution of BDT has investigated [9] for constant density parameter and one which is proportion to the inverse of sixth power of scale factor.

The Brans Dicke Theory has been studied in perfect fluid [10], string cosmological Models [11], inflationary models, extended inflation, hyper extended inflation and extended chaotic inflation [12]. Recently Brans Dicke has been studied for inflationary models [13], graceful exit problem [14] and extended chaotic inflations [15]. Recently, Brans Dicke Theory has been linked with conformal relativity or Hoyle Narlikar theory [16] which is invariant under conformal transformation of spacetime. It is shown that a particular value of $\omega = -3/2$ provides conformal relativity, not only so, we can also derive low energy effective superstring model at $\omega = -1$. The time dependency of Hubble parameter and Gravitational constant stacked out in [17] in Quantum creation era under Brans Dicke theory. The aspect of Brans Dicke theory has been extended upto quantum level dynamics and string based approximations. Cosmic acceleration has been studied with various views. Recently a strong

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constraints on cosmological parameters of BDT has been searched using CMB data from Plank [18]. He showed in his article that ω must be bounded below by 692 when initial condition of the scalar field is fixed and second type the initial condition for the scalar is a free parameter leading to a somewhat stronger constraint of $\omega > 890$. Brans Dicke theory has modified upto the self creation cosmology theory of gravity that allows mass creation in the universe.

Field equations of Brans Dicke theory are

$$R_{ij} - \frac{1}{2} g_{ij} R - \lambda g_{ij} = -\frac{8\pi}{\phi} T_{ij} - \frac{\omega}{\phi^2} (\phi_{,i}\phi_{,j} - \frac{1}{2} g_{ij} \phi_{,\sigma}\phi^{,\sigma}) - \frac{1}{\phi}(\phi_{,ij} - g_{ij} \Box\phi) \quad (1)$$

$$\Box\Phi - \frac{2\lambda}{3+2\omega}\Phi = \frac{8\pi}{3+2\omega} T^i_j \quad (2)$$

Mach’s principle consequently tells that static mass distribution of matter should cause only a static geometry. But, Gödel [19] has shown that there may exist solutions of Einstein’s theory which has rotation although matter distribution is static. Therefore, it is interesting to find solution of Brans Dicke theory in a framework of Gödel spacetimes.

2. FIELD EQUATIONS IN GÖDEL SPACETIME

We use Brans Dicke theory to cosmology where we consider the universe is of Gödel type. Gödel spacetime is a rotating universe whose angular velocity depends only on square root of mean density of matters [20, 21]. Gödel spacetime is defined by

$$ds^2 = a^2(t)(dt^2 - dr^2 + e^{2r} 2d\phi^2 - dz^2 + 2e^r dt\, d\phi) \quad (3)$$

We consider this spacetime to find the solution of Brans Dicke theory. Energy momentum tensor is taken that of perfect fluid

$$T_{ij} = -p(t) g_{ij} + [\rho(t) + p(t)] u_i u_j \quad (4)$$

Where ρ is the mass density, p is the pressure and $u_i = g_{ij}u^j = g_{ij}dx^i ds$. For our convenience, we take unit vector u_i in the direction of time lines, that is, $u_i = (-1, 0, 0, 0)$ After calculating Ricci tensor and scalar curvature of the metric (3), equation (2) reduces

$$\ddot{\Phi} - \frac{2\lambda}{3+2\omega}\Phi = \frac{8\pi}{3+2\omega}\left(\frac{2p}{a} + \frac{2e^{-2r}p}{a^2} - \frac{\rho}{a^2}\right) \quad (5)$$

and non-vanishing component of (1) are

$$\frac{1}{2} - \lambda a^2 + \frac{5\dot{a}^2}{a^2} - \frac{4\ddot{a}}{a} = -\frac{8\pi\rho}{\Phi} - (1-a^2)\frac{\ddot{\Phi}}{\Phi} - \frac{3}{2}\omega(\frac{\dot{\Phi}}{\Phi})^2 + 3\frac{\dot{a}}{a}\frac{\dot{\Phi}}{\Phi} \quad (6)$$

$$\frac{1}{2} + \lambda a^2 - \frac{\dot{a}^2}{a^2} + \frac{2\ddot{a}}{a} = \frac{8\pi\rho}{\Phi} - (1+a^2)\frac{\ddot{\Phi}}{\Phi} - \frac{1}{2}\omega(\frac{\dot{\Phi}}{\Phi})^2 + 3\frac{\dot{a}}{a}\frac{\dot{\Phi}}{\Phi} \quad (7)$$

$$\frac{e^{2r}}{2}\left(\frac{3}{2}-\lambda a^2+\frac{\dot{a}^2}{a^2}-\frac{2\ddot{a}}{a}\right)=\frac{8\pi\rho}{\Phi}-(1-\frac{e^{2r}}{2}a^2)\frac{\ddot{\Phi}}{\Phi}-(1-\frac{e^{2r}}{4})\omega(\frac{\dot{\Phi}}{\Phi})^2+3\frac{\dot{a}}{a}\frac{\dot{\Phi}}{\Phi} \tag{8}$$

$$e^{r}\left(\frac{1}{2}-\lambda a^2+\frac{\dot{a}^2}{a^2}-\frac{2\ddot{a}}{a}\right)=-(1-e^{r}a^2)\frac{\ddot{\Phi}}{\Phi}-(1-\frac{e^{r}}{2})\omega(\frac{\dot{\Phi}}{\Phi})^2+3\frac{\dot{a}}{a}\frac{\dot{\Phi}}{\Phi} \tag{9}$$

for $\lambda \neq 0$. The continuity equation is

$$\dot{\rho}+(p+\rho)\frac{3\dot{a}}{a}=0 \tag{10}$$

which is in agreement with the principle of equivalence. The gravitational constant is given by

$$G=\left[\frac{2\omega+4}{2\omega+3}\right]\frac{1}{\Phi} \tag{11}$$

It must be stressed that the role of scalar field in the Brans Dicke theory is confined to its effect on gravitational field equations.

3. SOLUTION TO FIELD EQUATIONS

Proceeding to solve these equations, addition of (6) and (7) yields

$$\frac{\ddot{a}}{a}=\frac{1}{2}-2H^2+\frac{1+2H^2}{e^{-r}(1-2e^{-r})}-\frac{16\pi p}{e^{r}(1-2e^{-r})}\frac{1}{\Phi}-\frac{4\pi}{\Phi}(p-\rho) \tag{12}$$

and subtracting (9) from (8), we get

$$\frac{\ddot{\Phi}}{\Phi}+\omega(\frac{\dot{\Phi}}{\Phi})^2-3H\frac{\dot{\Phi}}{\Phi}=\frac{1+2H^2}{e^{-r}(1-2e^{-r})}-\frac{16\pi p}{e^{r}(1-2e^{-r})}\frac{1}{\Phi} \tag{13}$$

where H is Hubble constant. We can find Φ from (13) and hence after substituting its value in equation (12), a can be determined. Subtracting (7) from (6), we get

$$\lambda=\frac{1}{2}\omega\left(\frac{\dot{\Phi}}{\Phi}\right)^2-\frac{\ddot{\Phi}}{\Phi}+\frac{1}{a^2}\left[3\left(H^2-\frac{\ddot{a}}{a}\right)+\frac{4\pi}{\Phi}(p+\rho)\right] \tag{14}$$

Since Φ, H, p and ρ are known by previous calculations, we can calculate the cosmological constant of Einstein field equation for a given value of coupling constant ω. If $\lambda = 0$, we can find Φ and a subject to condition

$$\frac{\ddot{\Phi}}{\Phi}-\frac{1}{2}\omega\left(\frac{\dot{\Phi}}{\Phi}\right)^2=\frac{1}{a^2}\left[3\left(H^2-\frac{\ddot{a}}{a}\right)+\frac{4\pi}{\Phi}(p+\rho)\right] \tag{15}$$

4. PARTICULAR CASE OF MATTER DOMINANT UNIVERSE

Consider a case of matter dominant universe where pressure can be ignored. This gives (10)

$$\rho a^3 = \rho_0 a_0^3 \tag{16}$$

where $\rho_0 a_0^3$ is a constant of integration. This equation signifies the fact that density decreases inversely cube of the scale factor in case of matter dominant universe. Since $a(t)$ increases with the expansion of the universe and therefore the density decreases drastically. In this particular case,(13) reduce to

$$\frac{\ddot{\Phi}}{\Phi} + \omega\left(\frac{\dot{\Phi}}{\Phi}\right)^2 - 3H\frac{\dot{\Phi}}{\Phi} = \frac{1+2H^2}{e^{-r}(1-2e^{-r})} \tag{17}$$

and equation(12) becomes

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2\left[1 - \frac{e^r}{(1-2e^{-r})}\right] = \frac{1}{2} + \frac{4\pi}{\Phi}\rho + \frac{e^r}{(1-2e^{-r})} \tag{18}$$

Equation (17) and (18) are second order differential equations which can be easily solved for Φ and a in case of matter dominant universe.

5. BRANS DICKE THEORY IN CYLINDRICALLY SYMMETRIC SPACETIME

Consider a cylindrically symmetric spacetime analogous to Gödel type spacetime

$$ds^2 = (dt + A(r)d\phi)^2 - dr^2 - B^2(r)d\phi^2 - dr^2 - dz^2 \tag{19}$$

equivalently,

$$ds^2 = dt^2 - dr^2 - (B^2 - A^2)d\phi^2 - dz^2 + 2A(r)dtd\phi \tag{20}$$

Solving for this spacetime,(1) and (2) yields

$$\frac{3A'^2 - 8BB''}{4B^2} + \lambda = \frac{8\pi}{\Phi}p + \frac{\omega}{\Phi^2}(\Phi_{,r}\Phi_{,r} + \frac{1}{2}\Phi_{,\alpha}\Phi^{,\alpha}) - \frac{1}{\Phi}\left(\Phi_{,r,r} + \mathrm{W}\Phi\right) \tag{21}$$

$$\frac{A'^2 - 4BB''}{4B^2} - \lambda = \frac{8\pi}{\Phi}p + \frac{\omega}{\Phi^2}(\Phi_{,t}\Phi_{,t} - \frac{1}{2}\Phi_{,\alpha}\Phi^{,\alpha}) - \frac{1}{\Phi}\left(\Phi_{,t,t} - \mathrm{W}\Phi\right) \tag{22}$$

$$\frac{1}{4B^2}[A^2A'^2 + 3B^2A'^2 - 4ABA'B' + 4AB^2A'' - 4B^2B'' + 4BA^2B'' - 4B^3B''] - \lambda(A^2 - B^2)$$
$$= -\frac{8\pi}{\Phi}p(A^2 - B^2) + \frac{\omega}{\Phi^2}\left[\Phi_{,\phi}\Phi_{,\phi} - \frac{1}{2}(A^2 - B^2)\Phi_{,\alpha}\Phi^{,\alpha}\right] - \frac{1}{\Phi}\left[\Phi_{,\phi;\phi} - (A^2 - B^2)\mathrm{W}\Phi\right] \tag{23}$$

$$\frac{1}{2B^2}\left[4ABB'' - BA'B' + B^2A''\right] - 2A\lambda$$
$$= -\frac{8\pi}{\Phi}pA + \frac{\omega}{\Phi^2}(\Phi_{,\phi}\Phi_{,t} + \frac{1}{2}pA\Phi_{,\alpha}\Phi^{,\alpha}) - \frac{1}{\Phi}(\Phi_{,t;\phi} + pA\mathrm{W}\Phi) \tag{24}$$

Φ is a scalar field of Brans Dicke theory. In the term $\Phi_{,\alpha}\Phi^{,\alpha}$, α represents the coordinates with which we differentiate scalar field Φ and then, take the sum. Box W represents d'Alembertian operator. Prime ' denotes the derivative with respect to r, that is, radial coordinates.

Addition of (22) and (21) yields

$$\frac{\omega}{\Phi^2}\left(\Phi_{,r}^2-\Phi_{,t}^2\right)-\frac{1}{\Phi}(\Phi_{,r;r}+\Phi_{,t;t})+\frac{8\pi(p+\rho)}{\Phi}=\frac{A'^2-BB''}{B^2} \tag{25}$$

There is no breakdown of casuality when $A'(r)^2$„ $B(r)B''(r)$ implies that right side of equation (25) will always be negative. We just take an example to find the value of Φ.

5.1. Case: $\frac{A'^2-BB''}{B^2}=b(constant)$

Equation (25) reduces to

$$\frac{\omega}{\Phi^2}\left(\Phi_{,r}^2-\Phi_{,t}^2\right)-\frac{1}{\Phi}(\Phi_{,r;r}+\Phi_{,t;t})+\frac{8\pi(p+\rho)}{\Phi}=b \tag{26}$$

We can find Φ once the equation of state is known. More specifically, if $\rho+p=0$ and hence,

$$\Phi(r,t)=A\exp\left[\sqrt{\frac{a}{\omega-1}}r+\sqrt{\frac{b-a}{\omega-1}}t\right] \tag{27}$$

when $\omega\neq 1$ and

$$\Phi(r,t)=A\exp\left[\frac{ar^2}{2}+\frac{b-a}{2}t^2+c\,r+d\,t\right] \tag{28}$$

Here, A,c,d are constants of integration, a and b are constants arised from solving equation (25) by seperation of variables.

6. DISCUSSIONS

We applied Brans-Dicke scalar-tensor theory to cosmology where the universe has considered as Gödel type with non-vanishing cosmological constant. This theory has been studied for two type of space-time, namely, space-time given be Gödel with a scale factor a and general cylindrical symmetric spacetime. Here it is worth mentioning that we have not considered any dependence of scale factor of the universe on the scalar field.

For the space-time with scale factor a, we have found coupled equations (12), (13) and (14). These equations provide a set of values for scale factor a, scalar field Φ and cosmological constant λ that satisfy the equations of the theory. Also, a particular case of matter dominated universe is discussed. Brans-Dicke theory reduced in a set of four equations for cylindrical symmetric space-time. An expression for scalar field Φ is given by equation (25). One can obtained equations (13) and (14) by adjusting the scale factor $a(t)$ as constant and values of functions $A(r)$ and $B(r)$ are specified as $A(r)=e^r$ and $B(r)=e^r/2$.

It may also think what would the theory predict for locally flat point of the gravitational field? The theory becomes most dramatic when we test very strong Principle of Equivalence. At a point in a gravitational field of Gödel space-time, where metric is locally flat, the following relation of scalar field ϕ and Newtonian potential Φ is given

$$\Phi + (\omega + 2)(\chi\phi - 1) = 0$$

and hence, universal constant of gravitation G have to replace with G_{eff} as

$$G_{eff} = G\left[1 + \frac{\Phi}{\omega + 2}\right]$$

Since Gödel space-time does not describe real universe, results of Brans-Dicke theory can be verified theoretically and also the gravitational constant G may have much differ value than that we use frequently.

7. CONCLUSION

Identities have been estabilished under which a special type of cylindrical symmetric spacetime would be a solution of Brans-Dicke theory (scalar-tensor theories). Under specific condition, Gödel space-time is a solution of Brans-Dicke theory.

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