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APPLICATIONS OF LOCAL FRACTIONAL DECOMPOSITION METHOD TO INTEGRAL EQUATIONS

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Abstract. *In this letter, we develop analytic solutions for integral equations within local fractional derivative operators by using the proposed local fractional Adomain's decomposition method. The iteration procedure is based on local fractional derivative. Two examples are given to elucidate the solution procedure, and the results are compared with the obtained results. The results reveal that the methodology is very efficient and simple tool for solving fractal phenomena arising in mathematical physics and engineering.*

Keywords: *Local fractional calculus; fractional differential equation; local fractional Adomian's decomposition method.*

1. INTRODUCTION

The local fractional calculus theory was applied to model and process the non-differentiable phenomena in fractal physical phenomena [1–12]. Here are some local fractional models, such as the local fractional Fokker-Planck equation [1], the local fractional stress-strain relations [2], the local fractional heat conduction equation [9], wave equations on the Cantor sets [11], local fractional Laplace equation [12], Newtonian mechanics on fractals subset of real-line [13], and the local fractional Helmholtz equation [14]. There are exist some analytical methods widely applied to solve non-linear problems includes variational iteration method [15], the homotopy perturbation method [16], the heat-balance integral method [17], the complex transform method [18], the homotopy analysis method [19], the fractional sub-equation method [20] and the fractional variational iteration method [21] and more details seen in [22].

Recently, the application of Adomian decomposition method for solving the linear and nonlinear fractional partial differential equations in the fields of the physics and engineering had been established in [23,24]. Adomian decomposition method was applied to handle the time- fractional Navier-Stokes equation [25], fractional space diffusion equation [26], fractional KdV-Burgers equation [27], linear and nonlinear fractional diffusion and wave equations [28], fractional Burgers' equation [29]. The Adomian decomposition method, as one of efficient tools for solving the linear and nonlinear differential equations, was extended to find the solutions for local fractional differential equations [30-33] and non-differentiable solutions were obtained.

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In this paper, our main aim is to apply the local fractional Adomain's decomposition method [34] for solving fractional integral equations in the sense of local fractional derivative. To illustrate the validity and advantages of the method, we will apply it to the local fractional integral equations.

2. PRELIMINARY RESULTS AND DEFINITIONS

In this section, we present few mathematical fundamentals of local fractional calculus and introduce the basic notions of local fractional continuity, local fractional derivative, and local fractional integral of non-differential functions.

Definition 1. If there exists the relation [7, 36]

$$|f(x)-f(x_0)|<\varepsilon^{\alpha},\ 0<\alpha\leq 1, \tag{1}$$

with $|x-x_0|<\delta,$ for $\varepsilon,\ \delta>0$ and $\varepsilon,\ \delta\in\Re.$ Now $f(x)$ is called local fractional continuous at $x=x_0$, denoted by $\lim\limits_{x\to x_0} f(x)=f(x_0)$ Then $f(x)$ is called local fractional continuous on the interval (a,b), denoted by

$$f(x)\in C_{\alpha}(a,b). \tag{2}$$

Definition 2. A function $f(x)$ is called a non-differentiable function of exponent α, $0<\alpha\leq 1$, which satisfy Hölder function of exponent α , then for , $x,y\in X$ such that [7, 36]

$$|f(x)-f(y)|<C|x-y|^{\alpha}. \tag{3}$$

Definition 3. A function $f(x)$ is called to be continuous of $\alpha,\ 0<\alpha\leq 1,$ or shortly α continuous, when we have the following relation [7, 36] $|f(x)-f(x_0)|<\varepsilon^{\alpha}$,

$$f(x)-f(x_0)=o\left((x-x_0)^{\alpha}\right). \tag{4}$$

Compared with (4), (1) is standard definition of local fractional continuity. Here (3) is unified local fractional continuity.

Definition 4. Setting $f(x)\in C_{\alpha}(a,b)$, local fractional derivative of $f(x)$ of order α at $x=x_0$, is defined [7, 36]

$$f^{(\alpha)}(x_0)=\left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_o}=\lim_{x\to x_0}\frac{\Delta^{\alpha}(f(x)-f(x_0))}{(x-x_0)^{\alpha}},\ 0<\alpha\leq 1 \tag{5}$$

where $\Delta^{\alpha}\left(f(x)-f(x_0)\right)\cong\Gamma(1+\alpha)\Delta\left(f(x)-f(x_0)\right)$. For any $x\in(a,b)$, there exists $f^{(\alpha)}(x)=D_x^{\alpha}f(x)$, denoted by $f(x)=D_x^{\alpha}f(a,b)$.

Local fractional derivative of high order is written in the form

$$f^{(k\alpha)}(x)=\overbrace{D_x^{\alpha}\ldots D_x^{\alpha}}^{k\ times}f(x),$$

and local fractional partial derivative of high order

$$\frac{\partial^{k\alpha}f(x)}{dx^{k\alpha}}=\overbrace{\frac{\partial^{\alpha}}{\partial x^{\alpha}}\cdots\frac{\partial^{\alpha}}{\partial x^{\alpha}}}^{k\ times}f(x).$$

Definition 5. Setting $f(x)\in C_{\alpha}(a,b)$, local fractional integral of $f(x)$ of order α in the interval $[a,b]$ is defined [36]

$$ {}_aI_b^{\alpha}f(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^{\alpha},\ 0<\alpha\le 1, \tag{6}$$

where $\Delta t_j=t_{j+1}-t_j$, $\Delta t=\max\{\Delta t_1,\Delta t_2,\Delta t_3,\ldots\}$ and $\lfloor\Delta t_j,\Delta t_{j+1}\rfloor$, $j=0,1,\ldots,N-1$, $t_0=a, t_N=b$, is a partition of the interval $[a,b]$. For any $x\in(a,b)$, there exists ${}_aI_x^{\alpha}f(x)$, denoted by $f(x)\in I_x^{(\alpha)}(a,b)$.

If $f(x)=D_x^{\alpha}f(a,b)$, or $I_x^{(\alpha)}(a,b)$, we have $f(x)\in C_{\alpha}(a,b)$.

Here, it follows that

$$\begin{aligned}
&{}_aI_a^{\alpha}f(x)=0, \text{ if } a=b.\\
&{}_aI_b^{\alpha}f(x)=-{}_bI_a^{\alpha}f(x), \text{ if } a<b.\\
&{}_aI_a^{0}f(x)=f(x).
\end{aligned}$$

For any $f(x)\in C_{\alpha}(a,b)$, $0<\alpha\le 1$, we have local fractional multiple integrals

$$ {}_{x_0}I_x^{(k\alpha)}f(x)=\overbrace{{}_{x_0}I_x^{(\alpha)}\ldots{}_{x_0}I_x^{(\alpha)}}^{k\ times}f(x),$$

For $0<\alpha\le 1$, $f^{((k\alpha))}(x)\in C^k{}_{\alpha}(a,b)$, then we have

$$\left({}_{x_0}I_x^{(k\alpha)}f(x)\right)^{(k\alpha)}=f(x),$$

where ${}_{x_0}I_x^{(k\alpha)}f(x)=\overbrace{{}_{x_0}I_x^{(\alpha)}\ldots{}_{x_0}I_x^{(\alpha)}}^{k\ times}f(x)$, and $f^{(k\alpha)}(x)=\overbrace{D_x^{\alpha}\ldots D_x^{\alpha}}^{k\ times}f(x)$.

Definition 6. Mittag-Leffer function in fractal space is defined by

$$E_{\alpha}\left(x^{\alpha}\right)=\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)},\ 0<\alpha\leq 1. \tag{7}$$

The following rules hold

$$\cos_{\alpha}\left(x^{\alpha}\right)=\sum_{k=0}^{\infty}(-1)^{k}\ \frac{x^{2k\alpha}}{\Gamma(1+k2\alpha)},\ 0<\alpha\leq 1. \tag{8}$$

$$\sin_{\alpha}\left(x^{\alpha}\right)=\sum_{k=0}^{\infty}(-1)^{k}\ \frac{x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)},\ 0<\alpha\leq 1. \tag{9}$$

Suppose that $f(x)$ is local fractional continuous on the interval $[a,b]$. Then we have

$${}_{a}I_{x}^{(\alpha)}\ {}_{a}I_{x}^{(\alpha)}\ f(x)={}_{a}I_{x}^{(\alpha)}\ \frac{(x-t)^{\alpha}\ f(x)}{\Gamma(1+\alpha)}. \tag{10}$$

$${}_{0}I_{x}^{(\alpha)}\ {}_{0}I_{x}^{(\alpha)}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}=\frac{x^{(k+2)\alpha}}{\Gamma(1+(k+2)\alpha)}. \tag{11}$$

3. LOCAL FRACTIONAL DERIVATIVES AND INTEGRALS

Some useful formulas and results of local fractional derivative were summarized [7, 37].

$$\frac{d^{\alpha}x^{k\alpha}}{dx^{\alpha}}=\frac{\Gamma(1+k\alpha)x^{(k-1)\alpha}}{\Gamma(1+(k-1)\alpha)}. \tag{12}$$

$$\frac{d^{\alpha}E_{\alpha}\left(x^{\alpha}\right)}{dx^{\alpha}}=E_{\alpha}\left(x^{\alpha}\right). \tag{13}$$

$$\frac{d^{\alpha}E_{\alpha}\left(k\,x^{\alpha}\right)}{dx^{\alpha}}=kE_{\alpha}\left(k\,x^{\alpha}\right). \tag{14}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}E_{\alpha}\left(x^{\alpha}\right)(dx)^{\alpha}=E_{\alpha}\left(b^{\alpha}\right)-E_{\alpha}\left(a^{\alpha}\right). \tag{15}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}\sin_{\alpha}\left(x^{\alpha}\right)(dx)^{\alpha}=\cos_{\alpha}\left(a^{\alpha}\right)-\cos_{\alpha}\left(b^{\alpha}\right). \tag{16}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b x^{k\alpha}(dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha} - b^{(k+1)\alpha}\right) \tag{17}$$

4. ANALYSIS OF LOCAL FRACTIONAL ADOMAIN'S DECOMPOSITION METHOD

For seek out of clarity of the explanation, the local fractional decomposition method will be briefly outlined. For integral equations a compact recurrence scheme has been developed [7-30]. The local fractional Volterra integral equation is written in the form [26, 30]

$$u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)}\int_0^x k(x,t)u(x)(dt)^\alpha, 0 < \alpha \le 1. \tag{18}$$

Suppose that there exists the solution in the following local fractional Adomian's series form

$$u(x) = \sum_{n=0}^{K} u_n(x). \tag{19}$$

The initial approximation in this case is

$$u_0(x) = f(x),$$

Hence, we can determine a few terms in the series such as $u(x) = \sum_{n=0}^{K} u_n(x)$ by truncating the series at certain term.

Substituting Eq. (19)) into Eq. (18) implies

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)}\int_a^b k(x,t)\sum_{n=0}^{\infty} u_n(t)(dt)^\alpha.$$

Or equivalently,

$$u_0(x) = f(x),$$

$$u_1(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)}\int_a^b k(x,t)u_0(t)(dt)^\alpha,$$

$$u_2(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)}\int_a^b k(x,t)u_1(t)(dt)^\alpha,$$

$$u_3(x)=\frac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_a^b k(x,t)u_2(t)(dt)^{\alpha},$$
$$\vdots$$
$$u_n(x)=\frac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_a^b k(x,t)u_{n-1}(t)(dt)^{\alpha},$$

and so on. We can write the compact recurrence relation as

$$u_0(x)=f(x), \tag{20}$$

$$u_{n+1}(x)=\frac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_a^b k(x,t)u_n(t)(dt)^{\alpha}, n\geq 0. \tag{21}$$

Hence, we give the local fractional series solution $u(x)=\sum_{n=0}^{\infty}u_n(x)$. The above processes were discussed in [37, 38].

5. NUMERICAL APPLICATION

Example 5.1. Consider the following local fractional Fredholm integral equation

$$u(x)=\Gamma(1+\alpha)+\frac{1}{\Gamma(1+\alpha)}\int_0^1 x^{\alpha}u(t)(dt)^{\alpha},\ 0<\alpha\leq 1. \tag{22}$$

The recurrence relation reads as

$$u_0(x)=\Gamma(1+\alpha),$$
$$u_{n+1}(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^1 x^{\alpha}u_n(t)(dt)^{\alpha},\ n\geq 0. \tag{23}$$

The zeroth component of the solution can be determine from initial conditions as

$$u_0(x)=\Gamma(1+\alpha), \tag{24}$$

Applying the recursive relation (23) and, we get the following results

$$u_1(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^1 x^{\alpha}u_0(t)(dt)^{\alpha}=x^{\alpha},$$

$$u_2(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^1 x^{\alpha}u_1(t)(dt)^{\alpha}=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{\alpha},$$

$$u_3(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^1 x^{\alpha}u_2(t)(dt)^{\alpha}=\frac{\Gamma^2(1+\alpha)}{\Gamma^2(1+2\alpha)}x^{\alpha},$$

$$u_4(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^1 x^{\alpha}u_3(t)(dt)^{\alpha}=\frac{\Gamma^3(1+\alpha)}{\Gamma^3(1+2\alpha)}x^{\alpha},$$

$$u_5(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^1 x^{\alpha}u_4(t)(dt)^{\alpha}=\frac{\Gamma^4(1+\alpha)}{\Gamma^4(1+2\alpha)}x^{\alpha},$$

$$\vdots,$$

and so on.

Thus the local fractional series solution of (22) is given by

$$u(x)=\Gamma(1+\alpha)+x^{\alpha}\sum_{n=0}^{\infty}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{n}.$$

Hence,

$$u(x)=\Gamma(1+\alpha)+x^{\alpha}\,\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)-\Gamma(1+\alpha)}. \tag{25}$$

The result is the same as the one which is obtained in [35].

Example 5.2. Consider the following local fractional Volterra equation

$$u(x)=\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(t-x)^{\alpha}}{\Gamma(1+\alpha)}u(t)(dt)^{\alpha},\ 0<\alpha\le 1. \tag{26}$$

According to local fractional Adomain decomposition method, the recurrence relation reads as

$$u_0(x)=\frac{x^{\alpha}}{\Gamma(1+\alpha)},$$

$$u_{n+1}(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{(t-x)^{\alpha}}{\Gamma(1+\alpha)}u_n(t)(dt)^{\alpha},\ n\ge 0. \tag{27}$$

The zeroth component of the solution can be determine from initial conditions as

$$u_0(x)=\frac{x^{\alpha}}{\Gamma(1+\alpha)}, \tag{28}$$

Proceeding in same manner, the other components are determined as

$$u_1(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{(t-x)^\alpha}{\Gamma(1+\alpha)}u_0(t)(dt)^\alpha=-\frac{1}{\Gamma(1+3\alpha)}x^{3\alpha},$$

$$u_2(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{(t-x)^\alpha}{\Gamma(1+\alpha)}u_1(t)(dt)^\alpha=\frac{1}{\Gamma(1+5\alpha)}x^{5\alpha},$$

$$u_3(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{(t-x)^\alpha}{\Gamma(1+\alpha)}u_2(t)(dt)^\alpha=-\frac{1}{\Gamma(1+7\alpha)}x^{7\alpha},$$

$$u_4(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{(t-x)^\alpha}{\Gamma(1+\alpha)}u_3(t)(dt)^\alpha=\frac{1}{\Gamma(1+9\alpha)}x^{9\alpha},$$

$$\vdots,$$

and so on.

Thus, the local fractional series solution is

$$u(x)=\frac{x^\alpha}{\Gamma(1+\alpha)}-\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}-\dots,$$

The closed form solution is

$$u(x)=\sin_\alpha\left(x^\alpha\right) \tag{29}$$

This result is the same as obtained [39].

6. CONCLUSIONS

In this paper, the non-differentiable solutions for the integral equations involving local fractional derivative operators in mathematical physics are determined by using the proposed local fractional Adomain's decomposition method. The analytically obtained results demonstrate the simplicity, accuracy, and approximately convergent solution, the reliability of the methodology and its wider applicability to local fractional differential equation arising in mathematical physics, engineering.

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