ORIGINAL PAPER

DIFFERENTIALS OF SOME USUAL STOCHASTIC PROCESSES

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Abstract. *In this paper, we obtain the stochastic differentials of some of the most common continuous stochastic processes using Itô's formula.*

Keywords: *Stochastic processes, Itô's formula,*

1. INTRODUCTION

Roughly speaking, by dynamics we mean evolution. In science, there are many preoccupations in the problem of discovering the evolution laws of of systems of particles, which interact among themselves and with the outside environment.

In the past few years, many researchers proposed new models that describe the best possible scenario and they try to anticipate the evolution of that phenomenon using modern mathematical theories and analysis, such as probabilities [1, 2], stochastic mathematics or the control theory. The most important part in this research is played by the stochastic differential calculus introduced by *Itô's* formula [3-8].

Let's remind the formula for calculating the stochastic differential for a stochastic process, $(Y_t)_t$ where $Y_t = g(t, x_t)$, differentiable function $g: R^2 \to R$, and $(X_t)_t$ is a stochastic process who's differential has the form

$$dX_t = udt + vdB_t \tag{1}$$

where $(B_t)_t$ is a one-dimensional Brownian movement.

Itô's formula for the stochastic differential of stochastic process (Y_t) is

$$dY_t = \frac{\partial g(t, x_t)}{\partial t} dt + \frac{\partial g(t, x_t)}{\partial x} dx_t + \frac{1}{2} \frac{\partial^2 g(t, x_t)}{\partial x^2} dx^2 \tag{2}$$

2. CONTENTS

***Theorem**:* If the corresponding *Itô's* stochastic equation for a stochastic process $(Y_t)_t$ of the form

$$Y_t = X_t^n \tag{3}$$

The stochastic differential is

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$$dy_t = nx_t^{n-1}dx_t + \frac{1}{2}n(n-1)x_t^{n-2}(dx_t)^2 \tag{4}$$

Replacing (1) in (4) we obtain

$$dY_t = nx_t^{n-1}udt + nx_t^{n-1}vdB_t + \frac{n(n-1)}{2}x_t^{n-2}v^2dt \tag{5}$$

or

$$dY_t = nx_t^{n-1}\left(ux_t + n(n-1)x^2\right)dt + nx_t^{n-1}vdB_t \tag{5'}$$

Theorem: If the corresponding *Itô's* stochastic equation for a stochastic process $(Y_t)_t$ of the form

$$Y_t = a^{x_t} \tag{6}$$

Then the stochastic differential will be

$$dY_t = a^{x_t}\ln(a)dx_t + \frac{1}{2}a^{x_t}\ln^2(a)(dx_t)^2 \tag{7}$$

Replacing (1) in (7) we obtain:

$$dy_t = a^{x_t}\ln(a)\left(u + \frac{v^2}{2}\ln(a)\right)dt + x\ln(a)a^{x_t}dB_t \tag{7'}$$

Theorem: If the corresponding *Itô's* stochastic equation for a stochastic process $(Y_t)_t$ of the form

$$y_t = \sin(x_t) \tag{8}$$

Then the stochastic equation will be

$$dy_t = \left(u\cos(x_t) - \frac{v^2}{2}\sin(x_t)\right)dt + v\cos(x_t)dB_t \tag{9}$$

Theorem: If the corresponding *Itô's* stochastic equation for a stochastic process $(Y_t)_t$ of the form

$$y_t = \cos(x_t) \tag{10}$$

We have

$$dy_t = \left(-u\sin(x_t) - \frac{x^2}{2}\cos(x_t)\right)dt - v\sin(x_t)dB_t \tag{11}$$

Theorem: If the corresponding *Itô's* stochastic equation for a stochastic process $(Y_t)_t$ of the form

$$y_t = \ln(x_t) \tag{12}$$

The Itô's stochastic differential is

$$dy_t = \left(\frac{u}{x_t} - \frac{v^2}{2x_t^2}\right)dt + \frac{v}{x_t}dB_t \tag{13}$$

In the case

$$y_t = \lg(x_t) \tag{14}$$

The Itô stochastic differential is

$$dy_t = \frac{1}{\cos^3 x_t}\left[\left(u\cos x_t + v^2 \sin x_t\right)dt + v\cos x_t dB_t\right] \tag{15}$$

***Theorem*:** For Trigonometric functions, the stochastic differentials are as follows

$$y_t = \text{ctg}(x_t) \tag{16}$$

The stochastic differential is

$$dy_t = \frac{1}{\sin^3 x_t}\left[\left(-u\sin x_t + v^2 \cos x_t\right)dt - v\sin x_t dB_t\right] \tag{17}$$

For

$$y_t = \arcsin(x_t), \text{ with } -1 \le x_t \le 1 \tag{18}$$

We have

$$dy_t = \frac{1}{\sqrt{1-x_t}}\left[\left(\left(1-x_t^2\right)u + \frac{1}{2}x_t v^2\right)dt - v\left(1-x_t^2\right)dB_t\right] \tag{19}$$

If

$$y_t = \arccos(x_t), \text{ with } -1 \le x_t \le 1 \tag{20}$$

$$dy_t = -\frac{1}{\sqrt{1-x_t}}\left[\left(\left(1-x_t^2\right)u + \frac{1}{2}x_t v^2\right)dt + v\left(1-x_t^2\right)dB_t\right] \tag{21}$$

If

$$y_t = \text{arctg}(x_t), \text{ with } -1 \le x_t \le 1 \tag{22}$$

$$dy_t = -\frac{1}{\sqrt{1+x_t}}\left[\left(\left(1+x_t^2\right)u - x_t v^2\right)dt + v\left(1+x_t^2\right)dB_t\right] \tag{23}$$

If

$$y_t = \frac{1}{x_t}, \quad x_t \ne 0 \tag{24}$$

we will obtain an Itô stochastic differential:

$$dy_t = -\frac{1}{x_t^2}dx_t + \frac{1}{2}\frac{2x_t}{x_t^2}\left(dx_t\right)^2$$

$$dy_t = -\frac{1}{x_t^2}dx_t + \frac{1}{x_t^3}\left(udt + vdB_t\right)^2$$

$$dy_t = -\frac{1}{x_t^2}\left(udt + vdB_t\right) + \frac{1}{x_t^3}v^2 dt$$

$$dy_t = \frac{1}{x_t^3}\left(-ux_t + v^2\right)dt - \frac{v^2}{x_t^2}dB_t$$

$$dy_t = \frac{1}{x_t^3}\left[\left(-ux_t + v^2\right)dt - vx_t dB_t\right] \tag{25}$$

For

$$y_t = \log_a x_t \tag{26}$$

$$dy_t = \frac{1}{x_t^2 \ln a}\left[\left(ux_t - v^2\right)dt - vx_t dB_t\right] \tag{27}$$

3. CONCLUSION

In the above calculus, several Itô stochastic differentials were obtain for a lot of stochastic processes, obtained by composing elementary functions with the stochastic process x_t.

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