ORIGINAL PAPER

INTRODUCTION TO QUATERNION LINEAR CANONICAL TRANSFORM

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Abstract. *Recently developed concepts of Quaternion Fourier transform and Fractional Quaternion Fourier transform based on Quaternion Algebra has been found to be useful for color image processing and signal analysis. In this paper we extend the concept of quaternion to Linear Canonical transform which will be more useful tool, because Linear Canonical transform is the generalization of Fractional Fourier transform and classical Fourier transform with three more degrees of freedom. First we shall define quaternion Linear Canonical transform and prove some of its properties.*

Keywords: *Quaternion Fourier transform, Fractional Quaternion Fourier, Linear Canonical transform.*

1. INTRODUCTION

Fractional Fourier transform and Linear Canonical transform proved their applicability in many areas such as signal processing and filtering etc. hence these topics are for interest of research of many researchers in recent years. The Linear Canonical Transform is an integral transform which is generalization of Fractional Fourier Transform with more parameters and more degrees of freedom than Fractional Fourier Transform. It was first introduced in 1970's and found to be more convenient tool in the area of signal analysis, signal processing, water marking etc. The generalized Linear Canonical transform is defined in [2] is given by,

$$[LCT[f(t)]](u) = F_A(u) = \langle f(t), K_A(t,u) \rangle$$

(1.1)

where the kernel is,

$$K_A\,(u, t)=\sqrt{\frac{1}{2\pi i b}}\cdot e^{i\pi\frac{d}{b}u^2}\cdot e^{-i\pi\frac{1}{b}ut}\cdot e^{i\pi\frac{a}{b}t^2}, \qquad b\neq 0$$

(1.2)

$$=\sqrt{d}\cdot e^{i\pi c d u^2}\cdot f(du), \qquad b=0$$

and a, b, c, d are real numbers with ad-bc = 1. The kernel can also be viewed as a 2×2 matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with determinant ad-bc=1.

We have studied some analytical aspects of Linear Canonical transform in [2]. Pie and Ding discussed Eigen functions of Linear Canonical transform in [5] and uncertainty principle in [4].Today there are number of papers involving Linear Canonical Transform still no one has involve the Linear Canonical Transform of Quaternion signals (or hyper-complex

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signals).The concept of Quaternion was introduced by Hamilton in1843 [3]. It is the generalization of a complex number. Quaternion signals are also called hyper-complex signals. Complex signals have two components: the real part and the imaginary parts. However a quaternion signal has four parts, one real component and three imaginary parts:
$q = q_r + iq_i + jq_j + kq_k = q_a + q_b j,$
where $q_i, q_j, q_k \in R$ and i, j, k are three imaginary units which satisfy following relationship.
$i^2 = j^2 = k^2 = -1$ and $ij = -ji = k, jk = -kj = i, ki = -ik = j.$

The paper is organized as follows. In section II we propose the definition of Quaternion Linear Canonical transform. In section III we have proved some properties of Quaternion Linear Canonical Transform. Convolution of Quaternion Linear Canonical transform is introduced in section IV. Section V concludes.

2. QUATERNION LINEAR CANONICAL TRANSFORM (QLCT)

2.1. DEFINITION

For any given quaternion signal $f(x,y) = f_r(x,y) + if_i(x,y) + jf_j(x,y) + kf_k(x,y)$, $(f_r(x,y),\ f_i(x,y), f_j(x,y), f_k(x,y) are\ real\ ones)$, the QLCT of $f(x,y)$ denoted by $F_{i,j}^{P_1,P_2}(u,v)$; is defined as

$$F_{i,j}^{A_1,A_2}(u,v) = F_{i,j}^{A_1,A_2}\{f(x,y)\}(u,v)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_i^{P_1}(x,u) f(x,y) K_j^{P_2}(y,v)\,dx\,dy$$

where $K_i^{A_1}(x,u) = \sqrt{\frac{1}{2\pi i b}}\, e^{\frac{i}{2b}(du^2 + ax^2 - xu)}$, $K_j^{A_2}(y,v) = \sqrt{\frac{1}{2\pi j q}}\, e^{\frac{j}{2q}(sv^2 + py^2 - yv)}$ and
$A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A_2 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$

Note here that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{pmatrix}$ and $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} cos\beta & sin\beta \\ -sin\beta & cos\beta \end{pmatrix}$ the above definition reduced to that of fractional Fourier transform as in [1].

2.2. INVERSION THEOREM

If $F_{i,j}^{A_1^1,A_2^1}(u,v)$ denotes quaternion Linear Canonical transform of a quaternion $f(x,y)$ then it is possible to reconstruct $f(x,y)$ from $F_{i,j}^{A_1^{-1},A_2^{-1}}(u,v)$

Proof:

$$F_{i,j}^{A_1^{-1},A_2^{-1}}\left[F_{i,j}^{A_1^1,A_2^1}(u,v)\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_i^{A_1^{-1}}(u,s) F_{i,j}^{A_1,A_2}(u,v) K_j^{A_2^{-1}}(v,w)\,du\,dv$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_i^{A_2^{-1}}(u,s)\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_i^{A_1}(x,u) f(x,y) K_j^{A_2}(y,v)\,dx\,dy\right\} K_j^{A_2^{-1}}(v,w)\,du\,dv$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K_i^{A_2^{-1}}(u,s)\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K_i^{A_1}(x,u)[\,f_r(x,y)+if_i(x,y)+jf_j(x,y)+kf_k]K_j^{A_2}(y,v)dxdy\}$$

$$\cdot K_j^{A_2^{-1}}(v,w)dudv$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_r(x,y)\left\{\left[\int_{-\infty}^{\infty}K_i^{A_1^{-1}}(u,s)K_i^{A_1}(x,u)du\right]\times\left[\int_{-\infty}^{\infty}K_j^{A_2^{-1}}(v,w)K_j^{A_2}(y,v)dv\right]\right\}dxdy$$
$$+i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_i(x,y)\left\{\left[\int_{-\infty}^{\infty}K_i^{A_1^{-1}}(u,s)K_i^{A_1}(x,u)du\right]\times\left[\int_{-\infty}^{\infty}K_j^{A_2^{-1}}(v,w)K_j^{A_2}(y,v)dv\right]\right\}dxdy$$
$$+j\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_j(x,y)\left\{\left[\int_{-\infty}^{\infty}K_i^{A_1^{-1}}(u,s)K_i^{A_1}(x,u)du\right]\times\left[\int_{-\infty}^{\infty}K_j^{A_2^{-1}}(v,w)K_j^{A_2}(y,v)dv\right]\right\}dxdy$$
$$+k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_k(x,y)\left\{\left[\int_{-\infty}^{\infty}K_i^{A_1^{-1}}(u,s)K_i^{A_1}(x,u)du\right]\times\left[\int_{-\infty}^{\infty}K_j^{A_2^{-1}}(v,w)K_j^{A_2}(y,v)dv\right]\right\}dxdy$$

(1)

Now consider,

$$\int_{-\infty}^{\infty}K_i^{A_1^{-1}}(u,s)\,K_i^{A_1}(x,u)\,du$$
$$=\sqrt{\frac{-1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{-i}{2b}(au^2+ds^2)+\frac{i}{b}us}\cdot\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2b}(dx^2+au^2)-\frac{i}{b}xu}du$$
$$=\frac{1}{2\pi b}e^{\frac{id}{2b}(x^2-s^2)}\int_{-\infty}^{\infty}e^{\frac{i}{b}(s-x)u}\,du$$
$$=e^{\frac{id}{2b}(x^2-s^2)}\cdot\delta(s-x)$$

So,

$$\int_{-\infty}^{\infty}K_i^{A_1^{-1}}(u,s)\,K_i^{A_1}(x,u)\,du=e^{\frac{id}{2b}(x^2-s^2)}\cdot\delta(s-x) \qquad (2)$$

Similarly,

$$\int_{-\infty}^{\infty}K_j^{A_2^{-1}}(v,w)\,K_j^{A_2}(y,v)dv=e^{\frac{id}{2b}(y^2-w^2)}\cdot\delta(w-y) \qquad (3)$$

Substituting (2) and (3) in (1) we get,

$$F_{i,j}^{A_1^{-1},A_2^{-1}}\left[F_{i,j}^{A_1^{1},A_2^{1}}(u,v)\right]$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_r(x,y)\left\{e^{\frac{id}{2b}(x^2-s^2)}\cdot\delta(s-x)\times e^{\frac{id}{2b}(y^2-w^2)}\cdot\delta(w-y)\right\}dxdy$$
$$+i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_i(x,y)\left\{e^{\frac{id}{2b}(x^2-s^2)}\cdot\delta(s-x)\times e^{\frac{id}{2b}(y^2-w^2)}\cdot\delta(w-y)\right\}dxdy$$
$$+j\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_j(x,y)\left\{e^{\frac{id}{2b}(x^2-s^2)}\cdot\delta(s-x)\times e^{\frac{id}{2b}(y^2-w^2)}\cdot\delta(w-y)\right\}dxdy$$
$$+k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_k(x,y)\left\{e^{\frac{id}{2b}(x^2-s^2)}\cdot\delta(s-x)\times e^{\frac{id}{2b}(y^2-w^2)}\cdot\delta(w-y)\right\}dxdy$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\left\{e^{\frac{id}{2b}(x^2-s^2)}\cdot\delta(s-x,w-y)\cdot e^{\frac{id}{2b}(y^2-w^2)}\right\}dxdy$$

3. PROPERTIES OF QUATERNION LINEAR CANONICAL TRANSFORM

3.1 For any given quaternion function $f_n(x,y)\,(n\in\aleph)$, the following relationship is true:

$$F_{i,j}^{A_1,A_2}\sum a_n\{f_n(x,y)\}=\sum a_n F_{i,j}^{A_1,A_2}\{f_n(x,y)\}\ (a_n\in\aleph).$$

Proof: Since LCT is one of the linear operator, property 3.1 is obtained directly from Definition 1.2.

3.2 Odd Even Invariant Property:

$$F_{i,j}^{A_3,A_4} F_{i,j}^{A_1,A_2} = F_{i,j}^{A_1,A_2} F_{i,j}^{A_3,A_4} = F_{i,j}^{A_1,A_3} F_{i,j}^{A_2,A_4}$$

Proof: For any one quaternion signal $f(x,y)$, from definition, we can obtain

$$F_{i,j}^{A_3,A_4} F_{i,j}^{A_1,A_2}\{f(x,y)\}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_i^{A_3}(u,s)\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_i^{A_1}(x,u) f(x,y) \times K_j^{A_2}(y,v)dxdy\right\} K_j^{A_4}(v,w)dudv$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\left[\int_{-\infty}^{\infty} K_i^{A_3}(u,s)K_i^{A_1}(x,u)du\right] \times f(x,y) \times \left[\int_{-\infty}^{\infty} K_j^{A_2}(y,v)K_j^{A_4}(v,w)dv\right]\right\}dxdy$$

… … (1)

For 1 D signal the $\int_{-\infty}^{\infty} K^{A_1}(x,t)\cdot K^{A_2}(t,y)dt = K^{A_1\cdot A_2}(x,y)$

Consider,

$$\int_{-\infty}^{\infty} K_A(x,t)\cdot K_B(t,y)dt \qquad \text{where } A = \begin{pmatrix} a & b \\ c & d\end{pmatrix}, B = \begin{pmatrix} p & q \\ r & s\end{pmatrix}$$
$$= \int_{-\infty}^{\infty}\sqrt{\frac{1}{2\pi ib}}\, e^{\frac{i}{2b}(dx^2+at^2)-\frac{i}{b}xt}\cdot\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{i}{2q}(st^2+py^2)-\frac{i}{b}ty}\, dt$$
$$= \sqrt{\frac{1}{2\pi ib\cdot 2\pi iq}}\, e^{\frac{i}{2}\left(\frac{d}{b}x^2+\frac{p}{q}y^2\right)}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}+\frac{s}{q}\right)t^2}\cdot e^{-i\left(\frac{x}{b}+\frac{y}{q}\right)t}\, dt$$
$$= \sqrt{\frac{-1}{4\pi^2 bq}}\, e^{\frac{i}{2bq}(dqx^2+bpy^2)}\sqrt{\frac{2\pi bq}{-i(aq+bs)}}\, e^{\left(\frac{i(qx+by)}{bq}\right)^2\times\frac{-2bq}{i(aq+bs)}}$$
$$= \sqrt{\frac{1}{2\pi i(aq+bs)}}\, e^{\frac{i}{2}\left(\frac{cq+ds}{aq+bs}\right)x^2+\frac{i}{2}\left(\frac{rb+ap}{aq+bs}\right)y^2-\frac{i}{aq+bs}xy}$$
$$= K_{A\cdot B}(x,y)$$
$$\int_{-\infty}^{\infty} K_A(x,t)\cdot K_B(t,y)dt = K_{A\cdot B}(x,y)$$

So equation (1) becomes

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{K_i^{A_1\cdot A_3}(x,s)f(x,y)K_j^{A_2\cdot A_4}(y,w)\right\}dxdy$$
$$= F_{i,j}^{A_1\cdot A_3,A_2\cdot A_4}\{f(x,y)\} \tag{2}$$

We have,

$$F_{i,j}^{A_3,A_4} F_{i,j}^{A_1,A_2}\{f(x,y)\} = F_{i,j}^{A_1\cdot A_3,A_2\cdot A_4}\{f(x,y)\} \tag{3}$$

The result can be obtained similarly

$$= F_{i,j}^{A_1,A_2} F_{i,j}^{A_3,A_4}\{f(x,y)\} = F_{i,j}^{A_1\cdot A_3,A_2\cdot A_4}\{f(x,y)\} \tag{4}$$

3.3 If $F_{i,j}^{A_1,A_2}\{f(x,y)\} = F_{i,j}^{A_1,A_2}\{f(u,v)\}$, then $F_{i,j}^{A_1,A_2}\{f(x,y)\} = F_{i,j}^{A_1,A_2}\{f(-u,-v)\}$ $F_{i,j}^{A_1,A_2}\{f(-x,y)\} = F_{i,j}^{A_1,A_2}\{f(-u,v)\}$ and $F_{i,j}^{A_1,A_2}\{f(x,-y)\} = F_{i,j}^{A_1,A_2}\{f(u,-v)\}$

Proof:

$$F_{i,j}^{A_1,A_2}\{f(x,y)\} = \sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi jq}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{id}{2b}u^2+\frac{ia}{2b}x^2-\frac{i}{b}xu}\cdot f(-x,-y)\cdot e^{\frac{is}{2q}v^2+\frac{ip}{2q}y^2-\frac{i}{q}vy}dxdy$$

$$= \sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi jq}}\, e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{s}{q}v^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{ia}{2b}s^2-\frac{i}{b}(-u)s} f(s,z)\, e^{\frac{ip}{2q}z^2-\frac{i}{q}(-v)z}\, dsdz$$

$$F_{i,j}^{A_1,A_2}\{f(x,y)\} = F_{i,j}^{A_1,A_2}\{f(-u,-v)\}$$

The result can be obtained similarly

$$F_{i,j}^{A_1,A_2}\{f(-x,y)\} = F_{i,j}^{A_1,A_2}\{f(-u,v)\} \text{ and}$$

$$F_{i,j}^{A_1,A_2}\{f(x,-y)\} = F_{i,j}^{A_1,A_2}\{f(u,-v)\}$$

3.4 $F_{i,j}^{A_3,A_4}F_{i,j}^{A_1,A_2} = F_{i,j}^{A_1,A_2}F_{i,j}^{A_3,A_4},\ F_{i,j}^{A_5,A_6}\left(F_{i,j}^{A_3,A_4}F_{i,j}^{A_1,A_2}\right) = \left(F_{i,j}^{A_5,A_6}F_{i,j}^{A_3,A_4}\right)F_{i,j}^{A_1,A_2}$

Proof: This property is obtained from Property (3.2) directly.

4. CONVOLUTION IN QLCT

Convolution of any integral transform plays an important role in its development and applicability. Here we have developed convolution in quaternion Linear Canonical transform where convolute for which we have proved.

Theorem 1. For any real, scalar or complex signal $f(x,y)$ and convolution kernel $h(x,y)$ and $g(x,y) = f(x,y) \star h(x,y) \triangleq$
$\sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{-\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\left[e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)} f(x,y) * h(x,y) e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\right]$

where $\star$ is the Linear Canonical Transform convolution operator, then

$$F^{A_1,A_2}\{g(x,y)\} = e^{-\frac{i}{2}\left(\frac{d}{b}u^2+\frac{s}{q}v^2\right)} F^{A_1,A_2}\{f(x,y)\} F^{A_1,A_2}\{h(x,y)\}$$

Proof:

$$F^{A_1,A_2}\{g(x,y)\}(u,v)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^{A_1,A_2}(x,y,u,v)\, g(x,y)\, dxdy$$

$$= \sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{s}{q}v^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)-i\left(\frac{1}{b}xu+\frac{1}{q}yv\right)}\, g(x,y)\, dxdy$$

$$= \left(\sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\right)^2 e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{s}{q}v^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\left(\frac{1}{b}xu+\frac{1}{q}yv\right)}\left[e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)} f(x,y) * h(x,y) e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\right]$$

$$\times\, dxdy$$

$$= \left(\sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\right)^2 e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{s}{q}v^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau,\eta)\cdot e^{\frac{i}{2}\left(\frac{d}{b}\tau^2+\frac{p}{q}\eta^2\right)}$$

$$\times\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\tau,\ y-\eta)\, e^{\frac{i}{2}\left(\frac{a}{b}(x-\tau)^2+\frac{p}{q}(y-\eta)^2\right)}\cdot e^{-i\left(\frac{1}{b}xu+\frac{1}{q}yv\right)}\, dxdy\right\} d\tau d\eta$$

$$= \sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{s}{q}v^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\tau^2+\frac{p}{q}\eta^2\right)-i\left(\frac{1}{b}\tau u+\frac{1}{q}v\eta\right)} f(\tau,\eta)\, d\tau d\eta$$

$$\times \sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{s}{q}v^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}s^2+\frac{p}{q}z^2\right)-i\left(\frac{1}{b}su+\frac{1}{q}zv\right)} h(s,z)\, dsdz \cdot e^{-\frac{i}{2}\left(\frac{d}{b}u^2+\frac{a}{b}v^2\right)}$$

$$= e^{-\frac{i}{2}\left(\frac{d}{b}u^2+\frac{a}{b}v^2\right)}\{F^{A_1,A_2}\{f(\tau,\eta)\}\cdot F^{A_1,A_2}\{h(s,z)\}\}$$

Next theorem we prove for the quaternion functions $f(x,y), h(x,y)$.

Theorem 2: For two given quaternion functions $f(x,y) = f_a(x,y) + f_b(x,y)j$ and $h(x,y) = h_a(x,y) + h_b(x,y)j$ where $f_a(x,y) = f_r(x,y) + if_i(x,y)$ and $f_b(x,y) = f_j(x,y) + if_k(x,y)$, $h_a(x,y) = h_r(x,y) + ih_i(x,y)$, $h_b(x,y) = h_j(x,y) + ih_k(x,y)$ and defining

$$g(x,y) = f(x,y) \star h(x,y) = (f \star h)(x,y) \triangleq$$
$$\sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{-i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\left[\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)f(x,y) * h(x,y)\,\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)\right]$$

where $\star$ is the Linear Canonical convolution operator, then

$$F^{A_1,A_2}\{g(x,y)\} = e^{-\frac{i}{2}\left(\frac{d}{b}u^2+\frac{a}{b}v^2\right)}\left\{F^{A_1,A_2}\{f_a(\tau,\eta)\}\cdot F^{A_1,A_2}\{h_a(s,z)\} - F^{A_1,A_2}\{f_b(\tau,\eta)\}\cdot F^{A_1,A_2}\{\overline{h_b}(s,z)\}\right\} + e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{a}{b}v^2\right)}\left\{F^{A_1,A_2}\{f_a(\tau,\eta)\}\cdot F^{-A_1,-A_2}\{h_b(-s,-z)\} + F^{A_1,A_2}\{f_b(\tau,\eta)\}\cdot F^{-A_1,-A_2}\{\overline{h_a}(-s,-z)\}\right\}j$$

Proof: Substituting the values of $f(x,y)$ and $h(x,y)$ in $g(x,y)$, we get

$$\sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{-i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\left[e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}[f_a(x,y) + f_b(x,y)j\,] * [h_a(x,y) + h_b(x,y)j\,]e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\right]$$

$$= \sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{-i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\left[e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}f_a(x,y) * h_a(x,y)e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\right]$$
$$+\sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{-i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\left[e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}f_a(x,y) * \left(h_b(x,y)e^{\frac{-i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\right)j\right]$$
$$+\sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{-i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\left[e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}f_b(x,y) * \left(\overline{h_a}(x,y)e^{\frac{-i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\right)j\right]$$
$$-\sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{-i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\left[e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}f_b(x,y) * \overline{h_b}(x,y)e^{-\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\right]$$
$$= I_1 + I_2 + I_3 + I_4 \qquad (1)$$

Now,

$$I_1 = \sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{-i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\left[e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}f_a(x,y) * h_a(x,y)e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\right]$$

Using theorem 1, we get

$$= \sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{s}{q}v^2\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\tau^2+\frac{p}{q}\eta^2\right)-i\left(\frac{1}{b}\tau u+\frac{1}{q}\eta v\right)}f_a(\tau,\eta)\,d\tau d$$
$$\times\sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{s}{q}v^2\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}s^2+\frac{p}{q}z^2\right)-i\left(\frac{1}{b}su+\frac{1}{q}zv\right)}h_a(s,z)dsdz\cdot e^{-\frac{i}{2}\left(\frac{d}{b}u^2+\frac{a}{b}v^2\right)}$$
$$= e^{-\frac{i}{2}\left(\frac{d}{b}u^2+\frac{a}{b}v^2\right)}\{F^{A_1,A_2}\{f_a(\tau,\eta)\}\cdot F^{A_1,A_2}\{h_a(s,z)\}\}$$

$$I_2 = \sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{-i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)} \left[e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)} f_a(x,y) * \left(h_b(x,y)e^{\frac{-i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\right)j\right]$$

$$= \sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{s}{q}v^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)-i\left(\frac{1}{b}xu+\frac{1}{q}yv\right)} \sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\, e^{\frac{-i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}$$

$$\times e^{\frac{i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)} f_a(x,y) * \left(h_b(x,y)e^{\frac{-i}{2}\left(\frac{a}{b}x^2+\frac{p}{q}y^2\right)}\right) j\,dxdy$$

$$= j\left(\sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\right)^2 e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{s}{q}v^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_a(\tau,\eta)\cdot e^{\frac{i}{2}\left(\frac{d}{b}\tau^2+\frac{p}{q}\eta^2\right)}$$

$$\times\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_b(x-\tau,\ y-\eta)e^{-\frac{i}{2}\left(\frac{a}{b}(x-\tau)^2+\frac{p}{q}(y-\eta)^2\right)}\cdot e^{-i\left(\frac{1}{b}xu+\frac{1}{q}yv\right)} dxdy\right\} d\tau d\eta$$

Substituting $x-\tau=-s$ and $y-\eta=-z$ in above equation, we get

$$= j\left(\sqrt{\frac{1}{2\pi ib}}\sqrt{\frac{1}{2\pi iq}}\right)^2 e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{s}{q}v^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_a(\tau,\eta)\cdot e^{\frac{i}{2}\left(\frac{d}{b}\tau^2+\frac{p}{q}\eta^2\right)}$$

$$\times\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_b(-s,\ -z)e^{-\frac{i}{2}\left(\frac{a}{b}s^2+\frac{p}{q}z^2\right)}\cdot e^{-i\left(\frac{1}{b}(-s+\tau)u+\frac{1}{q}(-z+\eta)v\right)} dsdz\right\} d\tau d\eta$$

$$= e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{a}{b}v^2\right)}\left\{F^{A_1,A_2}\{f_a(\tau,\eta)\}\cdot F^{-A_1,-A_2}\{h_b(-s,-z)\}\right\}j$$

Similarly we can find I_3 and I_4 using the theorem 1

$$I_3 = e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{a}{b}v^2\right)}\left\{F^{A_1,A_2}\{f_b(\tau,\eta)\}\cdot F^{-A_1,-A_2}\{\overline{h_a}(-s,-z)\}\right\}j \text{ and}$$

$$I_4 = e^{-\frac{i}{2}\left(\frac{d}{b}u^2+\frac{a}{b}v^2\right)}\left\{F^{A_1,A_2}\{f_b(\tau,\eta)\}\cdot F^{A_1,A_2}\{\overline{h_b}(s,z)\}\right\}$$

Substituting the values of I_1, I_2, I_3, I_4 in equation (1) we get,

$$F^{A_1,A_2}\{g(x,y)\} = e^{-\frac{i}{2}\left(\frac{d}{b}u^2+\frac{a}{b}v^2\right)}\left\{F^{A_1,A_2}\{f_a(\tau,\eta)\}\cdot F^{A_1,A_2}\{h_a(s,z)\} - F^{A_1,A_2}\{f_b(\tau,\eta)\}\cdot F^{A_1,A_2}\{\overline{h_b}(s,z)\}\right\} + e^{\frac{i}{2}\left(\frac{d}{b}u^2+\frac{a}{b}v^2\right)}\left\{F^{A_1,A_2}\{f_a(\tau,\eta)\}\cdot F^{-A_1,-A_2}\{h_b(-s,-z)\} + F^{A_1,A_2}\{f_b(\tau,\eta)\}\cdot F^{-A_1,-A_2}\{\overline{h_a}(-s,-z)\}\right\}j$$

5. CONCLUSION

This paper gives the extension of fractional quaternion Fourier transform and gives its convolution generalization to linear canonical transform for the quaternion function along with its inverse. Then the fractional quaternion convolution is defined and its properties are discussed lastly. It is proved that fractional convolution of two quaternion functions in LCT domain is the sum of the product of its LCT of their components, conjugated operators or sign change variable functions along with product by chirp function.

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