ORIGINAL PAPER

ON A CLASS OF TRILATERAL GENERATING FUNCTIONS BY GROUP THEORETIC METHOD

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Abstract. *In this article we have presented a novel result in connection with the unification of a class of trilateral generating relations for certain special functions with Tchebycheff polynomial by group theoretic method. A good number of applications of our result are also given in section 3 of this paper.*

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1. INTRODUCTION

Theories in connection with the unification of bilateral or trilateral generating relations for various special functions are of greater importance in the study of special functions. For previous works in this direction, one can see the works [1-8] and [9-16] in connection with the unification of bilateral and mixed trilateral generating relations.

In this paper, we have made an attempt to present a novel result in connection with the unification of trilateral generating relations for certain special functions by group theoretic method, of course when suitable continuous transformations group can be constructed for the special function under consideration, with Tchebycheff polynomials. In fact, this method [17] is based on the theory of one parameter group of continuous transformations by means of which any unilateral generating relation involving a special function can be transformed in to a bilateral generating relation and then into a trilateral generating relation with Tchebycheff polynomial by means of the relation [18]:

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right] \tag{1.1}$$

Furthermore, we would like to mention that in course of application of our result, we get a good number of theorems on generating relations for various special functions.

The detailed discussion is given below:

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2. GROUP-THEORETIC DISCUSSION

Let us first consider the following unilateral generating relation,

$$G(x,w)=\sum_{n=0}^{\infty} a_n \ p_{n+r}^{(\alpha)}(x)\, w^n \tag{2.1}$$

where $p_n^{(\alpha)}(x)$ is a special function of degree n and of parameter α and a_n is independent of x, w.

Replacing w by vwz and then multiplying both sides of (2.1) by y^{α} we get,

$$y^{\alpha} G(x,vwz)=\sum_{n=0}^{\infty} a_n \left(p_{n+r}^{(\alpha)}(x)\, y^{\alpha} z^n \right) (vw) \tag{2.2}$$

We now suppose that for the above special function, it is possible to define a linear partial differential operator *R*, which generates a continuous transformations group as follows:

$$R=\xi(x,y,z)\frac{\partial}{\partial x}+\eta(x,y,z)\frac{\partial}{\partial y}+\zeta(x,y,z)\frac{\partial}{\partial z}+\ \theta(x,y,z,r)$$

such that

$$R\ \left(p_{n+r}^{(\alpha)}(x) y^{\alpha} z^n \right)=\rho_{n+r}\ p_{n+r+1}^{(\alpha-1)}(x) y^{\alpha-1} z^{n+1} \tag{2.3}$$

and

$$e^{wR}\ f(x,y,z)=\Omega(x,y,z,w)\, f\left(\ g(x,y,z,w), h(x,y,z,w)\ , k(x,y,z,w)\ \right) \tag{2.4}$$

Operating both sides of (2.2) by e^{wR}, we get

$$e^{wR}\left(y^{\alpha} G(x,vwz)\ \right)=e^{wR}\left(\sum_{n=0}^{\infty} a_n \ (p_{n+r}^{(\alpha)}(x) y^{\alpha} z^n)\ (vw)^n \right). \tag{2.5}$$

The left number of (2.5), with the help of (2.4), becomes

$$\Omega(x,y,z,w)\left(h(x,y,z,w)\right)^{\alpha}\ G\left(\ g(x,y,z,w), vwk(x,y,z,w)\ \right). \tag{2.6}$$

The right number of (2.5), with the help of (2.3), becomes

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n \frac{w^k}{k!}\ \rho_{n+r}\rho_{n+r+1}\cdots\cdots\cdots\rho_{n+r+k-1}\ p_{n+r+k}^{(\alpha-k)}(x)\ y^{\alpha-k} z^{n+k}\ (vw)^n. \tag{2.7}$$

Now equating (2.6) and (2.7) and then putting $y=z=1$, we get

$$\Omega(x,1,1,w)\left(h(x,1,1,w)\right)^{\alpha}\ G\left(\ g(x,1,1,w),vwk(x,1,1,w)\right) \tag{2.8}$$
$$=\sum_{n=0}^{\infty}\ \sum_{k=0}^{\infty}a_n\frac{w^{n+k}}{k!}\rho_{n+r}\rho_{n+r+1}\cdots\rho_{n+r+k-1}\ p_{n+r+k}^{(\alpha-k)}(x)\ v^n$$
$$=\sum_{n=0}^{\infty}\ \sum_{k=0}^{n}a_{n-k}\frac{w^{n}}{k!}\rho_{n+r-k}\rho_{n+r-k+1}\cdots\rho_{n+r-1}\ p_{n+r}^{(\alpha-k)}(x)\ v^{n-k}$$
$$=\sum_{n=0}^{\infty}\ w^n\sum_{k=0}^{n}a_k\frac{\rho_{k+r}\rho_{k+r+1}\cdots\rho_{n+r-1}}{(n-k)!}\ p_{n+r}^{(\alpha-n+k)}(x)\ v^k$$
$$=\sum_{n=0}^{\infty}\ w^n\sum_{k=0}^{n}\ a_k\frac{\prod_{i=k}^{n-1}\rho_{i+r}}{(n-k)!}\ p_{n+r}^{(\alpha-n+k)}(x)\ v^k$$
$$=\sum_{n=0}^{\infty}\ w^n\ \sigma_n(x,v)$$

where

$$\sigma_n(x,v)=\sum_{k=0}^{n}\ a_k\ \frac{\prod_{i=k}^{n-1}\rho_{i+r}}{(n-k)!}\ \ p_{n+r}^{(\alpha-n+k)}(x)\,v^k \tag{2.9}$$

Now to convert the above bilateral generating relation into a trilateral generating relation with Tchebycheff polynomial, we notice that

$$\sum_{n=0}^{\infty}\ \sigma_n(x,v)\ T_n(u)\,w^n$$
$$=\sum_{n=0}^{\infty}\ \sigma_n(x,v)\ \frac{1}{2}\left[(u+\sqrt{u^2-1})^n+(u-\sqrt{u^2-1})^n\right]\ w^n$$
$$=\frac{1}{2}\left[\sum_{n=0}^{\infty}\ \sigma_n(x,v)\left\{w\,(u+\sqrt{u^2-1})\ \right\}^n+\sum_{n=0}^{\infty}\ \sigma_n(x,v)\left\{w\,(u-\sqrt{u^2-1})\ \right\}^n\right]$$
$$=\frac{1}{2}\left[\sum_{n=0}^{\infty}\ \sigma_n(x,v)\,\rho_1{}^n+\sum_{n=0}^{\infty}\ \sigma_n(x,v)\,\rho_2{}^n\right]$$
$$=\frac{1}{2}\Big[\Omega(x,1,1,\rho_1)\left(\ h(x,1,1,\rho_1)\right)^{\alpha}\ G\left(\ g(x,1,1,\rho_1),v\rho_1k(x,1,1,\rho_1)\ \right)$$
$$+\ \Omega(x,1,1,\rho_2)\left(\ h(x,1,1,\rho_2)\right)^{\alpha}\ G\left(\ g(x,1,1,\rho_2),v\rho_2k(x,1,1,\rho_2)\ \right)\Big]$$

where

$$\rho_1=\ w\,(u+\sqrt{u^2-1})\quad\text{and}\quad\rho_2=w\,(u-\sqrt{u^2-1})$$

Thus we have prove the following theorem.

Theorem 1. If there exists a unilateral generating relation of the form:

$$G(x,w)=\sum_{n=0}^{\infty}a_n\,p_{n+r}^{(\alpha)}(x)\ w^n$$

then

$$\sum_{n=0}^{\infty} \sigma_n(x,v)\ T_n(u)\ w^n$$
$$= \frac{1}{2}\Big[\Omega(x,1,1,\rho_1)\left(h(x,1,1,\rho_1)\right)^{\alpha} G\left(g(x,1,1,\rho_1), v\rho_1 k(x,1,1,\rho_1)\right)$$
$$+\Omega(x,1,1,\rho_2)\left(h(x,1,1,\rho_2)\right)^{\alpha} G\left(g(x,1,1,\rho_2), v\rho_2 k(x,1,1,\rho_2)\right)\Big]$$

where

$$\sigma_n(x,v) = \sum_{k=0}^{n} a_k \frac{\prod_{i=k}^{n-1}\rho_{i+r}}{(n-k)!}\ p_{n+r}^{(\alpha-n+k)}(x)\ v^k$$

$$\rho_1 = w\left(u+\sqrt{u^2-1}\right) \quad \text{and} \quad \rho_2 = w\left(u-\sqrt{u^2-1}\right)$$

The above theorem does not seem to have appeared in the earlier works.

Corollary 1. If we put $r = 0$, we get the result found derived in [16].

We now proceed to give a good number of applications of our result.

3. APPLICATIONS

Below we give some applications of our result.

Application 1: At first we take

$$p_{n+r}^{(\alpha)}(x) = L_{a,\,b,\,m,\,n+r}(x) \ \text{ with } \ \alpha = m \tag{3.1}$$

Now we consider the following operator R:

$$R = bxy^{-1}z\frac{\partial}{\partial x} + zb\frac{\partial}{\partial y} - (ax+b)y^{-1}z$$

such that

$$R\left(L_{a,\,b,\,m,\,n+r}(x)\ y^m z^n\right) = (n+r+1)\ L_{a,\,b,\,m-1,\,n+r+1}(x)\ y^{m-1}z^{n+1} \tag{3.2}$$

and

$$e^{wR}\ f(x,y,z) = \left(1+\frac{bwz}{y}\right)^{-1} \exp\left(-\frac{awxz}{y}\right) f\left(x\left(1+\frac{bwz}{y}\right), y\left(1+\frac{bwz}{y}\right), z\right). \tag{3.3}$$

So by comparing (3.2), (3.3) with (2.3),(2.4), we get

$$\rho_{n+r} = (n + r + 1)\,,$$

$$\Omega(x, y, z, w) = \left(1 + \frac{bwz}{y}\right)^{-1} \exp\left(-\frac{awxz}{y}\right),$$

$$g(x, y, z, w) = x\left(1 + \frac{bwz}{y}\right),$$

$$h(x, y, z, w) = y\left(1 + \frac{bwz}{y}\right),$$

$$k(x, y, z, w) = z\,.$$

Therefore, by the application of our theorem, we get the following result on trilateral generating relations with Tchebycheff polynomials.

Theorem 2: If

$$G(x, w) = \sum_{n=0}^{\infty} a_n L_{a,\,b,\,m,\,n+r}(x)\;\; w^n \tag{3.4}$$

then

$$\sum_{n=0}^{\infty} T_n(u)\,\sigma_n(x,v)\;\; w^n = \frac{1}{2}\Big[(1+b\rho_1)^{-1+m} \exp\left(-a\rho_1 x\right) G\left(x(1+b\rho_1), v\rho_1\right) + (1+b\rho_2)^{-1+m} \exp\left(-a\rho_2 x\right) G\left(x(1+b\rho_2), v\rho_2\right)\Big] \tag{3.5}$$

where

$$\sigma_n(x,v) = \sum_{k=0}^{n} a_k \binom{n+r}{k+r} L_{a,\,b,\,m-n+k,\,n+r}(x)\;\; v^k\,.$$

$$\rho_1 = w\,(u + \sqrt{u^2 - 1}) \quad \text{and} \quad \rho_2 = w\,(u - \sqrt{u^2 - 1})$$

which seems to be new.

Corollary 2: If we put $r = 0$ in the above theorem, we get the following result.

$$G(x, w) = \sum_{n=0}^{\infty} a_n L_{a,\,b,\,m,\,n}(x)\;\; w^n$$

then

$$\sum_{n=0}^{\infty} T_n(u)\,\sigma_n(x,v)\;\; w^n = \frac{1}{2}\Big[(1+b\rho_1)^{-1+m} \exp\left(-a\rho_1 x\right) G\left(x(1+b\rho_1), v\rho_1\right) + (1+b\rho_2)^{-1+m} \exp\left(-a\rho_2 x\right) G\left(x(1+b\rho_2), v\rho_2\right)\Big] \tag{3.5}$$

where

$$\sigma_n(x,v) = \sum_{k=0}^{n} a_k \binom{n+r}{k+r} L_{a,\,b,\,m-n+k,\,n+r}(x)\;\; v^k\,.$$

$$\rho_1 = w\,(u + \sqrt{u^2 - 1}) \quad \text{and} \quad \rho_2 = w\,(u - \sqrt{u^2 - 1})$$

3.1. SOME SPECIAL CASES OF INTEREST

On specializing the parameters $a = b = 1$ and $m = 1 + \alpha$ in the Theorem 2, we get the following result on Laguerre polynomials:

Result. If $G(x, w) = \sum_{n=0}^{\infty} a_n L_{n+r}^{(\alpha)}(x) \; w^n$ then

$$\sum_{n=0}^{\infty} T_n(u)\, \sigma_n(x, v) \; w^n$$

$$= \frac{1}{2}\Big[(1+\rho_1)^{\alpha} \exp\left(-\rho_1 x\right) G\left(x(1+\rho_1), v\rho_1\right) + (1+\rho_2)^{\alpha} \exp\left(-\rho_2 x\right) G\left(x(1+\rho_2), v\rho_2\right)\Big]$$

where

$$\sigma_n(x, v) = \sum_{k=0}^{n} a_k \binom{n+r}{k+r} L_{n+r}^{\alpha-n+k}(x) \; v^k ,$$

$$\rho_1 = w\,(u + \sqrt{u^2 - 1}) \quad \text{and} \quad \rho_2 = w\,(u - \sqrt{u^2 - 1}),$$

which does not seem to have appeared in the earlier works.

Corollary 3: If we put $r = 0$ in the above result, we get the result found derived in [19].

Application 2: Now we take

$$p_{n+r}^{(\alpha)}(x) = f_{n+r}^{\beta}(x) \text{ with } \alpha = \beta, \tag{3.6}$$

and we consider the following partial differential operator R [20] :

$$R = xy^{-1}z\,\frac{\partial}{\partial x} - z\,\frac{\partial}{\partial y} - (x-1)y^{-1}z$$

such that

$$R\left(f_{n+r}^{\beta}(x)\, y^{\beta} z^{n} \right) = -\,(n+r+1)\; f_{n+r+1}^{\beta-1}(x) y^{\beta-1} z^{n+1} \tag{3.7}$$

and

$$e^{wR}\; f(x, y, z) = \left(\frac{y}{y - wz}\right) \exp\left(\frac{-\,xwz}{y - zw}\right)\; f\left(\frac{xy}{y - zw}, y - zw, z\right) \tag{3.8}$$

Then by comparing (3.7), (3.8) with (2.3), (2.4), and finally applying our theorem (Theorem 1), we get the following result on trilateral generating relations with Tchebycheff polynomials:

Theorem 3: If

$$G(x, w) = \sum_{n=0}^{\infty} a_n\, f_{n+r}^{\beta}(x)\; w^n \tag{3.9}$$

then

$$\sum_{n=0}^{\infty} \sigma_n(x,v)\ T_n(u)\ w^n$$
$$= \frac{1}{2}\left[(1+\rho_1)^{\beta-1} \exp\left(\frac{x\rho_1}{1+\rho_1}\right) G\left(\frac{x}{1+\rho_1}, v\rho_1\right) + (1+\rho_2)^{\beta-1} \exp\left(\frac{x\rho_2}{1+\rho_2}\right) G\left(\frac{x}{1+\rho_2}, v\rho_2\right)\right] \tag{3.10}$$

where

$$\sigma_n(x,v) = \sum_{k=0}^{n} a_n \binom{n+r}{k+r} f_{n+r}^{(\beta-n+k)}(x)\ v^k,$$
$$\rho_1 = w\,(u+\sqrt{u^2-1}) \quad \text{and} \quad \rho_2 = w\,(u-\sqrt{u^2-1})\ ,$$

which does not seem to appear before.

Corollary 4: If we put $r=0$ in the above theorem, we get the result found derived in [19].

Application 3: We now take

$$p_{n+r}^{(\alpha)}(x) = Y_{n+r}^{(\alpha)}(x)$$

Then from [21], we notice that

$$R = x^2 y^{-1} z \frac{\partial}{\partial x} + xz\frac{\partial}{\partial y} + x\ y^{-1} z^2 \frac{\partial}{\partial z} + \left((r-1)x+\beta\right) y^{-1} z$$

such that

$$R\ \left(Y_{n+r}^{(\alpha)}(x)\ y^{\alpha}\ z^n\right) = \beta\, Y_{n+r+1}^{(\alpha-1)}(x)\ y^{\alpha-1}\ z^{n+1} \tag{3.11}$$

and

$$e^{wR}\ f(x,y,z) = (1-wxy^{-1}z)\exp(\beta\, wy^{-1}z) \times f\left(\frac{x}{1-wxy^{-1}z}, \frac{y}{1-wxy^{-1}z}, \frac{z}{1-wxy^{-1}z}\right) \tag{3.12}$$

Then by comparing (311), (3.12) with (2.3), (2.4), and finally applying our theorem (Theorem 1), we get the following result on trilateral generating relations with Tchebycheff polynomials:

Theorem 4: If

$$G(x,w) = \sum_{n=0}^{\infty} a_n\ Y_{n+r}^{(\alpha)}(x)\ \ w^n \tag{3.13}$$

then

$$\sum_{n=0}^{\infty} \sigma_n(x,v) T_n(u)\ w^n$$
$$= \frac{1}{2}\left[\exp(\beta\rho_1)(1-\rho_1 x)^{1-\alpha-r} G\left(\frac{x}{1-\rho_1 x}, \frac{v\rho_1}{1-\rho_1 x}\right) + \exp(\beta\rho_2)(1-\rho_2 x)^{1-\alpha-r} G\left(\frac{x}{1-\rho_2 x}, \frac{v\rho_2}{1-\rho_2 x}\right)\right] \tag{3.14}$$

where

$$\sigma_n(x,v) = \sum_{k=0}^{n} a_k \frac{\beta^{n-k}}{(n-k)!} Y_{n+r}^{(\alpha-n+k)}(x)\, v^k$$
$$\rho_1 = w\,(u+\sqrt{u^2-1}) \quad \text{and} \quad \rho_2 = w\,(u-\sqrt{u^2-1}) \tag{3.15}$$

which does not seem to appear before.

Corollary 5: If we put $r = 0$ in the above theorem, we get the result found derived in [19].

Application 4: We now take

$$p_{n+r}^{(\alpha)}(x) = C_{n+r}^{\lambda}(x) \quad \text{with } \alpha = \lambda .$$

Then from [22], we notice that

$$R = (x^2-1)y^{-1}z\frac{\partial}{\partial x} + 2x\,z\frac{\partial}{\partial y} - xy^{-1}z$$

such that

$$R\left(C_{n+r}^{\lambda}(x)y^{\lambda}z^{n} \right) = \frac{(n+r+2\lambda-1)(n+r+1)}{2(\lambda-1)} C_{n+r+1}^{\lambda-1}(x)y^{\lambda-1}z^{n+1} \tag{3.16}$$

and

$$e^{wR}\, f(x,y,z) = \left\{1+2wxy^{-1}z+(x^2-1)w^2y^{-2}z^2\right\}^{-1/2}$$
$$\times f\left(x+w(x^2-1)y^{-1}z,\, y\{1+2wxy^{-1}z+(x^2-1)w^2y^{-2}z^2\},\, z \right). \tag{3.17}$$

Then by comparing (3.16), (3.17) with (2.3), (2.4), and finally applying our theorem (Theorem 1), we get the following result on trilateral generating relations with Tchebycheff polynomials:

Theorem 5: If

$$G(x,w) = \sum_{n=0}^{\infty} a_n C_{n+r}^{\lambda}(x)\, w^n \tag{3.18}$$

then

$$\sum_{n=0}^{\infty} \sigma_n(x,v)T_n(u)\left(\frac{w}{2}\right)^n$$
$$= \frac{1}{2}\Big[\left\{1+2\rho_1 x+\rho_1^2(x^2-1)\right\}^{\lambda-1/2} G\left(x+\rho_1(x^2-1), \rho_1 v \right) \tag{3.19}$$
$$+\left\{1+2\rho_2 x+\rho_2^2(x^2-1)\right\}^{\lambda-1/2} G\left(x+\rho_2(x^2-1), \rho_2 v \right)\Big]$$

where

$$\sigma_n(x,v) = \sum_{k=0}^{n} a_k \binom{n+r}{k+r} \frac{(-2\lambda-r-k+1)_{n-k}}{(-\lambda+1)_{n-k}} C_{n+r}^{\lambda-n+k}(x)\,(2v)^k\ ,$$

$$\rho_1 = \frac{w}{2}\ (u + \sqrt{u^2 - 1}) \quad \text{and} \quad \rho_2 = \frac{w}{2}\ (u - \sqrt{u^2 - 1}),$$

which does not seem to have appeared in the earlier works

Corollary 6: If we put $r = 0$ in the above theorem, we get result found derived in [19]

Application 5: We now take

$$p_{n+r}^{(\alpha)}(x) = P_{n+r}^{(\alpha,\,\beta)}(x)$$

Then we consider the operator R [23], where

$$R = (1 - x^2)y^{-1}z\frac{\partial}{\partial x} - (1 - x)z\frac{\partial}{\partial y} + (1 - x)\ y^{-1}z^2\frac{\partial}{\partial z} + (1 - x)y^{-1}z(1 + \beta + r)$$

such that

$$R\left(P_{n+r}^{(\alpha,\,\beta)}(x)\ y^{\alpha}\ z^{n}\right) = -2(n + r + 1)\ P_{n+r+1}^{(\alpha-1,\,\beta)}(x)\ y^{\alpha-1}\ z^{n+1} \tag{3.20}$$

and

$$\begin{aligned} e^{wR}\ f(x,y,z) &= \{1 + w(x-1)y^{-1}z\}^{-1-\beta-r} \\ &\times f\left(\frac{x - w(x-1)y^{-1}z}{1 + w(x-1)y^{-1}z}, \frac{y(1 - 2wy^{-1}z)}{1 + w(x-1)y^{-1}z}, \frac{z}{1 + w(x-1)y^{-1}z}\right). \end{aligned} \tag{3.21}$$

Then by comparing (3.20), (3.21) with (2.3), (2.4), and finally applying our theorem (Theorem 1), we get the following result on trilateral generating relations with Tchebycheff polynomials:

Theorem 6: If

$$G(x,w) = \sum_{n=0}^{\infty} a_n P_{n+r}^{(\alpha,\,\beta)}(x)\ w^n \tag{3.22}$$

then

$$\begin{aligned} &\sum_{n=0}^{\infty} \sigma_n(x,v) T_n(u)\, t^n \\ &= \frac{1}{2}\Bigg[\{1 - (1+x)\frac{\rho_1}{2}\}^{-1-\alpha-\beta-r}\ (1-\rho_1)^{\beta}\ G\left(\frac{x - (1+x)\frac{\rho_1}{2}}{1 - (1+x)\frac{\rho_1}{2}}, \frac{\rho_1 v}{1 - (1+x)\frac{\rho_1}{2}}\right) \\ &+ \{1 - (1+x)\frac{\rho_2}{2}\}^{-1-\alpha-\beta-r}\ (1-\rho_2)^{\beta}\ G\left(\frac{x - (1+x)\frac{\rho_2}{2}}{1 - (1+x)\frac{\rho_2}{2}}, \frac{\rho_2 v}{1 - (1+x)\frac{\rho_2}{2}}\right)\Bigg] \end{aligned} \tag{3.23}$$

where

$$\sigma_n(x,v) = \sum_{k=0}^{n} a_k \binom{n+r}{k+r}\ P_{n+r}^{(\alpha,\,\beta-n+k)}(x)\ v^k,$$

$$\rho_1 = t\ (u + \sqrt{u^2 - 1}) \quad \text{and} \quad \rho_2 = t\ (u - \sqrt{u^2 - 1}).$$

which does not seem to have appeared in the earlier works.

Corollary 7: If we put $r=0$ in the above theorem, we immediately get the result found derived in [19].

Now if in place of *R*, we consider the following operator R_1 [23]:

$$R_1=(1-x^2)y^{-1}z\frac{\partial}{\partial x}+(1-x)z\frac{\partial}{\partial y}-(1+x)\ y^{-1}z^2\frac{\partial}{\partial z}-(1+\alpha+r)(1+x)y^{-1}z.$$

such that

$$R_1\left(P_{n+r}^{(\alpha,\,\beta)}(x)\ y^{\beta}\ z^n\right)=-2(n+r+1)\ P_{n+r+1}^{(\alpha,\,\beta-1)}(x)\ y^{\beta-1}\ z^{n+1} \tag{3.24}$$

and

$$e^{wR_1}f(x,y,z)=\left\{1+w(1+x)y^{-1}z\right\}^{-1-\alpha-r}\times f\left(\frac{x+w(1+x)y^{-1}z}{1+w(1+x)y^{-1}z},\frac{y(1+2wy^{-1}z)}{1+w(1+x)y^{-1}z},\frac{z}{1+w(1+x)y^{-1}z}\right). \tag{3.25}$$

Then by the application of our theorem, we get the following result (analogous to Theorem 6) on bilateral generating relation involving Jacobi polynomial.

Theorem 7: If

$$G(x,w)=\sum_{n=0}^{\infty}a_nP_{n+r}^{(\alpha,\ \beta)}(x)\ w^n \tag{3.26}$$

then

$$\begin{aligned}\sum_{n=0}^{\infty}\ \sigma_n(x,v)T_n(u)\ w^n &=\frac{1}{2}\Bigg[\{1-(x-1)\frac{\rho_1}{2}\}^{-1-\alpha-\beta-r}\ (1+\rho_1)^{\alpha}\ G\left(\frac{x+(x-1)\frac{\rho_1}{2}}{1-(x-1)\frac{\rho_1}{2}},\frac{\rho_1 v}{1-(x-1)\frac{\rho_1}{2}}\right)\\ &+\{1-(x-1)\frac{\rho_2}{2}\}^{-1-\alpha-\beta-r}\ (1+\rho_2)^{\alpha}\ G\left(\frac{x+(x-1)\frac{\rho_2}{2}}{1-(x-1)\frac{\rho_2}{2}},\frac{\rho_2 v}{1-(x-1)\frac{\rho_2}{2}}\right)\Bigg]\end{aligned} \tag{3.27}$$

where

$$\sigma_n(x,v)=\sum_{k=0}^{n}a_k\binom{n+r}{k+r}\ P_{n+r}^{(\alpha-n+k,\,\beta)}(x)\ v^k$$

$$\rho_1=\ t\ (u+\sqrt{u^2-1})\quad \text{and}\quad \rho_2=t\ (u-\sqrt{u^2-1})$$

Corollary 8: If we put $r=0$ in the above theorem, we get the result found derived in [19]. It may be pointed out that the Theorem 7 can be directly obtained from Theorem 6 by using the symmetry relation [17]

$$P_n^{(\alpha,\beta)}(-x)=(-1)^n\,P_n^{(\beta,\alpha)}(x).$$

Application 6: Finally, we take

$$p_{n+r}^{(\alpha)}(x) = {}_2F_1(\overline{-n+r}, \beta\,; v\,; x) \text{ with } \alpha = v.$$

Then we consider the operator R, where

$$R = x(1-x)y^{-1}z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - (x\beta+1)y^{-1}z$$

such that

$$R\left({}_2F_1(\overline{-n+r}, \beta\,; v\,; x)y^v z^n \right) = (v-1)\ {}_2F_1(\overline{-n+r+1}, \beta\,; v-1\,; x)y^{v-1}z^{n+1} \tag{3.28}$$

and

$$\begin{aligned} e^{wR}\ f(x,y,z) &= (1+wy^{-1}z)^{-1}\ (1+xwy^{-1}z)^{-\beta} \\ &\times f\left(\frac{x(1+wy^{-1}z)}{(1+wxy^{-1}z)}, y(1+wy^{-1}z)\ , z\right). \end{aligned} \tag{3.29}$$

Then by comparing (3.28), (3.29) with (2.3), (2.4) and finally applying our theorem (Theorem 1), we get the following result on trilateral generating relations with Tchebycheff polynomials

Theorem 8: If

$$G(x,w) = \sum_{n=0}^{\infty} a_n\ {}_2F_1(\overline{-n+r}, \beta\,; v-n+k;\, x)\ \ w^n \tag{3.30}$$

then

$$\begin{aligned} &\sum_{n=0}^{\infty} \sigma_n(x,v)\,T_n(u)\,w^n \\ &= \frac{1}{2}\left[(1-\rho_1)^{v-1}\ (1-x\rho_1)^{-\beta}\ G\left(\frac{x(1-\rho_1)}{1-\rho_1 x}, v\rho_1\right)\right] \\ &+ \frac{1}{2}\left[(1-\rho_2)^{v-1}\ (1-x\rho_2)^{-\beta}\ G\left(\frac{x(1-\rho_2)}{1-\rho_{21} x}, v\rho_2\right)\right] \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} \sigma_n(x,v) &= \sum_{k=0}^{n} a_k \frac{(-v+1)_{n-k}}{(n-k)!}\ {}_2F_1(\overline{-n+r}, \beta\,; v\,; x)\ v^k\ , \\ \rho_1 &= w\,(u+\sqrt{u^2-1}) \quad \text{and} \quad \rho_2 = w\,(u-\sqrt{u^2-1}) \end{aligned} \tag{3.32}$$

which does not seem to have appeared in the earlier works.

Corollary 9: If we put $r = 0$ in the above theorem, we get the result found derived in [19].

4. CONCLUSION

From the above discussion, it is clear that one may apply Theorem-1 in the case of other polynomials and functions existing in the field of special functions subject to the condition of construction of continuous transformations group for the said special function. Furthermore, the importance of the above theorems (2-8) lies in the fact that whenever one knows a unilateral generating relation of the form (3.4, 3.9 etc.) then the corresponding trilateral generating functions can at once be written down from (3.5, 3.10 etc.). Thus, one can get a large number of trilateral generating functions with Tchebycheff polynomials by attributing different suitable values to a_n.

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