ORIGINAL PAPER

A NEW GENERALIZATION OF NESBITT'S INEQUALITY

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Abstract. *This paper presents a retrospective of our refinements, extensions and generalizations of Nesbitt's inequality. Also we present a new generalization for this remarkable inequality.*

Keywords: *Radon's inequality, Bergström's inequality, Jensen's inequality, Nesbitt's inequality.*

1. INTRODUCTION

In this section, we will do a retrospective of our results on Nesbitt's inequality:

1.1. ON NESBITT'S INEQUALITY FOR TWO VARIABLES

If $x_1, x_2 \in R_+^*$, then:

$$A(x_1, x_2) = \frac{x_1}{x_2} + \frac{x_2}{x_1} \geq 2, \text{ (see for e.g. [6, 11])}$$

1.1.1. If $a, b, x_1, x_2 \in R_+^*, X_2 = x_1 + x_2$ and $aX_2 > b\max\{x_1, x_2\}$, then:

$$B(x_1, x_2) = \frac{x_1}{aX_2 - bx_1} + \frac{x_2}{aX_2 - bx_2} \geq \frac{2}{2a - b}, \text{ (see [6, 11]).}$$

1.2. ON NESBITT'S INEQUALITY FOR THREE VARIABLES

If $x_1, x_2, x_3 \in R_+^*$, then:

$$C(x_1, x_2, x_3) = \frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_1} + \frac{x_3}{x_1 + x_2} \geq \frac{3}{2}, \text{ (Nesbitt,1903, see [13, 11]).}$$

1.2.1. If $a, b, x_1, x_2, x_3 \in R_+^*$, then:

$$D(x_1, x_2, x_3) = \frac{x_1}{ax_2 + bx_3} + \frac{x_2}{ax_3 + bx_1} + \frac{x_3}{ax_1 + bx_2} \geq \frac{3}{a + b}, \text{ (see [3, 11]).}$$

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1.2.2. If $a_k, b_k, x_k \in R_+^*, k = \overline{1,3}$ and $a_1 + b_2 = a_2 + b_3 = a_3 + b_1 = t$, then:

$$E(x_1, x_2, x_3) = \frac{x_1}{a_1 x_2 + b_1 x_3} + \frac{x_2}{a_2 x_3 + b_2 x_1} + \frac{x_3}{a_3 x_1 + b_3 x_2} \geq \frac{3}{t}, \text{ (see [3, 11]).}$$

1.2.3. If $a, b, x_1, x_2, x_3 \in R_+^*$, $x_1 + x_2 + x_3 = X_3$ and $aX_3 > b\max\{x_1, x_2, x_3\}$, then:

$$F(x_1, x_2, x_3) = \frac{x_1}{aX_3 - bx_1} + \frac{x_2}{aX_3 - bx_2} + \frac{x_3}{aX_3 - bx_3} \geq \frac{3}{3a - b}, \text{ (see [3, 11]).}$$

1.2.4. If $a, b, x_1, x_2, x_3 \in R_+^*$, $x_1 + x_2 + x_3 = X_3$, such that $aX_3 > b\max\{x_1, x_2, x_3\}$, and $m, p \in [1, \infty)$, then:

$$G(x_1, x_2, x_3) = \frac{x_1^m}{(aX_3 - bx_1)^p} + \frac{x_2^m}{(aX_3 - bx_2)^p} + \frac{x_3^m}{(aX_3 - bx_3)^p} \geq \frac{3^{-m+p+1}}{(3a - b)^p} \cdot X_3^{m-p}, \text{ (see [3, 11]).}$$

1.2.5. If $a, b, x_1, x_2, x_3 \in R_+^*$, $x_1 + x_2 + x_3 = X_3$ and $m \in R_+$, then:

$$H(x_1, x_2, x_3) = \frac{x_1}{(aX_3 - bx_1)^m} + \frac{x_2}{(aX_3 - bx_2)^m} + \frac{x_3}{(aX_3 - bx_3)^m} \geq \frac{3^m}{(3a - b)^m \cdot X_3^{m-1}}, \text{ (see [3]).}$$

1.2.6. If $a, b, c, x, y, z \in R_+^*$, then:

$$I(x, y, z) = \frac{ax}{y + z} + \frac{by}{z + x} + \frac{cz}{x + y} \geq \frac{(ax + by + cz)^2}{(a + b)xy + (b + c)yz + (c + a)zx} \geq$$
$$\geq \frac{3(abxy + bcyz + cazx)}{(a + b)xy + (b + c)yz + (c + a)zx}, \text{ (see [1, 4]).}$$

1.2.7. If $a, b, x, y, z \in R_+^*$ and $m \in R_+$, then:

$$J(x, y, z) = \frac{x^{m+1}}{(ay + bz)^{2m+1}} + \frac{y^{m+1}}{(az + bx)^{2m+1}} + \frac{z^{m+1}}{(ax + by)^{2m+1}} \geq \frac{3^{m+1}}{(a + b)^{2m+1}(x + y + z)^m}, \text{ (see [2]).}$$

1.3. ON NESBITT'S INEQUALITY FOR n VARIABLES:

If $x_k \in R_+^*, \forall k = \overline{1, n}$, $X_n = \sum_{k=1}^{n} x_k$, then:

$$N_n = \sum_{k=1}^{n} \frac{x_k}{X_n - x_k} \geq \frac{n}{n - 1}, \text{ (see for e.g [14], the case } n = 4 \text{ and also [11]).}$$

1.3.1. If $n \in N^* - \{1,2\}$, $m, p \in [1,\infty)$, $a, b, x_k \in R_+^*$, $\forall k = \overline{1,n}$, $X_n = \sum_{k=1}^{n} x_k$, and $aX_n > b \max_{1 \le k \le n} x_k$, then:

$$K_n = \sum_{k=1}^{n} \frac{x_k^n}{(aX_n - bx_k)^p} \ge \frac{n^{-m+p+1}}{(an-b)^p} \cdot X_n^{m-p}, \text{ (see [2]).}$$

1.3.2. If $a, b, x_k \in R_+^*, c, y_k \in R_+, \forall k = \overline{1,n}, n \in N^* - \{1\}$,

$X_n = \sum_{k=1}^{n} x_k$ and $aX_n > b \max_{1 \le k \le n} x_k$, $y_k \in \left[0, \frac{1}{n} X_n\right]$, $\forall k = \overline{1,n}$ and $m \in R_+$, then:

$$L_n = \sum_{k=1}^{n} \frac{x_k^{m+1}}{(aX_n - bx_k + cy_k)^{2m+1}} \ge \frac{n^{m+1}}{(an-b+c)^{2m+1} X_n^m}, \text{ (see [2]).}$$

1.3.3. If $n \in N^* - \{1,2\}$, $a \in R_+, b, c, d, x_k \in R_+^*$, $\forall k = \overline{1,n}$, $X_n = \sum_{k=1}^{n} x_k$, and $cX_n > d \max_{1 \le k \le n} x_k$, then:

$$M_n = \sum_{k=1}^{n} \frac{aX_n + bx_k}{cX_n - dx_k} \ge \frac{(an+b) \cdot n}{cn-d}, \text{ (see [5]).}$$

1.3.4. If $a \in R_+, b, c, d, x_k \in R_+^*$, $\forall k = \overline{1,n}$, $X_n = \sum_{k=1}^{n} x_k$, $m \in [1,\infty)$ and $cX_n^m > d \max_{1 \le k \le n} x_k^m$, then:

$$U_n = \sum_{k=1}^{n} \frac{aX_n + bx_k}{cX_n^m - dx_k^m} \ge \frac{(an+b)n^m}{cn^m - d} X_n^{1-m}, \text{ (see [7] and also [10]).}$$

1.3.5. If $a, m \in R_+, b, c, d, x_k \in R_+^*$, $\forall k = \overline{1,n}$, $X_n = \sum_{k=1}^{n} x_k$, $p \in [1,\infty)$ and $cX_n^m > d \max_{1 \le k \le n} x_k^m$, then:

$$V_n = \sum_{k=1}^{n} \frac{aX_n + bx_k}{(cX_n^m - dx_k^m)^p} \ge \frac{(an+b)n^{mp}}{(cn^m - d)^p} X_n^{1-mp}, \text{ (see [8]).}$$

1.3.6. If $n \in N^* - \{1\}$, $a \in R_+$, $b, c, d, x_k \in R_+^*$, $\forall k = \overline{1,n}$, $X_n = \sum_{k=1}^{n} x_k$, $m, p, r, s \in [1,\infty)$, such that $cX_n^m > d \max_{1 \le k \le n} x_k^m$, then:

$$W_n = \sum_{k=1}^{n} \frac{(aX_n^r + bx_k^r)^s}{(cX_n^m - dx_k^m)^p} \ge \frac{(an^r + b)^s}{(cn^m - d)^p} n^{mp-rs+1} X_n^{rs-mp}, \text{ (see [9, 12]).}$$

1.3.7. If $n \in N^* - \{1\}$, $m, p \in R_+^*$, $x_k \in R_+^*, \forall k = \overline{1,n}$, and we denotes
$X_{n,m} = \sum_{k=1}^{n} x_k^m$, $X_{n,p} = \sum_{k=1}^{n} x_k^p$, then:

$$Y_n = \sum_{k=1}^{n} \frac{x_k^m}{X_{n,p} - x_k^p} \geq \frac{n}{n-1} \cdot \frac{X_{n,m}}{X_{n,p}},$$ (will apper in math Journal *Recreaţii Matematice*).

1.3.8. If $n \in N^* - \{1\}$, $a \in R_+$, $b,c,d,m,p \in R_+^*$, $x_k \in R_+^*$, $k = \overline{1,n}$, $X_{n,m} = \sum_{k=1}^{n} x_k^m$,
$X_{n,p} = \sum_{k=1}^{n} x_k^p$, such that $c \cdot X_{n,p} > d \cdot \max_{1 \le k \le n} x_k^p$, then

$$Z_n = \sum_{k=1}^{n} \frac{a \cdot X_{n,m} + b \cdot x_k^m}{c \cdot X_{n,p} - d \cdot x_k^p} \geq \frac{n \cdot (an+b)}{cn-d} \cdot \frac{X_{n,m}}{X_{n,p}},$$ (will apper in math Journal *Recreaţii Matematice*).

2. MAIN RESULTS

Theorem. If $n \in N^* - \{1\}$, $a,v \in R_+$, $b,c,d,m,p \in R_+^*$, $x_k \in R_+^*$, $k = \overline{1,n}$, $t \in [1,\infty)$,
$X_{n,m} = \sum_{k=1}^{n} x_k^m$, $X_{n,p} = \sum_{k=1}^{n} x_k^p$, such that $c \cdot X_{n,p} > d \cdot \max_{1 \le k \le n} x_k^p$, then

$$\sum_{k=1}^{n} \frac{(a \cdot X_{n,m} + b \cdot x_k^m)^t}{(c \cdot X_{n,p} - d \cdot x_k^p)^v} \geq \frac{n^{v-t} \cdot (an+b)^t}{cn-d} \cdot \frac{X_{n,m}^t}{X_{n,p}^v} \tag{1}$$

Proof. WLOG, we can assume that:

$$x_1 \geq x_2 \geq \ldots \geq x_n,$$

and then:

- the sequence $\left((a \cdot X_{n,m} + d \cdot x_k^m)^t\right)_{1 \le k \le n}$ is decreasing,
- the sequence $\left((c \cdot X_{n,p} - d \cdot x_k^p)^v\right)_{1 \le k \le n}$ is increasing,
- and the sequence $\left(\frac{1}{(c \cdot X_{n,p} - d \cdot x_k^p)^v}\right)_{1 \le k \le n}$ is decreasing.

By Chebyshev's inequality for decreasing sequences $\left((a \cdot X_{n,m} + d \cdot x_k^m)^t\right)_{1 \le k \le n}$, respecvtively $\left(\frac{1}{(c \cdot X_{n,p} - d \cdot x_k^p)^v}\right)_{1 \le k \le n}$ we obtain that:

$$U = \sum_{k=1}^{n} \frac{(a \cdot X_{n,m} + b \cdot x_k^m)^t}{(c \cdot X_{n,p} - d \cdot x_k^p)^v} \geq \frac{1}{n}\left(\sum_{k=1}^{n} (aX_{n,m} + bx_k^m)^t\right)\left(\sum_{k=1}^{n} \frac{1}{(cX_{n,p} - dx_k^p)^v}\right) \tag{2}$$

By J. Radon's inequality we have that:

$$\sum_{k=1}^{n}\left(aX_{n,m}+bx_k^m\right)^t \geq \frac{\left(\sum_{k=1}^{n}(aX_{n,m}+bx_k^m)\right)^t}{n^{t-1}} = \frac{(an+b)^t X_{n,m}^t}{n^{t-1}} \tag{3}$$

Also by J. Radon's inequality we deduce that:

$$\sum_{k=1}^{n}\frac{1}{(cX_{n,p}-dx_k^p)^v} \geq \frac{n^{v+1}}{\left(\sum_{k=1}^{n}(cX_{n,p}-dx_k^p)\right)^v} = \frac{n^{v+1}}{(cn-d)^v X_{n,p}^v} \tag{4}$$

By (2), (3) and (4) we obtain that:

$$U \geq \frac{1}{n}\cdot\frac{(an+b)^t X_{n,m}^t}{n^{t-1}}\cdot\frac{n^{v+1}}{(cn-d)^v X_{n,p}^v} = \frac{(an+b)^t n^{v-t+1}}{(cn-d)^v}\cdot\frac{X_{n,m}^t}{X_{n,p}^v},$$

and the proof is complete.

Remarks.

- If we take in (1) $t=v=1$, then

$$\sum_{k=1}^{n}\frac{a\cdot X_{n,m}+b\cdot x_k^m}{c\cdot X_{n,p}-d\cdot x_k^p} \geq \frac{n\cdot(an+b)}{cn-d}\cdot\frac{X_{n,m}}{X_{n,p}} \tag{5}$$

i.e. we reobtain **1.3.8.**

- If we take in (5) $a=0, b=1$, then:

$$\sum_{k=1}^{n}\frac{x_k^m}{c\cdot X_{n,p}-d\cdot x_k^p} \geq \frac{n}{cn-d}\cdot\frac{X_{n,m}}{X_{n,p}} \tag{6}$$

- If we take in (6) $m=n=p=1$, then we obtain:

$$\sum_{k=1}^{n}\frac{x_k}{c\cdot X_n-d\cdot x_k} \geq \frac{n}{cn-d} \tag{7}$$

where $X_n = X_{n,1} = \sum_{k=1}^{n} x_k$.

- Finally if in (7) we take $c=d=1$, then we obtain:

$$\sum_{k=1}^{n}\frac{x_k}{X_n-x_k} \geq \frac{n}{n-1} \tag{N}$$

i.e. Nesbitt's inequality for n variables.

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