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A PROBLEM IN NONLINEAR OPTIMIZATION

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In [4], Ragget, Hempson and Jakes solved the following problem called "A student optimal control problem"

P1

$$\begin{array}{l} min \sum_{i=1}^{n} a_i x_i^2 \\ \text{subject to} \sum_{i=1}^{n} b_i x_i = S, \quad x_i \geq 0 \\ \text{for given } S > 0, a_i > 0, b_i > 0, i = 1, 2, ..., n. \end{array}$$

In [3], Muntean, Vornicescu added restrictions $x_i \leq B$, with $B > 0$.

In [6] Vornicescu considered the continuous case:

P2

$$\begin{array}{l} min \int_0^1 a(t)u^2(t)dt \\ \text{subject to } \int_0^1 b(t)u(t)dt = S,\ u \in C[0,1] \text{ and } 0 \leq u(t) \leq B \text{ for } t \in [0,1], \\ \text{for given positive functions } a, b \in C[0,1] \text{ and for given } B, S > 0. \end{array}$$

In this paper is considered the following case:

P3

$$\begin{array}{l} min \int_0^1 a(t)u^m(t)dt, \quad m \geq 1 \\ \text{subject to} \int_0^1 b(t)u(t)dt = S,\ u \in C[0,1] \text{ and } 0 \leq u(t) \leq B \text{ for } t \in [0,1], \\ \text{for given positive functions } a, b \in C[0,1] \text{ and for given } B, S > 0. \end{array} \tag{1}$$

Remark that the problem cannot be solved using the classical methods of the calculus of variations.

A function $u : [0, 1] \to R$ is said to be an **admissible strategy** for the problem P3 if it verifies conditions (1).

For an admissible strategy u, let us denote $J[u] = \int_0^1 a(t)u^m(t)dt$

An admissible strategy u^* is said to be an **optimal strategy** for P3 if for each admissible strategy u we have $J[u^*] \leq J[u]$.

An admissible strategy u^* is said to be a **locally optimal strategy** for P3 if there exists $\delta > 0$ such that, if u is an admissible strategy verifying $max\{|u(t) - u^*(t)| : t[0, 1]\} < \delta$, then $J[u^*] \leq J[u]$.

We need to consider four cases, discussed in Propositions 1, 2, 3 and in Theorem 1.

Proposition 1. If $\frac{S}{B} > \int_0^1 b(t)dt$, then the Problem P3 has no solution.

Proof. $\int_0^1 b(t)u(t)dt \leq B \int_0^1 b(t)dt < S$ for each function u verifying conditions (1).

Proposition 2. If $\frac{S}{B} = \int_0^1 b(t)dt$, then the unique admissible strategy is $u = B$.

The proof is straightforward.

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Proposition 3. If

$$\frac{S}{B} \leq min\{a^{\frac{1}{m-1}}(s)b^{-\frac{1}{m-1}}(s), s[0,1]\} \int_0^1 a^{-\frac{m}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt \int_0^1 b(t)dt,$$

then the optimal strategy is

$$u^*(t) = S\left(\int_0^1 a^{-\frac{1}{m-1}}(s)b^{\frac{m}{m-1}}(s)ds\right)^{-1} a^{-\frac{1}{m-1}}(t)b^{\frac{1}{m-1}}(t)$$

and

$$J[u^*] = S^m\left(\int_0^1 b^{\frac{m}{m-1}}(t)a^{\frac{-1}{m-1}}(t)dt\right)^{-m+1}$$

Proof. Function u^* is easily seen to be an admissible strategy.

Let u be an arbitrary admissible strategy. Using Holder's inequality with $p = m$ and $q = \frac{m}{m-1}$ we obtain

$$S = \int_0^1 a^{\frac{1}{m}}(t)u(t) \cdot a^{-\frac{1}{m}}(t)b(t)dt \leq \left(\int_0^1 a(t)u^m(t)dt\right)^{\frac{1}{m}} \left(\int_0^1 a^{-\frac{1}{m-1}}(t)b^{\frac{m}{m-1}}(t)ds\right)^{\frac{m-1}{m}},$$

whence $J[u] \geq S^m \left(\int_0^1 a^{-\frac{1}{m-1}}(t)b^{\frac{m}{m-1}}(t)ds\right)^{-m+1}$.

From $J[u^*] = S^m \left(\int_0^1 a^{-\frac{1}{m-1}}(t)b^{\frac{m}{m-1}}(t)ds\right)^{-m+1}$, we obtain that u^* is the optimal strategy.

It remains to study the case when

$$min\{a^{\frac{1}{m-1}}(s)b^{-\frac{1}{m-1}}(s), s[0,1]\} \int_0^1 a^{-\frac{m}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt < \frac{S}{B} < \int_0^1 b(t)dt \tag{2}$$

which is in fact the main case.

In this case we obtain a characterization of locally optimal strategy using the variation of the functional J.

Before giving the main result (Theorem 1), we need to prove the following lemma.

Lemma 1. If u^* is a locally optimal strategy for problem P3 and if there exist $t_1, t_2 \in (0,1)$ such that

$$a(t_1)b^{-1}(t_1)(u^*)^{m-1}(t_1) < a(t_2)b^{-1}(t_2)(u^*)^{m-1}(t_2), \tag{3}$$

then $u^*(t_1) = B$.

Proof. Suppose, contrary to our claim, that $u^*(t_1) < B$.

There exists $\alpha > 0$ such that

$$\begin{array}{l} [t_1-\alpha, t_1+\alpha] \subset [0,1], \\ [t_2-\alpha, t_2+\alpha] \subset [0,1] \\ [t_1-\alpha, t_1+\alpha] \cup [t_2-\alpha, t_2+\alpha] = \varnothing \\ a(c_1)b^{-1}(c_2)(u^*)^{m-1}(c_1) < a(c_3)b^{-1}(c_4)(u^*)^{m-1}(c_3) \end{array} \tag{4}$$

for all $c_1, c_2 \in [t_1-\alpha, t_1+\alpha], c_3, c_4 \in .[t_2-\alpha, t_2+\alpha]$

Let us denote

$$P = \int_{t_1-\alpha}^{t_1+\alpha} b(t)[\alpha^2 - (t-t_1)^2]dt,$$
$$Q = \int_{t_2-\alpha}^{t_2+\alpha} b(t)[\alpha^2 - (t-t_2)^2]dt.$$

By applying the mean value theorem, we obtain that there exist $c_2 \in [t_1-\alpha, t_1+\alpha], c_4 \in [t_2-\alpha, t_2+\alpha]$ such that

$$\begin{aligned} P &= \frac{4}{3}\alpha^3 b(c_2), \\ Q &= \frac{4}{3}\alpha^3 b(c_4). \end{aligned}$$

Let $\varepsilon_0 > 0$ be sufficiently small such that for $0 < \varepsilon \leq \varepsilon_0$ we have:

$$u^*(t) + \frac{\varepsilon}{P}[2-(t-t_1)^2] < B \quad \text{for } t \in [t_1-\alpha, t_1+\alpha],$$

$$u^*(t) - \frac{\varepsilon}{Q}[2-(t-t_2)^2] > 0 \quad \text{for } t \in [t_2-\alpha, t_2+\alpha].$$

For $0 < \varepsilon \leq \varepsilon_0$ the function $u_\varepsilon : [0,1] \to [0,1]$, defined by

$$u_\varepsilon(t) = \begin{cases} u^*(t), \ t \in [0,1] \backslash ([t_1-\alpha, t_1+\alpha] \cup [t_2-\alpha, t_2+\alpha]) \\ u^*(t) + \frac{\varepsilon}{P}[\alpha^2 - (t-t_1)^2], \ t \in [t_1-\alpha, t_1+\alpha] \\ u^*(t) - \frac{\varepsilon}{Q}[\alpha^2 - (t-t_2)^2], \ t \in [t_2-\alpha, t_2+\alpha] \end{cases}$$

is an admissible strategy for the problem P3.

We define the function $L : [0, \varepsilon_0] \to R, \quad L(\varepsilon) = \int_0^1 a(t) u_\varepsilon^m(t) dt$. We have

$$L'(\varepsilon) = \left(\int_{t_1-\alpha}^{t_2+\alpha} a(t) u_\varepsilon^m(t) dt + \int_{t_2-\alpha}^{t_2+\alpha} a(t) u_\varepsilon^m(t) dt \right)' =$$

$$= m\left(\frac{1}{P} \int_{t_1-\alpha}^{t_2+\alpha} a(t) u_\varepsilon^{m-1}(t)(\alpha^2-(t-t_1)^2) dt - \frac{1}{Q} \int_{t_2-\alpha}^{t_2+\alpha} a(t) u_\varepsilon^{m-1}(t)(\alpha^2-(t-t_2)^2) dt \right)$$

and

$$\lim_{\varepsilon \to 0_+} L'(\varepsilon) =$$

$$m\left(\frac{1}{P} \int_{t_1-\alpha}^{t_1+\alpha} a(t)(u^*)^{m-1}(t)(\alpha^2-(t-t_1)^2) dt - \frac{1}{Q} \int_{t_2-\alpha}^{t_2+\alpha} a(t)(u^*)^{m-1}(t)(\alpha^2-(t-t_2)^2) dt \right)$$

We have that there exist $c_1 \in [t_1-\alpha, t_1+\alpha], c_3 \in [t_2-\alpha, t_2+\alpha]$ such that

$$\lim_{\varepsilon \to 0_+} L\prime(\varepsilon) = m[a(c_1)b(c_2)(u^*)^{m-1}(c_1) - a(c_3)b(c_4)(u^*)^{m-1}(c_3)] < 0$$

which contradicts the local optimality of $L[0]$.

Theorem 1. If inequalities in (2) are verified and if u^* is a locally optimal strategy for problem P3 then there exists $t_0 \in [0,1]$ such that

$$u^*(t) = \begin{cases} B, \quad if \ a(t)b^{-1}(t) < a(t_0)b^{-1}(t_0) \\ Ba^{\frac{1}{m-1}}(t_0)b^{-\frac{1}{m-1}}(t_0)a^{-\frac{1}{m-1}}(t)b^{\frac{1}{m-1}}(t), \quad if \ a(t)b^{-1}(t) \geq a(t_0)b^{-1}(t_0) \end{cases} \tag{4}$$

Proof. Denote $C = max\{a(t)b^{-1}(t)(u^*)^{m-1}(t) | t \in [0,1]\}$, $I1 = \{ta(t)b^{-1}(t)(u^*)^{m-1}(t) = C, t[0,1]\}$ and $I2 = [0,1] \backslash I1$.

First we will show that $I2 \neq \varnothing$.

Supposing the contrary, we have $u^*(t) = C^{\frac{1}{m-1}} a^{\frac{-1}{m-1}}(t) b^{\frac{1}{m-1}}(t)$ for $t \in [0,1]$. Then (1) implies

$$S = C^{\frac{1}{m-1}} \int_0^1 a^{\frac{-1}{m-1}}(t) b^{\frac{m}{m-1}}(t) dt = a^{\frac{1}{m-1}}(t) b^{\frac{-1}{m-1}}(t) u^*(t) \int_0^1 a^{\frac{-1}{m-1}}(t) b^{\frac{m}{m-1}}(t) dt \leq$$

$$\leq Ba^{\frac{1}{m-1}}(t)b^{\frac{-1}{m-1}}(t)\int_0^1 a^{\frac{-1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt$$

whence $\frac{S}{B} \leq a^{\frac{1}{m-1}}(t)b^{\frac{-1}{m-1}}(t)\int_0^1 a^{\frac{-1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt$ which contradicts (2).

In conclusion $I2$ is a nonempty open set in induced topology on interval $[0,1]$.

If $t \in I2$ then we have $a(t)b^{-1}(t)(u^*)^{m-1}(t) < C$ and from Lemma 1 we find $u^*(t) = B$.

Let I be a maximal open interval included in $I2$. Since $I1 \neq \varnothing$, $I \neq [0,1]$ and I has one of the following forms:$[0,t_1), (to, t_1)$ or $(t_0, 1]$.

Suppose, for example, that $t_0 \notin I_2$. Thus $t_0 \in I_1$, $u^*(t_0) = B$ and $a(t_0)b^{-1}(t_0)B^{m-1} = C$. In the case when $a(t)b^{-1}(t) < a(t_0)b^{-1}(t_0)$ then $a(t)b^{-1}(t)(u^*)^{m-1}(t) < a(t_0)b^{-1}(t_0)(u^*)^{m-1}(t_0)$, whence $u^*(t) = B$.Let t be such that $a(t)b^{-1}(t) > a(t_0)b^{-1}(t_0)$.

If we suppose that $a(t)b^{-1}(t)(u^*)^{m-1}(t) < C$ then $u^*(t) = B$ and $a(t)b^{-1}(t)(u^*)^{m-1}(t) > a(t_0)b^{-1}(t_0)(u^*)^{m-1}(t_0)$, which contradicts optimality of C. Hence $a(t)b^{-1}(t)(u^*)^{m-1}(t) = C$ and

$$u^*(t) = C^{\frac{1}{m-1}}a^{\frac{-1}{m-1}}(t)b^{\frac{1}{m-1}}(t) = Ba^{\frac{1}{m-1}}(t_0)b^{\frac{-1}{m-1}}(t_0)a^{\frac{-1}{m-1}}(t)b^{\frac{1}{m-1}}(t)$$

We will say that an admissible strategy u is an **extremal strategy** if there exists $t_0 \in [0,1]$ such that u is given by (4). The Theorem 1 states that a locally optimal strategy is an extremal strategy.

Define the function $F : [0,1] \to \Re$,

$$F(t) = \int_0^t b(s)ds + a^{\frac{1}{m-1}}(t)b^{-\frac{1}{m-1}}(t)\int_t^1 a^{-\frac{1}{m-1}}(s)b^{\frac{m}{m-1}}(s)ds \tag{5}$$

Corollary 1. If inequalities (2) hold and if $a^{\frac{1}{m-1}}b^{-\frac{1}{m-1}}$ is an increasing function, then there exists a unique $t_0 \in [0,1]$ such that $F(t_0) = \frac{S}{B}$ and the function $u : [0,1] \to \Re$ given by

$$u(t) = \begin{cases} B, \quad if\ 0 \leq t < t_0 \\ Ba^{\frac{1}{m-1}}(t_0)b^{-\frac{1}{m-1}}(t_0)a^{-\frac{1}{m-1}}(t)b^{\frac{1}{m-1}}(t), \quad if\ t_0 \leq t \leq 1 \end{cases} \tag{6}$$

is an extremal strategy.

Proof. Let $0 \leq t_1 < t_2 \leq 1$. Then

$$F(t_2) - F(t_1) = \int_{t_1}^{t_2} a^{-\frac{1}{m-1}}(s)b^{\frac{m}{m-1}}(s)(a^{\frac{1}{m-1}}(s)b^{-\frac{1}{m-1}}(s) - a^{\frac{1}{m-1}}(t_1)b^{-\frac{1}{m-1}}(t_1))ds +$$

$$+\int_{t_2}^1 a^{-\frac{1}{m-1}}(s)b^{\frac{m}{m-1}}(s)(a^{\frac{1}{m-1}}(t_2)b^{-\frac{1}{m-1}}(t_2) - a^{\frac{1}{m-1}}(t_1)b^{-\frac{1}{m-1}}(t_1))ds.$$

Hence F is an increasing function and

$$\begin{aligned} F(0) &= a^{\frac{1}{m-1}}(0)b^{-\frac{1}{m-1}}(0)\int_0^1 a^{-\frac{1}{m-1}}(s)b^{\frac{m}{m-1}}(s)ds < \frac{S}{B}, \\ F(1) &= \int_0^1 b(t)ds > \frac{S}{B}. \end{aligned}$$

Therefore there exists a unique$t_0 \in [0,1]$ such that $F(t_0) = \frac{S}{B}$. Function u given by (6) is easily seen to be an extremal strategy.

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