Journal of Science and Arts 1(2009), 21-23
Valahia University of Târgoviște

ABOUT SOME FUNCTIONAL INTEGRAL EQUATION IN SPACE WITH PERTURBATED METRIC

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Abstract: *In this paper we study, in space with perturbated metric, the following functional integral equation:*

$$x(t) = g(t, h(x)(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K(t, s, x(s))ds,$$

where $\theta \in (0,1)$, $t \in [-T, T]$, $T > 0$.

Keywords: *functional integral equations, fixed points, perturbated metrics.*
2000 Mathematics subject classification: *34K10*

1 Introduction

Let (X, d) be a complete metric space. We denote by P the set of functions $g : \mathbb{R}_+ \to \mathbb{R}_+$ which is strictly increasing, continuous and surjective. By Φ we denote the set of function introduced by

Definition 1.1. *amsfonts We say that the function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to the* Φ *class if the following conditions are met:*

(1) φ *is increasing;*

(2) $\varphi^n(t) \to 0$, *for all* $t \in \mathbb{R}_+$.

Example 1. *The function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = \frac{t}{t+1}$, *belong to the set* Φ.

Example 2. *The function* $g : \mathbb{R}_+ \to \mathbb{R}_+$, $g(t) = t^2$, *belong to the set* P.

Proposition 1.1. *[3] Let* (X, d) *be a complete metric space,* $f : X \to X$ *an operator and* $\varphi \in \Phi$, $g \in P$ *such that:*

(i) $g(d(f(x), f(y))) \leq \varphi(g(d(x, y)))$, *for all* $x, y \in X$;

Then f *has a unique fixed point, which is the limit of successively approximations sequence.*

Paper presented at The VI-th International Conference on Nolinear Analysis and Applied Mathematics (ICNAAM), Târgovişte, 21-22 nov, 2008

2 Main result

We consider the following functional integral equation:

$$x(t) = g(t, h(x)(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K(t, s, x(s))ds, \tag{1}$$

where :

(H_1) $\theta \in (0, 1)$, $t \in [-T, T]$, $T > 0$;

(H_2) $K \in C([-T, T] \times [-T, T] \times X)$, $g \in C([-T, T] \times X^3)$, $(X, \|\cdot\|)$ Banach space $h : C([-T, T], X) \to C([-T, T], X)$;

(H_3) $g(0, h(x)(0), x(0), x(0)) = x(0)$.

Proposition 2.1. *We suppose that:*

(i) $\|h(x)(t) - h(y)(t)\| \le \|x(t) - y(t)\|$, *for all* $t \in [-T, T]$, $x, y \in X$;

(ii) there exists $a, b > 0$, $\varphi \in \Phi$ *such that*

$$\|g(t, u_1, u_2, w) - g(t, v_1, v_2, w)\|^2 \le a\varphi(\|u_1 - u_2\|^2) + b\varphi(\|v_1 - v_2\|^2),$$

for all $t \in [-T, T]$, $u_1, u_2, v_1, v_2 \in X$;

(iii) there exists a integrable function $l(t, \cdot)$ *such that*

$$\|K(t, s, u) - K(t, s, v)\|^2 \le l(t, s)\varphi(\|u - v\|^2),$$

for all $t \in [-T, T]$, $u, v \in X$

(iv) $\sup\limits_{t\in[-T,T]} 2(a + b + 2T \int_{-\theta t}^{\theta t} l(t, s)ds) \le 1.$

Then the equation (1) has a unique solution in $(C([-T, T], X), \|\cdot\|_\infty)$.

Proof: We consider the operator

$$A : C([-T, T], X) \to C([-T, T], X),$$

$$A(x)(t) = g(t, h(x)(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K(t, s, x(s))ds$$

Then for all $x, y \in C([-T, T], X)$ we have

$$\|A(x)(t) - A(y)(t)\|^2 \le$$

$$\le \{\|g(t, h(x)(t), x(t), x(0)) - g(t, h(y)(t), y(t), y(0))\| + \int_{-\theta t}^{\theta t} \|K(t, s, x(s)) - K(t, s, y(s))\|ds\}^2$$

$$\le 2(\|g(t, h(x)(t), x(t), x(0)) - g(t, h(y)(t), y(t), y(0))\|^2 + \{\int_{-\theta t}^{\theta t} \|K(t, s, x(s)) - K(t, s, y(s))\|ds\}^2)$$

$$\leq 2(a\varphi(\|x(t)-y(t)\|^2)+b\varphi(\|x(t)-y(t)\|^2)+2T\int_{-\theta t}^{\theta t}\|K(t,s,x(s))-K(t,s,y(s))\|^2 ds)$$

$$\leq 2(a+b+2T\int_{-\theta t}^{\theta t} l(t,s)ds)\varphi(\|x-y\|_\infty^2)\leq \varphi(\|x-y\|_\infty^2).$$

It follow that

$$\|A(x)-A(y)\|_\infty^2 \leq \varphi(\|x-y\|_\infty^2)$$

and from Proposition 1.1 we obtain the conclusion.

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Manuscript received: 30.01.2009 / accepted: 30.05.2009