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ON THE CAUCHY PROBLEM OF NAVIER-STOKES FLOW

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Abstract: *Consider the Cauchy semilinear problem of the Navier-Stokes flow of incompressible fluids - one of the* Millenium Prize Problems *(see [1]). By standard arguments we can formulate the problem as an abstract equation and prove the existence and the uniqueness of the strong solution. The proof is constructive and it is based on the Fourier method developed in the energetical space of the Stokes operator (on the complete sequence of the eigenvectors of the duality map). Some open problems are also appended.*

1 Introduction

Consider the Navier-Stokes system for incompressible fluids filling all of $\mathbb{R}^N$, ($N = 2$ *or* 3):

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u - \nabla p + f, \tag{1}$$

$$\nabla \cdot u = 0,\ in\ \mathbb{R}^N,\ t \geq 0, \tag{2}$$

$$u(x,0) = u^0(x),\ x \in \mathbb{R}^N, \tag{3}$$

where $\nu \geq 0$ (*dynamical viscozity*), f (*body forces*) and u^0 (*initial velocity*) are given. These equations are to be solved for an unknown *velocity vector* $u : \mathbb{R}^N \times [0,\infty) \mapsto \mathbb{R}^N$ and the *presure* $p : \mathbb{R}^N \times [0,\infty) \mapsto \mathbb{R}$. The *Euler equations* can be obtained for $\nu = 0$ in (1)-(3).

For physically reasonable solutions, we ask that u does not grow large as $|x| \to \infty$. Hence we will restrict attention to data f and u^0 that satisfy:

$$|\partial_x^\alpha \partial_t^n f(x,t)| \leq C_{\alpha n K}(1 + |x| + t)^{-K},\ x \in \mathbb{R}^N, t \geq 0, \tag{4}$$

and

$$|\partial_x^\alpha u^0(x)| \leq C_{\alpha K}(1 + |x|)^{-K}, \tag{5}$$

for any α, n and K. A solution $[u,p]$ is physically reasonable only if it is enough smooth and *it has bounded energy*, i.e.,

$$\int_{\mathbb{R}^N} |u(x,t)|^2 dx < C, \forall t \geq 0. \tag{6}$$

Let u^0 be any smooth, divergence-free vector field (i.e., $\nabla \cdot u^0 = 0$) satifying (5). We have to prove that either there exists a smooth solution $[u,p] \in C^\infty(\mathbb{R}^N \times [0,\infty))$ that satisfy (1),(2),(3) and (6), or there exist no such solutions. Remark that these problems are not solved yet for $\nu > 0$, $f = 0$ and $N = 3$ (see [1]).

Let $\phi : \mathbb{R}^N \times [0,\infty) \mapsto \mathbb{R}^N$ be a compactly supported vector field. Then, multiplying (1) and (2) by ϕ, a formal integration by parts yields:

$$\iint_{\mathbb{R}^N \times \mathbb{R}_+} u \cdot \frac{\partial \phi}{\partial t} dxdt - b(u,u,\phi) =$$

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$$= \iint_{\mathbb{R}^N \times \mathbb{R}_+} \nabla u \cdot \nabla \phi dxdt + \iint_{\mathbb{R}^N \times \mathbb{R}_+} (f - \nabla p) \cdot \phi dxdt, \tag{7}$$

where

$$b(u, u, \phi) := \sum_{ij} \iint_{\mathbb{R}^N \times \mathbb{R}_+} u_i u_j \frac{\partial \phi_i}{\partial x_j} dxdt$$

and

$$\iint_{\mathbb{R}^N \times \mathbb{R}_+} u \cdot \nabla_x \phi dxdt = 0. \tag{8}$$

A solution of (7)-(8) is called a *weak solution* of Navier-Stokes system. Observe that these equations make sense for all $u \in L^2$ and $p \in L^1$.

2 Abstract Fourier Method

Define *the space of incompressible fluids*:

$$C_{0,\sigma}^{\infty} := \{y \in (C_0^{\infty}(\mathbb{R}^N))^N;\ \nabla \cdot u = 0\},$$

and let X be its completion with respect to the norm $\|\cdot\|_2$. Then X is a Hilbert space with the scalar product:

$$(y, w) := \int_{\mathbb{R}^N} y \cdot w = \sum_{i=1}^{N} \int_{\mathbb{R}^N} y_i w_i dx.$$

Let E be its subspace:

$$E := \{y \in X;\ y \in (W^{1,2}(\mathbb{R}^N)^N\}.$$

Since $W^{1,2}(\mathbb{R}^N) = W_0^{1,2}(\mathbb{R}^N)$ (see [2]), E can be viewed as the completion of $C_{0,\sigma}^{\infty}$ in the norm of $W_0^{1,2}$, namely:

$$E := \overline{C_{0,\sigma}^{\infty}}^{\|\cdot\|_{1,2}} = \{y \in (W_0^{1,2}(\mathbb{R}^N))^N;\ \nabla \cdot y = 0\}.$$

Remark that the scalar product in X is in fact the duality pairing between E^* and E.

Consider *the Stokes operator* $A \in L(E, E^*)$ defined by:

$$(Ay, w) := \sum_{i=1}^{N} \int_{\mathbb{R}^N} \nabla y_i \cdot \nabla w_i dx,\ \forall y, w \in E,$$

and define the three-linear form:

$$b(y, z, w) := \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} y_i D_i z_j w_j dx,$$

that determines the nonlinear operator $C : E \mapsto E^*$ by:

$$C(y, w) := b(y, y, w),\ \ \forall y, w \in E.$$

Then the equation (7) can be written as:

$$\frac{d}{dt}(u, v) = (f - \nu A u - C(u), v),\ \forall v \in E, \tag{9}$$

and we obtain the following weak formulation of Navier-Stokes system:
Given $f \in L^2(0, T; E^*)$ and $u^0 \in X$ find $u \in L^2(0, T; E)$ such that $u_t \in L^2(0, T; E^*)$ and:

$$\begin{cases} \frac{du}{dt} + \nu A u + C(u) = f \ \text{ on } (0, T), \\ u(0) = u^0. \end{cases} \tag{10}$$

Here $u_t := \frac{du}{dt}$ and $T > 0$ is any large real number. The weak solution that is enough smooth, i.e, $u \in C((0,T);E) \cap C^1((0,T);X)$, is called *the strong solution* of Navier-Stokes system.

As $A : E \mapsto E^*$ is symmetric $(Ay,w) = (y,Aw)$ and strongly monotone $(Ay,y) \geq \|y\|^2$ and E is its energetic space, there exist $[e_n,\lambda_n] \in E \times [0,\infty)$ solutions of the eigenvalue problem:

$$(Ae_n,v) = \lambda_n(e_n,v),\ \forall v \in E,$$

with $(e_i,e_j) = \delta_{ij}$ and $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \to \infty$. Then $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis in E, $(\sqrt{\lambda_n}e_n)_{n\in\mathbb{N}}$ is an orthonormal basis in X and $(\lambda_n e_n)_{n\in\mathbb{N}}$ is an orthonormal basis in E^*. Moreover, A is continuous and can be viewed as the duality mapping $J : E \mapsto E^*$, that is $\langle Ju,v\rangle = (u,v)_E$ (see [4]). Hence, $u \in L^2(0,T;E) \ \Rightarrow \ Au \in L^2(0,T;E^*)$ and we have the implications (see [8] p.281):

$$C(u) \in L^1(0,T;E^*) \ \Rightarrow \ f - \nu Au - C(u) \in L^1(0,T;E^*) \ \Rightarrow \ u_t \in L^1(0,T;E^*).$$

Therefore u is $(a.e. =)$ continuous from $[0,T)$ to E^*.

We try to find the weak solution of the Cauchy problem (10) as the Fourier series in E:

$$u(x,t) := \sum_{n=0}^{\infty} b_n(t)e_n(x) = \lim_{n\to\infty}\sum_{k=0}^{n} b_k(t)e_k(x). \tag{11}$$

To do this denote by $D_i v := \frac{\partial v}{\partial x_i}$ and describe in detail the convective term:

$$(v\cdot\nabla)v := (v_1D_1 + \cdots + v_ND_N)v = (v_1D_1v_k + \cdots + v_ND_Nv_k)_{1\leq k\leq N}.$$

Denoting $e_n := (e_n^m)1 \leq m \leq N$ and $v_n(x,t) := \sum_{p=0}^{n} b_p(t)e_p(x)$ for all $n \in \mathbb{N}$, the k^{th}-component, $(1 \leq k \leq N)$, of the vector $(v_n\cdot\nabla)v_n$ will be:

$$v_n^1D_1v_n^k + \cdots + v_n^ND_Nv_n^k = \sum_{m=1}^{N}\left(\sum_{p=0}^{n} b_pe_p^m \cdot \sum_{p=0}^{n} b_pD_me_p^k\right).$$

Then $(u\cdot\nabla)u = \lim(v_n\cdot\nabla)v_n$ by Mertens theorem and:

$$((v_n\cdot\nabla)v_n(\cdot,t),e_j) := \sum_{k=1}^{N}\int_{\mathbb{R}^N}(v_n^1D_1v_n^k + \cdots + v_n^ND_Nv_n^k)\cdot e_j^k(x)dx =$$

$$= \sum_{p=1}^{n}\sum_{q=1}^{p} b_{p-q}(t)b_q(t)\left[\int_{\mathbb{R}^N}\sum_{k=1}^{N}\left(\sum_{m=1}^{N} e_{p-q}^m(x)D_me_q^k(x)\right)\cdot e_j^k(x)dx\right] =:$$

$$=: c_j(b_1(t),\ldots b_n(t)),\ \ j \geq 1.$$

Taking (11) and $v := e_j$, $j \geq 1$, we formally obtain that $b_j(t)$ must satisfy the scalar Cauchy problem:

$$\begin{cases} b_j' + \nu\lambda_jb_j + c_j(b_1,\ldots,b_n) = f_j(t), \\ b_j(0) = u_j^0,\ 1 \leq j \leq n, \end{cases} \tag{12}$$

where u_j^0 and $f_j(t)$ are the Fourier coefficients in E of u^0 and $f(\cdot,t)$, respectively:

$$u_j^0 := (u^0,e_j)_E, \quad f_j(t) := (f(\cdot,t),e_j)_E, \quad \forall j \in \mathbb{N}.$$

Let $n \in \mathbb{N}$ be any fixed number. The homogeneuos linear system $b_j' + \nu\lambda_jb_j = 0$, $1 \leq j \leq n$, has the solution $b_j(t) := k_je^{-\nu\lambda_jt}$ with $k_j \in \mathbb{R}$, $1 \leq j \leq n$. By variation of constants $k_j := h_j(t)$ we deduce that h_j are solutions of the following system

$$\begin{cases} h_j' = e^{\nu\lambda_jt}f_j(t) - \lambda_je^{\nu\lambda_jt}c_j(e^{-\nu\lambda_1t}h_1(t),\ldots,e^{-\nu\lambda_nt}h_n(t)), \\ h_j(0) = u_j^0,\ 1 \leq j \leq n \end{cases},$$

with c_j some sums of quadratic terms in $h_1, \ldots, h_n$. Consider the vectors $h := (h_j)_{1\leq j\leq n}$ and

$$F(t,h) := (e^{\nu\lambda_j t} f_j(t) - \lambda_j e^{\nu\lambda_j t} c_j(e^{-\nu\lambda_1 t} h_1(t), \ldots, e^{-\nu\lambda_n t} h_n(t)))_{1\leq j\leq n}.$$

Then, in vectorial form, the above system can be expressed as the Cauchy problem

$$\begin{cases} h' = F(t,h), \\ h(0) = u^0, \end{cases} \tag{13}$$

with F continuous in $[t,h]$ and of class C^1 in h. Hence there exists a unique solution $h = h(t)$ and $h \in C^1([0,\tau])$, with $0 < \tau \leq T$, by Picard theorem.

Consequently, the Cauchy problem (12) has a unique solution, that determine the coefficients of the Fourier series (11) and so $u(x,t)$. To prove that this is the weak solution of the Navier-Stokes system (10) we have to prove that:

(a) The series $\sum\limits_{n=1}^{\infty} b_n(t)e_n(x)$ converges uniformly to $u \in C([0,T];E)$;

(b) The series $\sum\limits_{n=1}^{\infty} b'_n(t)e_n(x)$ converges uniformly to $u_t \in C([0,T];E^*)$;

(c) This $u(x,t)$ is the unique weak solution of the Cauchy problem (10).

3 Existence, Uniqueness and Smoothness

The answers to the above three problems, (a),(b) and (c), depend on the convergence of the numerical series:

$$\sum \lambda_n^{-\frac{N+2}{2}}.$$

Suppose that this series is convergent in $\mathbb{R}$ and remark that

$$M_1 := \sup\{|h_n(t)|; n \in \mathbb{N},\ t \in [0,T]\} < +\infty,$$

$$M_2 := \sup\{|h'_n(t)|; n \in \mathbb{N},\ t \in [0,T]\} < +\infty.$$

Then

$$|b_n(t)|^2 = |e^{-\nu\lambda_n t} h_n(t)| = e^{-2\nu\lambda_n t}|h_n(t)|^2$$

and thus for any $\delta \in (0,T)$ there exists $n_\delta \in \mathbb{N}$ such that:

$$e^{-2\nu\lambda_n t} < e^{-2\nu\lambda_n \delta} < \frac{1}{\lambda_n^{\frac{N+2}{2}}},\ \ \forall n \geq n_\delta,\ t \in [\delta, T].$$

Hence

$$|b_n(t)|^2 \leq M_1^2 \frac{1}{\lambda_n^{\frac{N+2}{2}}},$$

that is, the function series $\sum |b_n(t)|^2$ is convergent for $t \in [\delta, T]$, with $\delta > 0$ arbitrarily choosen in $(0,T)$. Thus the series $\sum b_n(t)e_n$ is uniformly convergent to $u \in C([0,T];E)$ because

$$\|\sum b_n(t)e_n\|_E^2 = \sum |b_n|^2 < +\infty.$$

Further, $b'_n(t) = -\nu\lambda_n e^{-\nu\lambda_n t} h_n(t) + e^{-\nu\lambda_n t} h_n(t) h'_n(t)$ and thus:

$$|b'_n(t)|^2 \leq 2\nu^2\lambda_n^2 e^{-2\nu\lambda_n t}|h_n(t)|^2 + 2e^{-2\nu\lambda_n t}|h'_n(t)|^2 \leq 2(\nu^2\lambda_n^2 M_1^2 + M_2^2)e^{-2\nu\lambda_n t}.$$

Then

$$\left|\frac{b'_n(t)}{\lambda_n}\right|^2 \leq 2\nu^2 M_1^2 e^{-2\nu\lambda_n t} + 2M_2^2 \frac{e^{-2\nu\lambda_n t}}{\lambda_n^2}$$

and, similarly, we can conclude that:

$$\left|\frac{b_n'(t)}{\lambda_n}\right|^2 \leq 2\nu^2 M_1^2 \frac{1}{\lambda_n^{\frac{N+2}{2}}} + 2M_2^2 \frac{1}{\lambda_n^{\frac{N+2}{2}}},\ \forall n \geq n_\delta,\ t \in [\delta, T].$$

Consequently, the series $\sum b_n'(t)e_n$ converges absolutely to $\tilde{u} \in C((0,T); E^*)$ because

$$\|\sum b_n'(t)e_n\|_{E^*}^2 = \left\|\sum \frac{b_n'(t)}{\lambda_n}\lambda_n e_n\right\|_{E^*} = \sum \left|\frac{b_n'(t)}{\lambda_n}\right|^2 < +\infty.$$

We will show that $\tilde{u} = \frac{du}{dt}$ as a distribution on $(0,T)$. Indeed:

$$\int_0^1 \left(\sum_{n=1}^m b_n'(t)e_n\right)\phi(t)dt = \left[\left(\sum_{n=1}^m b_n(t)e_n\right)\phi(t)\right]_0^T - \int_0^T \left(\sum_{n=1}^m b_n(t)e_n\right)\phi'(t)dt =$$

$$= -\int_0^T \left(\sum_{n=1}^m b_n(t)e_n\right)\phi'(t)dt,\ \forall \phi \in C_0^\infty(0,T),\ m \in \mathbb{N},$$

that is $u' = \tilde{u}$ as a distribution from $(0,T)$ into E^*.

Let us show that $u := \sum b_n(t)e_n$ is a weak solution of the Cauchy problem (10): Indeed, for $t = 0$ we have:

$$u(x,0) = \sum b_n(0)e_n(x) = \sum u_n^0 e_n(x) = u^0(x).$$

On the other hand, remark that

$$\sum [b_j' + \nu\lambda_j b_j + \lambda_j c_j(b_1,\ldots,b_n)]e_j = \sum_{j=1}^n f_j e_j$$

and thus, multiplying in X by e_k, $1 \leq k \leq n$, we deduce:

$$(\sum_{j=1}^n b_j' e_j, e_k) + (\sum_{j=1}^n \nu\lambda_j b_j e_j, e_k) + (\sum_{j=1}^n \lambda_j c_j e_j, e_k) = (\sum_{j=1}^n f_j e_j, e_k).$$

Since $Ae_j = Je_j = \lambda_j e_j$ and $\{e_k;\ k \in \mathbb{N}\}$ is an orthonormal basis in E, letting $n \to \infty$ we deduce that:

$$(\frac{du}{dt}, w) + \nu(Au, w) + ((u \cdot \nabla)u, w) = (f, w),\ \forall w \in E,$$

that is, u is the weak solution of the problem (10). Remark that $u \in C([0,T]; E) \cap C^1((0,T); X)$, that is, u is the strong solution of the Cauchy problem.

To show the uniqueness of this strong solution, let u_1 and u_2 be two such solutions, i.e.,

$$u_1(x,t) := \sum p_n(t)e_n(x) = \lim v_{1n}(x,t),$$

$$u_2(x,t) := \sum q_n(t)e_n(x) = \lim v_{2n}(x,t),$$

where

$$v_{1n}(x,t) := \sum_{j=1}^n p_j(t)e_j(x), \quad v_{2n}(x,t) := \sum_{j=1}^n q_j(t)e_j(x).$$

Thus, for any $\varepsilon > 0$ there exist $n_\varepsilon', n_\varepsilon'' \in \mathbb{N}$ such that:

$$|(\frac{d}{dt}v_{1n} - \nu\Delta v_{1n} - (v_{1n} \cdot \nabla)v_{1n}, v) - (f, v)| < \varepsilon,\ \forall v \in E,\ n \geq n_\varepsilon',$$

and

$$|(\frac{d}{dt}v_{2n} - \nu\Delta v_{2n} - (v_{2n}\cdot\nabla)v_{2n}, v) - (f, v)| < \varepsilon,\ \forall v \in E,\ n \geq n''_\varepsilon.$$

Denote by $n_\varepsilon := \max\{n'_\varepsilon, n''_\varepsilon\}$ and take $v := e_j,\ 1 \leq j \leq n$. Then we obtain:

$$k_j := |p'_j(t) + \nu\lambda_j p_j(t) + \lambda_j c_j(p_1, \ldots, p_n) - f_j(t)| < \varepsilon,\ \forall n \geq n_\varepsilon,\ t \in (0, T)$$

and

$$r_j := |q'_j(t) + \nu\lambda_j q_j(t) + \lambda_j c_j(q_1, \ldots, q_n) - f_j(t)| < \varepsilon,\ \forall n \geq n_\varepsilon,\ t \in (0, T).$$

Let $p := (p_1, \ldots, p_n)$ and $q := (q_1, \ldots, q_n)$ be, respectively, the solutions of the following problems:

$$\begin{cases} b' = F_1(t, b) \\ b(0) = u^0 \end{cases}, \quad \begin{cases} b' = F_2(t, b) \\ b(0) = u^0 \end{cases},$$

where $F_1(t, b) := (-\lambda_j b_j - c_j(b) + f_j(t) + k_j(t))_{1\leq j\leq n}$ and $F_2(t, b) := (-\lambda_j b_j - c_j(b) + f_j(t) + r_j(t))_{1\leq j\leq n})$. As F_1 and F_2 are continuous in t and of class C^1 in b, the solutions depend continuously on the second term, that is:

$$|F_1 - F_2| := \max_{1\leq j\leq n} |F_{1j} - F_{2j}| = \max_{1\leq j\leq n} |k_j(t) - r_j(t)| \leq$$

$$\leq \max_{1\leq j\leq n} \{|k_j| + |r_j|\} < 2\varepsilon,\ \forall t \in (0, T).$$

It follows that:

$$|p(t) - q(t)| := \max_{1\leq j\leq n} |p_j(t) - q_j(t)| < g(\varepsilon),\ \forall t \in (0, T)$$

with $\lim_{\varepsilon\to 0} g(\varepsilon) = 0$, that is $p_j(t) = q_j(t)$ and so $u_1 = u_2$.

We can conclude:

Theorem: *If the series* $\sum \lambda_n^{-\frac{N+2}{2}}$ *is convergent, then there exists an unique (weak) strong solution of the Cauchy problem of Navier-Stokes system.*

Along to same line we mention that this series is convergent in the case of a bounded domain in $\mathbb{R}^N$. The proof is based on the Weyl's law:

$$N(\lambda) = \frac{\mu(\Omega)\omega_N}{(2\pi)^N}\lambda^N + R(\lambda),$$

where $N(\lambda) := card\{j \in \mathbb{N}; \sqrt{\lambda_j} \leq \lambda\}$, ω_N is the volume of the unit ball in $\mathbb{R}^N$ and $R(\lambda) = \mathcal{O}(\lambda^{N-1})$, which depends on the measure of the domain $\mu(\Omega)$ in $\mathbb{R}^N$ (see [3] for details). In the case of the unbounded domain, particularly when the domain is all of $\mathbb{R}^N$, this convergence is still unproved.

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