JOURNAL OF SCIENCE AND ARTS

Journal of Science and Arts 1(2009), 5-9
Valahia University of Târgoviște

A NOTE ON THE SCHURER'S CUBATURE FORMULA

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Abstract: *Starting with the Schurer's bivariate approximation formula, a cubature formula of Schurer type is constructed. When the approximated function is sufficiently differentiable, an upper bound estimation for the remainder term is established.*

Keywords: *Schurer's operator, Schurer's bivariate operator, Schurer's cubature formula, remainder term*

2000 Mathematics subject classification: *65D32, 41A10, 41A63*

1 Preliminaries

Let us denote $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $p \in \mathbb{N}_0$ is given, the Schurer's operator $\widetilde{B}_{m,p} : C([0, 1+p]) \to C([0, 1])$ is defined [15] for any positive integer m, any $f \in C([0, 1+p])$ and any $x \in [0, 1]$ by

$$(\widetilde{B}_{m,p}f)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k}{m}\right) \tag{1}$$

where $\widetilde{p}_{m,k}(x)$ denotes the Schurer's fundamental polynomials, i.e.

$$\widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k}. \tag{2}$$

It is well known [15] the following convergence property of the sequence $\{\widetilde{B}_{m,p}f\}_{m \ge 1}$

$$\lim_{m \to \infty} \widetilde{B}_{m,p}f = f \tag{3}$$

uniformly on $[0, 1]$, for any $f \in C([0, 1+p])$.

Considering the non-negative integers p, q and using the method of parametric extensions [1], [10], in [2] was constructed the bivariate Schurer's operator $\widetilde{B}_{m,p,n,q} : C([0, 1+p] \times [0, 1+q]) \to C([0, 1] \times [0, 1])$, defined for any positive integers m, n, any $f \in C([0, 1+p]) \times [0, 1+q])$ and any $(x, y) \in [0, 1] \times [0, 1]$ by

$$(\widetilde{B}_{m,p,n,q}f)(x, y) = \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x) \widetilde{p}_{n,j}(y) f\left(\frac{k}{m}, \frac{j}{n}\right). \tag{4}$$

Many approximations properties of the operator (4) can be found in [3].
Consider now the Schurer's bivariate approximation formula

$$f = \widetilde{B}_{m,p,n,q}f + \widetilde{R}_{m,p,n,q}f. \tag{5}$$

Regarding the remainder term of (5), in our recent paper [8] were proved the following results.

Paper presented at The VI-th International Conference on Nolinear Analysis and Applied Mathematics (ICNAAM), Târgoviște, 21-22 nov, 2008

Theorem 1.1. [8] *The remainder term of Schurer's bivariate approximation formula* (5) *can be represented under the form*

$$(\widetilde{R}_{m,p,n,q}f)(x,y) = S_1 + S_2 + S_3 \tag{6}$$

where

$$\begin{aligned} S_1 = &-\frac{px}{m}\sum_{k=0}^{m+p}\sum_{j=0}^{n+q}\widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y)\left[\begin{matrix} x,\frac{k}{m} \\ \frac{j}{n} \end{matrix}\; ;f\right] \\ &-\frac{x(1-x)(m+p)}{m^2}\sum_{k=0}^{m+p-1}\sum_{j=0}^{n+q}\widetilde{p}_{m-1,k}(x)\widetilde{p}_{n,j}(y)\left[\begin{matrix} x,\frac{k}{m},\frac{k+1}{m} \\ \frac{j}{n} \end{matrix}\; ;f\right]; \end{aligned} \tag{7}$$

$$\begin{aligned} S_2 = &-\frac{qy}{n}\sum_{k=0}^{m+p}\sum_{j=0}^{n+q}\widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y)\left[\begin{matrix} \frac{k}{m} \\ y,\frac{j}{n} \end{matrix}\; ;f\right] \\ &-\frac{y(1-y)(n+q)}{n^2}\sum_{k=0}^{m+p}\sum_{j=0}^{n+q-1}\widetilde{p}_{m,k}(x)\widetilde{p}_{n-1,j}(y)\left[\begin{matrix} \frac{k}{m} \\ y,\frac{j}{n},\frac{j+1}{n} \end{matrix}\; ;f\right]; \end{aligned} \tag{8}$$

$$\begin{aligned} S_3 = &\frac{xy(1-x)(1-y)}{mn}\sum_{k=0}^{m+p-1}\sum_{j=0}^{n+q-1}\widetilde{p}_{m-1,k}(x)\widetilde{p}_{n-1,j}(y)\left[\begin{matrix} x,\frac{k}{m},\frac{k+1}{m} \\ y,\frac{j}{n},\frac{j+1}{n} \end{matrix}\; ;f\right] \\ &+\frac{(m+p)q}{m^2n}xy(1-x)\sum_{k=0}^{m+p-1}\sum_{j=0}^{n+q}\widetilde{p}_{m-1,k}(x)\widetilde{p}_{n,j}(y)\left[\begin{matrix} x,\frac{k}{m},\frac{k+1}{m} \\ y,\frac{j}{n} \end{matrix}\; ;f\right] \\ &+\frac{(n+q)p}{mn^2}xy(1-y)\sum_{k=0}^{m+p}\sum_{j=0}^{n+q-1}\widetilde{p}_{m,k}(x)\widetilde{p}_{n-1,j}(y)\left[\begin{matrix} x,\frac{k}{m} \\ y,\frac{j}{n},\frac{j+1}{n} \end{matrix}\; ;f\right] \\ &+\frac{pq}{mn}xy\sum_{k=0}^{m+p}\sum_{j=0}^{n+q}\widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y)\left[\begin{matrix} x,\frac{k}{m} \\ y,\frac{j}{n} \end{matrix}\; ;f\right]. \end{aligned} \tag{9}$$

Note that in (7), (8) and (9) the brackets denote bivariate divided differences (for more details, see [5], [13], [14]).

Theorem 1.2. [8] *Let* $f:[0,1+p]\times[0,1+q]\to\mathbb{R}$ *be a function belonging to* $C^{(2,2)}([0,1+p]\times[0,1+q])$. *Then, there exists a constant* $M>0$ *depending on* f,p,q *such that for any* $(x,y)\in[0,1]\times[0,1]$, *any* $m,n\in\mathbb{N}$ *the following*

$$|(\widetilde{R}_{m,p,n,q}f)(x,y)| \le \left(\frac{9m+p}{8m^2}+\frac{9n+q}{8n^2}+\frac{(9m+p)(9n+q)}{64m^2n^2}\right)M. \tag{10}$$

holds.

Note that from (10) follows directly the convergence of the sequence $\{\widetilde{B}_{m,p,n,q}f\}_{m,n\ge 1}$ to f, uniformly on $[0,1]\times[0,1]$.

2 Main results

Starting with the approximation formula (5), by integration on $[0,1]\times[0,1]$ it follows the cubature formula

$$\int_0^1\int_0^1 f(x,y)dxdy = \sum_{k=0}^{m+p}\sum_{j=0}^{n+q}A_{k,j}f\left(\frac{k}{m},\frac{j}{n}\right)+R_{m,p,n,q}[f]. \tag{11}$$

The cubature formula (11) will be denoted the "Schurer's cubature" formula, because it is obtained by integrating the bivariate Schurer's bivariate approximation formula (4).

Theorem 2.1. *The coefficients* $A_{k,j}$ *of* (11) *can be expressed under the form*

$$A_{k,j} = \frac{1}{(m+p+1)(n+q+1)}, \tag{12}$$

for any $k \in \{0, 1, \ldots, m\}$ *and any* $j \in \{0, 1, \ldots, n\}$.

Proof. Taking (4) into account, we get

$$\begin{aligned} A_{k,j} &= \int_0^1 \int_0^1 \widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y)dxdy = \int_0^1 \widetilde{p}_{m,k}(x)dx \int_0^1 \widetilde{p}_{n,j}(y)dy \\ &= \binom{m+p}{k}\binom{n+q}{j} \int_0^1 x^k(1-x)^{m+p-k}dx \int_0^1 y^j(1-x)^{n+q-j}dy \\ &= \binom{m+p}{k}\binom{n+q}{j} B(k+1, m+p+1-k)B(j+1, n+q+1-j), \end{aligned} \tag{13}$$

where $B(k+1, m+p+1-k)$, $B(j+1, n+q+1-j)$ denote the Beta Euler's functions [17].

Taking the well known properties of Euler's functions Gamma and Beta into account, yields

$$B(k+1, m+p+1-k) = \frac{\Gamma(k+1)\Gamma(m+p-k+1)}{\Gamma(m+p+2)} = \frac{k!(m+p-k)!}{(m+p+1)!}, \tag{14}$$

$$B(j+1, n+q+1-j) = \frac{\Gamma(j+1)\Gamma(n+q-j+1)}{\Gamma(n+q+2)} = \frac{j!(n+q-j)!}{(n+q+1)!}. \tag{15}$$

Taking (13), (14) and (15) into account, it follows (12). □

Theorem 2.2. *Suppose* $f \in C^{(2,2)}([0, 1+p] \times [0, 1+q])$. *Then, there exists a constant* $M > 0$ *depending on* f, p, q *such that the remainder term of* (11) *verifies*

$$|R_{m,p,n,q}[f]| \le \left(\frac{9m+p}{8m^2} + \frac{9n+q}{8n^2} + \frac{(9m+p)(9n+q)}{64m^2n^2}\right) M. \tag{16}$$

Proof. One applies Theorem 1.2. Integrating (10), one arrives to the desired inequality (16). □

Remark 2.1. For more informations about the constant M, see [8].

Remark 2.2. For $p = q = 0$ the Schurer's bivariate approximation formula (4) reduces to the bivariate Bernstein's approximation formula

$$f = B_{m,n}f + R_{m,n}f \tag{17}$$

and, consequently, it follows the Bernstein's cubature formula

$$\int_0^1 \int_0^1 f(x,y)dxdy = \sum_{k=0}^{m} \sum_{j=0}^{n} A_{k,j} f\left(\frac{k}{m}, \frac{j}{n}\right) + R_{m,n}[f]. \tag{18}$$

Applying Theorem 2.1, it follows

Corollary 2.1. [21], [9] *The coefficients of Bernstein's cubature formula* (18) *can be represented under the form*

$$B_{k,j}=\frac{1}{(m+1)(n+1)}, \tag{19}$$

for any $k\in\{0,1,\ldots,m\}$ *and any* $j\in\{0,1,\ldots,n\}$.

Proof. In (12), one takes $p=q=0$. □

Corollary 2.2. *Suppose* $f\in C^{(2,2)}([0,1]\times[0,1])$. *Then, there exists a constant* $M_1>0$ *depending of* f *such that the remainder term of* (18) *verifies:*

$$|R_{m,n}[f]|\leq\left(\frac{9}{8m}+\frac{9}{8n}+\frac{81}{64mn}\right)M_1. \tag{20}$$

Proof. One applies (16) for $p=q=0$. □

Corollary 2.3. *Suppose* $f\in C^{(2,2)}([0,1+p]\times[0,1+q])$. *Then*

$$\lim_{m,n\to\infty}\frac{1}{(m+p+1)(n+q+1)}\sum_{k=0}^{m+p}\sum_{j=0}^{n+q}f\left(\frac{k}{m},\frac{j}{n}\right)=\int_0^1\int_0^1 f(x,y)dxdy \tag{21}$$

uniformly on $[0,1]\times[0,1]$.

Proof. The assertion follows from Theorem 2.2. □

Corollary 2.4. *Suppose* $f\in C^{(2,2)}([0,1]\times[0,1])$. *Then*

$$\lim_{m,n\to\infty}\frac{1}{(m+1)(n+1)}\sum_{k=0}^{m}\sum_{j=0}^{n}f\left(\frac{k}{m},\frac{j}{n}\right)=\int_0^1\int_0^1 f(x,y)dxdy \tag{22}$$

uniformly on $[0,1]\times[0,1]$.

Proof. One applies the Theorem 2.2 for $p=q=0$ (or Corollary 2.3 for $p=q=0$ or, directly, Corollary 2.2). □

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Manuscript received: 04.03.2009 / accepted: 18.05.2009