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NEW TECHNIQUE TO ACCELERATE THE CONVERGENCE OF THE SOLUTIONS OF FRACTIONAL ORDER BRATU-TYPE DIFFERENTIAL EQUATIONS

ALI KHALOUTA[1]

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Abstract. *There are many common combination methods for solving fractional differential equations. In this work, we propose a new technique called Adomian decomposition transform method (ADTM) in order to provide a new approximate series solution of fractional order Bratu-type differential equations. The fractional order derivative is described in the Caputo sense. The ADTM is a combination of two powerful methods, the Jafari transform method and Adomian decomposition method. For accelerating the convergence of ADTM when used for these equations, we replace the nonlinear terms by their Taylor expansion. To demonstrate the efficiency and validity of the proposed method, four numerical examples are presented and we compare our obtained results with the analytical results. Finally, the numerical results obtained are represented graphically using MATLAB software.*

Keywords: *Bratu-type differential equation; Caputo fractional derivative; Jafari transform; Adomian decomposition method; approximate series solution.*

1. INTRODUCTION

The Batu-type equations is one of the important differential equations in the modeling of many chemical and physical processes in science and engineering, as it is also used in a large variety of applied fields, such as modeling thermal reaction process in combustible non-deformable materials, including the solid fuel ignition model, the electrospinning process for production of ultra-fine polymer fibers, modeling some chemical reaction-diffusion, questions in geometry and relativity about the Chandrasekhar model, radiative heat transfer, and nanotechnology [1-5].

In the last few decades, fractional order differential equations have contributed many significant roles in various branches of mathematics, science and engineering for an instance, in physics, chemistry, astrophysics, electrodynamics, viscoelasticity, aerodynamics, control theory, financial models, quantum mechanics, and other applied sciences. In all these numerous applications, it is essential to obtain exact or in most cases approximate solutions of these fractional order differential equations which are generally more complex to calculate than the classical type, because the operators are defined by integral.

Nowadays, many authors have proposed and developed numerical and analytical techniques for fractional order differential equations, which are: the variational iteration transform method (VITM) [6], Sumudu Adomian decomposition method (SADM) [7], homotopy perturbation transform method (HPTM) [8], conformable fractional differential

[1] Ferhat Abbas Sétif University, Faculty of Sciences, Department of Mathematics, Laboratory of Fundamental and Numerical Mathematics, 19000 Sétif, Algeria. E-mail nadjibkh@yahoo.fr; ali.khalouta@univ-setif.dz.

transform method (CFDTM) [9], modified homotopy analysis method (MHAM) [10], natural reduced differential transform method (NRDTM) [11], Exp-function Method (EFM) [12], and so on.

The main objective of the present paper is to propose a new technique to accelerate the convergence of the solution of the fractional order Bratu type differential equation, which can be formally determined by an approximate analytical method known as the Adomian decomposition transform method (ADTM).

The fractional order Bratu type differential equation is given by

$$D^{\alpha}\psi(t)+\lambda\exp\big(\psi(t)\big)=0, \tag{1.1}$$

subject to the initial conditions

$$\psi(0)=C_0,\psi'(t)=C_1, \tag{1.2}$$

where $0<\lambda<1, \lambda\in\mathbb{R}$ and D^{α} is the Caputo fractional derivative operator of order α with $1<\alpha\leq 2$, defined as

$$D^{\alpha}\psi(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\psi^{(n)}(\tau)d\tau, \tag{1.3}$$

for $n-1<\alpha\leq n$ and $n\in\mathbb{N}$.

This paper is arranged as follows: In Section 1, we give an introduction and a review of the literature. In Section 2, we present our main results. In Sections 3 and 4, we explain the fundamental theorem of ADTM to solve the fractional order Bratu-type differential equations. Four numerical examples are provided in Section 5, to show the simplicity, efficiency and applicability of the proposed method. Finally, in Section 6, we give a conclusion of this work.

2. MAIN RESULTS

Recently, Hossein Jafari defined and developed a new general integral transform [13] called Jafari transform, which is applied to solve ordinary and partial differential equations, as follows

$$\mathbb{J}[\psi(t)]=\mathcal{J}(s)=p(s)\int_0^{\infty}\psi(t)\exp(-q(s)t)dt, \tag{2.1}$$

where $\mathcal{J}(s)$ denotes the Jafari transform of the function $\psi(t)$ and $p(s)\neq 0, q(s)$ are positive real functions. This section presents our main results related to the Jafari transform of the Caputo fractional derivative.

Theorem 2.1. If $\mathcal{J}(s)$ is the Jafari transform of the function $\psi(t)$, then the Jafari transform of Riemann-Liouville fractional integral of order $\alpha>0$, is

$$\mathbb{J}[I^{\alpha}\psi(t)]=\frac{1}{q^{\alpha}(s)}\mathcal{J}(s). \tag{2.2}$$

Proof: The Riemann-Liouville fractional integral for the function $\psi(t)$ defined by [14], can be expressed as the convolution

$$I^{\alpha}\psi(t)=\frac{1}{\Gamma(\alpha)}\,t^{\alpha-1}*\psi(t). \tag{2.3}$$

By applying, the Jafari transform to both sides of the equation (2.3), we get

$$\begin{aligned}\mathbb{J}[I^{\alpha}\psi(t)] &=\mathbb{J}\left[\frac{1}{\Gamma(\alpha)}\,t^{\alpha-1}*\psi(t)\right]=\frac{1}{p(s)}\,\mathbb{J}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]\mathbb{J}[\psi(t)]\\ &=\frac{1}{p(s)}\frac{p(s)}{q^{\alpha}(s)}\,\mathcal{J}(s)=\frac{1}{q^{\alpha}(s)}\,\mathcal{J}(s).\end{aligned} \tag{2.4}$$

The proof is complete.

Theorem 2.2. If $n\in\mathbb{Z}^{+}$ where $n-1<\alpha\leq n$ and $\mathcal{J}(s)$ be the Jafari transform of the function $\psi(t)$, then, the Jafari transform of the Caputo fractional derivative of order $\alpha>0$, is

$$\mathbb{J}[D^{\alpha}\psi(t)]=q^{\alpha}(s)\,\mathcal{J}(s)-p(s)\sum_{k=0}^{n-1}q^{\alpha-1-k}(s)\,\psi^{(k)}(0). \tag{2.5}$$

Proof: We put

$$v(t)=\psi^{(n)}(t). \tag{2.6}$$

Then, the Caputo fractional derivative defined by (1.3), can be expressed as follows

$$\begin{aligned}D^{\alpha}\psi(t)&=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\psi^{(n)}(\tau)d\tau\\ &=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}v(\tau)d\tau\\ &=I^{n-\alpha}v(t),\end{aligned} \tag{2.7}$$

Applying the Jafari transform on both sides of equation (2.7) and using the Theorem 2.1, we get

$$\mathbb{J}[D^{\alpha}\psi(t)]=\mathbb{J}[I^{n-\alpha}v(t)]=\frac{1}{q^{n-\alpha}(s)}\,\mathcal{V}(s), \tag{2.8}$$

where $\mathcal{V}(s)$ denotes the jafari transform of the function $v(t)$.

From the properties of the Jafari transform [13], we have

$$\mathbb{J}[v(t)]=\mathbb{J}\left[\psi^{(n)}(t)\right], \tag{2.9}$$

and

$$\mathcal{V}(s)=q^{n}(s)\,\mathcal{J}(s)-p(s)\sum_{k=0}^{n-1}q^{n-1-k}(s)\,\psi^{(k)}(0). \tag{2.10}$$

Therefore, the equation (2.8) becomes

$$\begin{aligned}\mathbb{J}[D^{\alpha}\psi(t)] &= \frac{1}{q^{n-\alpha}(s)}\left(q^{n}(s)\,\mathcal{J}(s) - p(s)\sum_{k=0}^{n-1} q^{n-1-k}(s)\,\psi^{(k)}(0)\right)\\ &= q^{\alpha}(s)\,\mathcal{J}(s) - p(s)\sum_{k=0}^{n-1} q^{\alpha-1-k}(s)\,\psi^{(k)}(0)\end{aligned} \tag{2.11}$$

The proof is complete.

Corollary 2.1.

■ If $p(s) = 1$ and $q(s) = s$, we get the Laplace transform of the Caputo fractional derivative as follows [15]

$$\mathbb{L}[D^{\alpha}\psi(t)] = s^{\alpha}\,\mathcal{L}(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k}\,\psi^{(k)}(0), \tag{2.12}$$

where $\mathcal{L}(s)$ denotes the Laplace transform of the function $\psi(t)$.

■ If $p(s) = s$ and $q(s) = \frac{1}{s}$, we get the Elzaki transform of the Caputo fractional derivative as follows [16]

$$\begin{aligned}\mathbb{E}[D^{\alpha}\psi(t)] &= \frac{1}{s^{\alpha}}\,\mathcal{E}(s) - \mathrm{s}\sum_{k=0}^{n-1} \frac{1}{s^{\alpha-1-k}}\,\psi^{(k)}(0)\\ &= \frac{1}{s^{\alpha}}\,\mathcal{E}(s) - \sum_{k=0}^{n-1} s^{2-\alpha+k}\,\psi^{(k)}(0),\end{aligned} \tag{2.13}$$

where $\mathcal{E}(s)$ denotes the Elzaki transform of the function $\psi(t)$.

■ If $p(s) = \frac{1}{s}$ and $q(s) = \frac{1}{s}$, we get the Aboodh transform of the Caputo fractional derivative as follows [17]

$$\begin{aligned}\mathbb{A}[D^{\alpha}\psi(t)] &= s^{\alpha}\,\mathcal{A}(s) - \frac{1}{s}\sum_{k=0}^{n-1} s^{\alpha-1-k}\,\psi^{(k)}(0)\\ &= s^{\alpha}\,\mathcal{A}(s) - \sum_{k=0}^{n-1} s^{\alpha-2-k}\,\psi^{(k)}(0),\end{aligned} \tag{2.14}$$

where $\mathcal{A}(s)$ denotes the Aboodh transform of the function $\psi(t)$.

■ If $p(s) = q(s) = \frac{1}{s}$, we get the Sumudu transform of the Caputo fractional derivative as follows [18]

$$\begin{aligned}\mathbb{S}[D^{\alpha}\psi(t)] &= \frac{1}{s^{\alpha}}\,\mathcal{S}(s) - \frac{1}{s}\sum_{k=0}^{n-1}\frac{1}{s^{\alpha-1-k}}\psi^{(k)}(0)\\ &= s^{-\alpha}\left[\mathcal{S}(s) - \sum_{k=0}^{n-1} s^{k}\,\psi^{(k)}(0)\right],\end{aligned} \tag{2.15}$$

where $\mathcal{S}(s)$ denotes the Sumudu transform of the function $\psi(t)$.

■ If $p(s) = \frac{1}{v}$ and $q(s) = \frac{s}{v}$, we get the natural transform of the Caputo fractional derivative as follows [19]

$$\begin{aligned}\mathbb{N}^{+}[D^{\alpha}\psi(t)] &= \left(\frac{s}{v}\right)^{\alpha}\,\mathcal{N}(s,v) - \frac{1}{v}\sum_{k=0}^{n-1}\left(\frac{s}{v}\right)^{\alpha-1-k}\psi^{(k)}(0)\\ &= \left(\frac{s}{v}\right)^{\alpha}\,\mathcal{N}(s,v) - \sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{v^{\alpha-k}}\psi^{(k)}(0),\end{aligned} \tag{2.16}$$

where $\mathcal{N}(s,v)$ denotes the natural transform of the function $\mathcal{L}(s)$.

■ If $p(s) = 1$ and $q(s) = \frac{s}{v}$, we get the Shehu transform of the Caputo fractional derivative as follows [20]

$$\mathbb{H}[D^{\alpha}\psi(t)] = \left(\frac{s}{v}\right)^{\alpha}\,\mathcal{H}(s,v) - \sum_{k=0}^{n-1}\left(\frac{s}{v}\right)^{\alpha-1-k}\psi^{(k)}(0). \tag{2.16}$$

where $\mathcal{H}(s,v)$ denotes the Shehu transform of the function $\psi(t)$.

3. METHODOLOGY OF THE ADTM

This section gives the methodology of the ADTM

Theorem 3.1. The fractional order Bratu-type differential equation (1.1) subject to the initial conditions (1.2) has the ADTM series solution in the form

$$\psi(t) = \sum_{n=0}^{\infty}\psi_n(t). \tag{3.1}$$

Proof: To prove this result, we consider the fractional order Bratu-type differential equation (1.1) subject to the initial conditions (1.2).

For accelerating the convergence of ADTM, we replace the nonlinear term in equation (1.1) by their Taylor expansion. To this end, we can consider $\exp\big(\psi(t)\big)$ as

$$\exp\big(\psi(t)\big) = 1 + \psi(t) + \frac{\psi^{2}(t)}{2!}. \tag{3.2}$$

Then, equation (1.1) can be written in the form

$$D^{\alpha}\psi(t)+\lambda\left(1+\ \psi(t)+\frac{\psi^{2}(t)}{2!}\right)=0. \tag{3.3}$$

Taking the Jafari transform of both sides of equation (3.3), we get

$$\mathbb{J}[D^{\alpha}\psi(t)]+\mathbb{J}\left[\lambda\left(1+\ \psi(t)+\frac{\psi^{2}(t)}{2!}\right)\right]=0. \tag{3.4}$$

Using the Theorem 2.2, we have

$$\mathbb{J}[\psi(t)]=\frac{p(s)}{q(s)}\psi(0)+\frac{p(s)}{q^{2}(s)}\psi'(0)-\frac{\lambda}{q^{\alpha}(s)}\mathbb{J}\left[1+\ \psi(t)+\frac{\psi^{2}(t)}{2!}\right]. \tag{3.5}$$

Applying the inverse Jafari transform both sides of equation (3.5), and using the initial conditions (1.2), we get

$$\psi(t)=C_{0}+C_{1}t-\lambda\mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}\left[1+\ \psi(t)+\frac{\psi^{2}(t)}{2!}\right]\right). \tag{3.6}$$

Now, assume the solution $\psi(t)$ in term of infinite series given by

$$\psi(t)=\sum_{n=0}^{\infty}\psi_{n}(t). \tag{3.7}$$

Also, the nonlinear term $\psi^{2}(t)$is decomposed in term of Adomian polynomials as

$$\psi^{2}(t)=\sum_{n=0}^{\infty}A_{n}(t), \tag{3.8}$$

where A_n is known as the Adomian polynomials [21] can be determined from the relation

$$A_{n}=\frac{1}{n!}\frac{d^{n}}{d\vartheta^{n}}\left[\left(\sum_{i=0}^{\infty}\vartheta^{i}\,\psi^{i}\right)^{2}\right]_{\vartheta=0},n=0,1,2,\ldots \tag{3.9}$$

The first Adomian Polynomials are given by

$$\begin{aligned}A_{0}&=\ \psi_{0}^{2},\\ A_{1}&=\ 2\psi_{0}\psi_{1},\\ A_{2}&=\ 2\psi_{0}\psi_{2}+\psi_{1}^{2},\\ A_{3}&=\ 2\psi_{0}\psi_{3}+\ 2\psi_{1}\psi_{2}.\end{aligned} \tag{3.10}$$

Substituting equations (3.7) and (3.8) into equation (3.6), we get

$$\sum_{n=0}^{\infty} \psi_n(t) = C_0 + C_1 t - \lambda \mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}\left[1 + \psi(t) + \frac{1}{2!}\sum_{n=0}^{\infty} A_n(t)\right]\right). \tag{3.11}$$

Comparing both sides of equation (3.11), we have the following recurrence relation

$$\begin{aligned}
\psi_0(t) &= C_0 + C_1 t - \lambda \mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}[1]\right),\\
\psi_1(t) &= -\lambda \mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}\left[\psi_0(t) + \frac{1}{2!}A_0(t)\right]\right),\\
\psi_2(t) &= -\lambda \mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}\left[\psi_1(t) + \frac{1}{2!}A_1(t)\right]\right),\\
\psi_3(t) &= -\lambda \mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}\left[\psi_2(t) + \frac{1}{2!}A_2(t)\right]\right),\\
&\vdots\\
\psi_{n+1}(t) &= -\lambda \mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}\left[\psi_n(t) + \frac{1}{2!}A_n(t)\right]\right).
\end{aligned} \tag{3.12}$$

Hence, the series solution of equations (1.1) and (1.2) is given by

$$\psi(t) = \sum_{n=0}^{\infty} \psi_n(t). \tag{3.13}$$

The proof is complete.

Remark 3.1. The approximate solution of m −order term for equations (1.1) and (1.2), is given by

$$\psi(t) = \sum_{n=0}^{m-1} \psi_n(t) = \psi_0(t) + \psi_1(t) + \psi_2(t) + \cdots + \psi_{m-1}(t). \tag{3.14}$$

4. CONVERGENCE AND ERROR ESTIMATION

This section introduces the convergence and sufficient condition of the convergence for the ADTM when it is applied to solve the fractional order Bratu type differential equation. Based on the sufficient condition of the convergence, an estimation of the maximum absolute truncated error of the solution is also studied.

Theorem 4.1. Let $\psi_n(t)$ and $\psi(t)$ be defined in Banach space $(C[0,1], \|.\|)$, then the ADTM series $\sum_{n=0}^{\infty} \psi_n(t)$ converges to the solution $\psi(t)$ of equation (1.1) if there exists $\xi, 0 < \xi < 1$ such that $\|\psi_{n+1}(t)\| \leq \xi, \forall n \in \mathbb{N}$.

Proof: Let $\{S_n(t)\}_{n\geq 0}$ be a sequence of partial sums of the series (3.1), defined by

$$S_n(t) = \sum_{k=0}^{n} \psi_k(t), \tag{4.1}$$

then

$$\|S_{n+1}(t) - S_n(t)\| = \|\psi_{n+1}(t)\| \leq \xi\|\psi_n(t)\| \leq \xi^2\|\psi_{n-1}(t)\| \leq \cdots \leq \xi^{n+1}\|\psi_0(t)\|. \tag{4.2}$$

For any $p, q \in \mathbb{N}, p > q$, by using (4.2) and triangle inequality successively, we have

$$\begin{aligned}
\left\|S_p(t) - S_q(t)\right\| &= \left\|S_p(t) - S_{p-1}(t) + S_{p-1}(t) - S_{p-2}(t) + \cdots + S_{q+1}(t) - S_q(t)\right\| \\
&\leq \left\|S_p(t) - S_{p-1}(t)\right\| + \left\|S_{p-1}(t) - S_{p-2}(t)\right\| + \cdots + \left\|S_{q+1}(t) - S_q(t)\right\| \\
&\leq \xi^p\|\psi_0(t)\| + \xi^{p-1}\|\psi_0(t)\| + \cdots + \xi^{q+1}\|\psi_0(t)\| \\
&= \xi^{q+1}(1 + \xi + \xi^2 + \cdots + \xi^p + \cdots)\|\psi_0(t)\| \\
&\leq \xi^{q+1}\left(\frac{1 - \xi^{p-q}}{1 - \xi}\right)\|\psi_0(t)\| .
\end{aligned} \tag{4.3}$$

Since $0 < \xi < 1$ we have $1 - \xi^{p-q} < 1$, then

$$\left\|S_p(t) - S_q(t)\right\| \leq \frac{\xi^{q+1}}{1 - \xi}\|\psi_0(t)\|. \tag{4.4}$$

So $\left\|S_p(t) - S_q(t)\right\| \xrightarrow[p,q\to\infty]{} 0$ as ψ_0 is bounded. Thus $\{S_n(t)\}$ is a Cauchy sequence in Banach space and hence convergent. Therefore, there exists $\psi \in B$ such that

$$\sum_{n=0}^{\infty} \psi_n(t) = \psi(t). \tag{4.5}$$

The proof is complete.

Theorem 4.2. If there exists $0 < \xi < 1$ such a way $\|\psi_{n+1}(t)\| \leq \xi\|\psi_n(t)\|, \forall n \in \mathbb{N}$, then the maximum absolute truncated error of the ADTM series solution (3.1) is estimated as

$$\left\| \psi(t) - \sum_{k=0}^{m} \psi_k(t) \right\| \leq \frac{\xi^{m+1}}{1 - \xi}\|\psi_0(t)\|. \tag{4.6}$$

Proof: Since $\sum_{k=0}^{n}\psi_k(t)$ is finite, this implies that $\sum_{k=0}^{n}\psi_k(t) < \infty$.

Consider.

$$\left\|\psi(t)-\sum_{k=0}^{m}\psi_k(t)\right\| = \left\|\sum_{k=n+1}^{\infty}\psi_n(t)\right\|$$

$$\begin{aligned} &= \sum_{k=m+1}^{\infty}\|\psi_n(t)\| \\ &\leq \sum_{k=m+1}^{\infty}\xi^m\,\|\psi_0(t)\| \\ &\leq \xi^{m+1}(1+\xi+\xi^2+\cdots)\|\psi_0(t)\| \\ &\leq \frac{\xi^{m+1}}{1-\xi}\|\psi_0(t)\|. \end{aligned} \tag{4.7}$$

The proof is complete.

5. NUMERICAL EXAMPLES

In this section, four numerical examples are studied to demonstrate the performance and efficiency of the ADTM. The results obtained by the proposed method are compared with the analytical solution and are found to be in good agreement with each other.

Example 5.1. Consider the following fractional order Bratu-type differential equation

$$D^\alpha\psi(t) - 2\exp\big(\psi(t)\big) = 0, \tag{5.1}$$

subject to the initial conditions

$$\psi(0) =, \psi'(t) = 0, \tag{5.2}$$

where $0 < t < 1$ and D^α is the Caputo fractional derivative operator of order $1 < \alpha \leq 2$.

The exact solution of equations (5.1) and (5.2) for $\alpha = 2$ is given by [5]

$$\psi(t) = -\ \ln(\cos t). \tag{5.3}$$

Using the mentioned method in Section 3, we can generate the following recurrence relation as

$$\psi_0(t) = 2\mathbb{J}^{-1}\left(\frac{1}{q^\alpha(s)}\mathbb{J}[1]\right),$$

$$\psi_1(t) = 2\mathbb{J}^{-1}\left(\frac{1}{q^\alpha(s)}\mathbb{J}\left[\psi_0(t)+\frac{1}{2!}A_0(t)\right]\right),$$

$$\psi_2(t) = 2\mathbb{J}^{-1}\left(\frac{1}{q^\alpha(s)}\mathbb{J}\left[\psi_1(t) + \frac{1}{2!}A_1(t)\right]\right),$$

$$\psi_3(t) = 2\mathbb{J}^{-1}\left(\frac{1}{q^\alpha(s)}\mathbb{J}\left[\psi_2(t) + \frac{1}{2!}A_2(t)\right]\right),$$

$$\vdots$$

$$\psi_{n+1}(t) = 2\mathbb{J}^{-1}\left(\frac{1}{q^\alpha(s)}\mathbb{J}\left[\psi_n(t) + \frac{1}{2!}A_n(t)\right]\right). \tag{5.4}$$

Now, from the recurrence relation (5.4) and the Adomian polynomials (3.9), we obtain the first few components of the solution for equations (5.1) and (5.2) as follows

$$\psi_0(t) = \frac{2}{\Gamma(\alpha+1)}t^\alpha,$$

$$\psi_1(t) = \frac{4}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{4\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha},$$

$$\psi_2(t) = \frac{8}{\Gamma(3\alpha+1)}t^{3\alpha} + \frac{8[2\Gamma(\alpha+1)\Gamma(3\alpha+1) + \Gamma^2(2\alpha+1)]}{\Gamma^2(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}t^{4\alpha}$$
$$+ \frac{16\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha}. \tag{5.5}$$

Therefore, the approximate solution of the 3 −order term for equation (5.1) and (5.2) is given by

$$\begin{aligned}\psi(t) = {} & \frac{2}{\Gamma(\alpha+1)}t^\alpha + \frac{4}{\Gamma(2\alpha+1)}t^{2\alpha} + 4\left(\frac{2\Gamma^2(\alpha+1) + \Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right)t^{3\alpha} \\ & + 8\left(\frac{2\Gamma(\alpha+1)\Gamma(3\alpha+1) + \Gamma^2(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}\right)t^{4\alpha} \\ & + \frac{16\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha}.\end{aligned} \tag{5.6}$$

The behavior of the exact solution and approximate solutions of the 3 −order term for different values of α for equations (5.1) and (5.2) are represented graphically in Fig. 1.

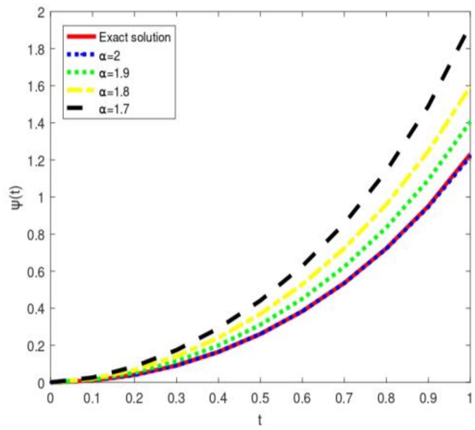

Figure 1. The graphs of the exact solution and approximate solutions using the ADTM for Example 5.1

Example 5.2. Consider the following fractional order Bratu-type differential equation

$$D^{\alpha}\psi(t) - \exp\big(2\psi(t)\big) = 0, \tag{5.7}$$

subject to the initial conditions

$$\psi(0) =, \psi'(t) = 0, \tag{5.8}$$

where $0 < t < 1$ and D^{α} is the Caputo fractional derivative operator of order $1 < \alpha \leq 2$.

The exact solution of equations (5.7) and (5.8) for $\alpha = 2$ is given by [22]

$$\psi(t) = \ln(\sec t). \tag{5.9}$$

Using the mentioned method in Section 3, we can generate the following recurrence relation as

$$\psi_0(t) = \mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}[1]\right),$$

$$\psi_1(t) = 2\mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}[\,\psi_0(t) + A_0(t)]\right),$$

$$\psi_2(t) = 2\mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}[\,\psi_1(t) + A_1(t)]\right),$$

$$\psi_3(t) = 2\mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}[\,\psi_2(t) + A_2(t)]\right),$$

$$\vdots$$

$$\psi_{n+1}(t) = 2\mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}[\,\psi_n(t) + A_n(t)]\right). \tag{5.10}$$

Now, from the recurrence relation (5.10) and the Adomian polynomials (3.9), we obtain the first few components of the solution for equations (5.7) and (5.8) as follows

$$\begin{aligned}
\psi_0(t) &= \frac{1}{\Gamma(\alpha+1)}t^{\alpha},\\
\psi_1(t) &= \frac{2}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha},\\
\psi_2(t) &= \frac{4}{\Gamma(3\alpha+1)}t^{3\alpha} + \frac{4[2\Gamma(\alpha+1)\Gamma(3\alpha+1) + \Gamma^2(2\alpha+1)]}{\Gamma^2(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}t^{4\alpha}\\
&\quad + \frac{8\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha}.
\end{aligned} \tag{5.11}$$

Therefore, the approximate solution of the $3-$order term for equation (5.7) and (5.8) is given by

$$\psi(t)=\frac{1}{\Gamma(\alpha+1)}t^{\alpha}+\frac{2}{\Gamma(2\alpha+1)}t^{2\alpha}+2\left(\frac{2\Gamma^{2}(\alpha+1)+\Gamma(2\alpha+1)}{\Gamma^{2}(\alpha+1)\Gamma(3\alpha+1)}\right)t^{3\alpha}$$
$$,$$
$$+4\left(\frac{2\Gamma(\alpha+1)\Gamma(3\alpha+1)+\Gamma^{2}(2\alpha+1)}{\Gamma^{2}(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}\right)t^{4\alpha}+\frac{8\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^{3}(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha}. \tag{5.12}$$

The behavior of the exact solution and approximate solutions of the $3-$order term for different values of α for equations (5.7) and (5.8) are represented graphically in Fig. 2.

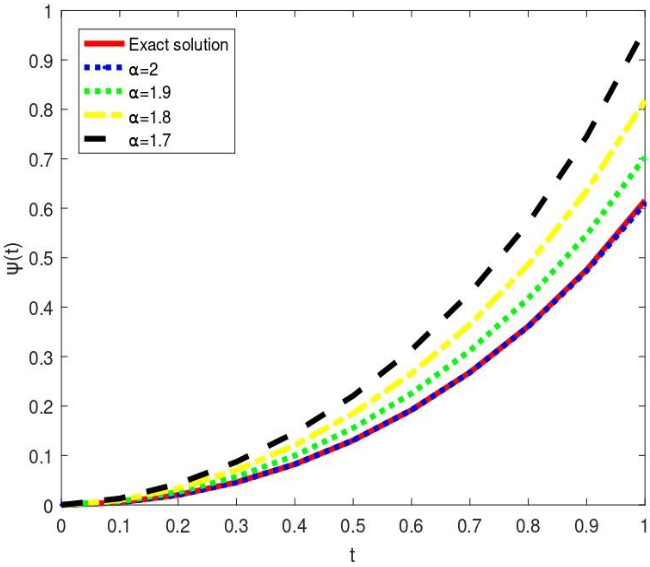

Figure 2. The graphs of the exact solution and approximate solutions using the ADTM for Example 5.2

Example 5.3. Consider the following fractional order Bratu-type differential equation

$$D^{\alpha}\psi(t)-\pi^{2}\exp\big(\psi(t)\big)=0, \tag{5.13}$$

subject to the initial conditions

$$\psi(0)=,\psi'(t)=\pi, \tag{5.14}$$

where $0<t<1$ and D^{α} is the Caputo fractional derivative operator of order $1<\alpha\leq 2$.

The exact solution of equations (5.13) and (5.14) for $\alpha=2$ is given by [22]

$$\psi(t)=\ -\ln(1-\sin\pi t). \tag{5.15}$$

Using the mentioned method in Section 3, we can generate the following recurrence relation as

$$\psi_{0}(t)=\pi t+\pi^{2}\mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}[1]\right),$$

$$\psi_{1}(t)=\pi^{2}\mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}\left[\,\psi_{0}(t)+\frac{1}{2!}A_{0}(t)\right]\right),$$

$$\psi_2(t) = \pi^2 \mathbb{J}^{-1}\left(\frac{1}{q^\alpha(s)}\mathbb{J}\left[\psi_1(t) + \frac{1}{2!}A_1(t)\right]\right),$$

$$\psi_3(t) = \pi^2 \mathbb{J}^{-1}\left(\frac{1}{q^\alpha(s)}\mathbb{J}\left[\psi_2(t) + \frac{1}{2!}A_2(t)\right]\right),$$

$$\vdots$$

$$\psi_{n+1}(t) = \pi^2 \mathbb{J}^{-1}\left(\tfrac{1}{q^\alpha(s)}\mathbb{J}\left[\psi_n(t) + \tfrac{1}{2!}A_n(t)\right]\right), \tag{5.16}$$

Now, from the recurrence relation (5.10) and the Adomian polynomials (3.9), we obtain the first few components of the solution for equations (5.13) and (5.14) as follows

$$\psi_0(t) = \pi t + \frac{\pi^2}{\Gamma(\alpha+1)}t^\alpha,$$

$$\psi_1(t) = \frac{\pi^3}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{\pi^4}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{\pi^4}{\Gamma(\alpha+1)}t^{\alpha+2} + \frac{\pi^5\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}t^{2\alpha+1} + \frac{\pi^6\Gamma(2\alpha+1)}{2\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}. \tag{5.17}$$

Therefore, the approximate solution of the $2-$order term for equation (5.13) and (5.14) is given by

$$\psi_1(t) = \pi t + \frac{\pi^2}{\Gamma(\alpha+1)}t^\alpha + \frac{\pi^3}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{\pi^4}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{\pi^4}{\Gamma(\alpha+1)}t^{\alpha+2} + \frac{\pi^5\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}t^{2\alpha+1} + \frac{\pi^6\Gamma(2\alpha+1)}{2\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}. \tag{5.18}$$

The behavior of the exact solution and approximate solutions of the $3-$order term for different values of α for equations (5.13) and (5.14) are represented graphically in Fig. 3.

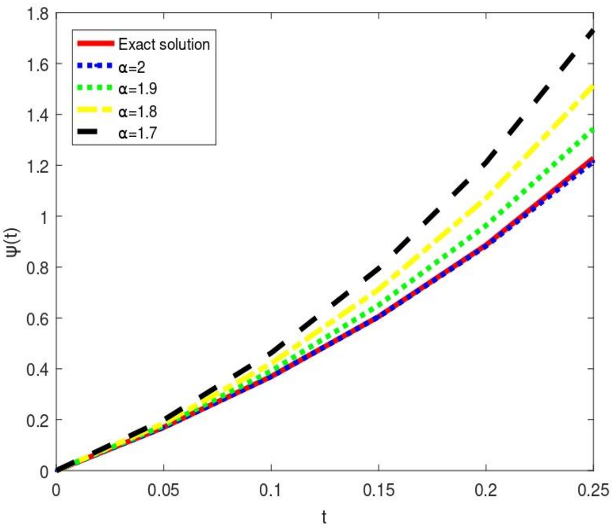

Figure 3. The graphs of the exact solution and approximate solutions using the ADTM for Example 5.3

Example 5.4. Consider the following fractional order Bratu-type differential equation

$$D^{\alpha}\psi(t)+\pi^{2}\exp\big(-\psi(t)\big)=0, \tag{5.19}$$

subject to the initial conditions

$$\psi(0)=,\psi'(t)=\pi, \tag{5.20}$$

where $0<t<1$ and D^{α} is the Caputo fractional derivative operator of order $1<\alpha\leq 2$.

The exact solution of equations (5.19) and (5.20) for $\alpha=2$ is given by [22]

$$\psi(t)=\ln(1+\sin\pi t). \tag{5.21}$$

Using the mentioned method in Section 3, we can generate the following recurrence relation as

$$\psi_0(t)=\pi t-\pi^{2}\mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}[1]\right),$$

$$\psi_1(t)=-\pi^{2}\mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}\left[-\psi_0(t)+\frac{1}{2!}A_0(t)\right]\right),$$

$$\psi_2(t)=-\pi^{2}\mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}\left[-\psi_1(t)+\frac{1}{2!}A_1(t)\right]\right),$$

$$\psi_3(t)=-\pi^{2}\mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}\left[-\psi_2(t)+\frac{1}{2!}A_2(t)\right]\right),$$

$$\vdots$$

$$\psi_{n+1}(t)=-\pi^{2}\mathbb{J}^{-1}\left(\frac{1}{q^{\alpha}(s)}\mathbb{J}\left[-\psi_n(t)+\frac{1}{2!}A_n(t)\right]\right). \tag{5.22}$$

Now, from the recurrence relation (5.22) and the Adomian polynomials (3.9), we obtain the first few components of the solution for equations (5.19) and (5.20) as follows

$$\psi_0(t)=\pi t-\frac{\pi^{2}}{\Gamma(\alpha+1)}t^{\alpha},$$

$$\psi_1(t)=\frac{\pi^{3}}{\Gamma(\alpha+2)}t^{\alpha+1}-\frac{\pi^{4}}{\Gamma(2\alpha+1)}t^{2\alpha}-\frac{\pi^{4}}{\Gamma(\alpha+1)}t^{\alpha+2}+\frac{\pi^{5}\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}t^{2\alpha+1}$$
$$+\frac{\pi^{6}\Gamma(2\alpha+1)}{2\Gamma^{2}(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}. \tag{5.23}$$

Therefore, the approximate solution of the $2-$order term for equation (5.19) and (5.20) is given by

$$\psi_1(t)=\pi t-\frac{\pi^{2}}{\Gamma(\alpha+1)}t^{\alpha}+\frac{\pi^{3}}{\Gamma(\alpha+2)}t^{\alpha+1}-\frac{\pi^{4}}{\Gamma(2\alpha+1)}t^{2\alpha}-\frac{\pi^{4}}{\Gamma(\alpha+1)}t^{\alpha+2}$$

$$+\frac{\pi^5\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}t^{2\alpha+1}+\frac{\pi^6\Gamma(2\alpha+1)}{2\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}. \tag{5.24}$$

The behavior of the exact solution and approximate solutions of the $3-$order term for different values of α for equations (5.19) and (5.20) are represented graphically in Fig. 4.

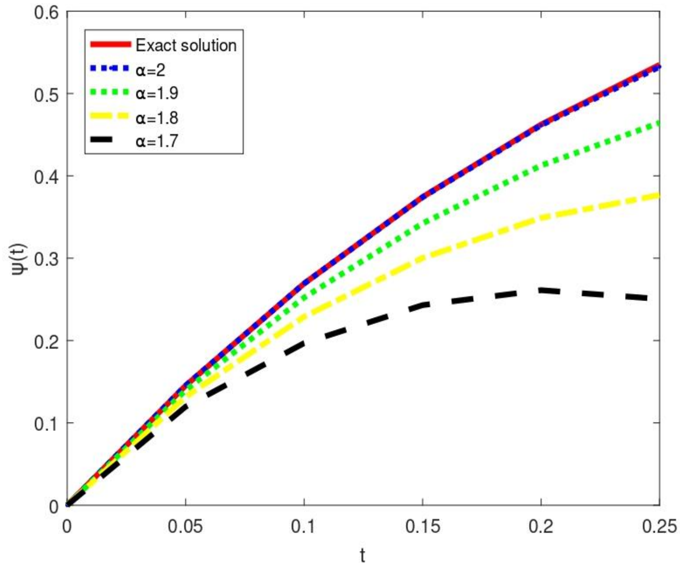

Figure 4. The graphs of the exact solution and approximate solutions using the ADTM for Example 5.4

Remark 5.1. In this work, only $3-$order term ADTM-approximate solution is used to calculate the numerical solution and ADTM can provide a more precise solution with less absolute error by calculating a higher order approximation.

4. CONCLUSION

In this work, the combination of the Adomian decomposition method and the Jafari transform in the sense of Caputo fractional derivative, proved very effective to solve fractional order Bratu-type differential equations. The proposed technique provides the solution in a series form that converges rapidly to the exact solution if it exists. We have applied the technique to different examples. From the obtained results, it is clear that the ADTM yields very accurate solutions using only a few iterates. Due to the efficiency and flexibility in the application as we have seen in the proposed examples, the conclusion that comes through this work is that ADTM can be applied to other fractional differential equations arising in science and engineering.

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