ORIGINAL PAPER

ON SUMS WITH THE r-DERANGEMENT NUMBERS

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Abstract. *In this paper, we derive some sums involving the r-derangement numbers, $D_r(n)$ and the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ by using the generating functions and Riordan arrays. For example, for $n, r \in \mathbb{Z}^+$ with $n \geq r$,*

$$\sum_{i=0}^{n-r}(-1)^i \binom{r+1}{i} \frac{D_r(n-i)}{(n-i)!} = \frac{(-1)^{n-r}}{(n-r)!}.$$

Keywords: *sums; generalized harmonic numbers; r-derangement numbers; generating function.*

1. INTRODUCTION

The harmonic numbers are defined by

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{i=1}^{n} \frac{1}{i} \quad \text{for } n \geq 1.$$

There exists integral representation in the form

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx.$$

Harmonic numbers and generalized harmonic numbers have been studied recently by many mathematicians [1-8]. In [7], for any $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the generalized harmonic numbers $H_n(\alpha)$ are defined by

$$H_0(\alpha) = 0 \quad \text{and} \quad H_n(\alpha) = \sum_{i=1}^{n} \frac{1}{i\alpha^i} \quad \text{for } n \geq 1.$$

For $\alpha = 1$, the usual harmonic numbers are $H_n(1) = H_n$ and the generating function of $H_n(\alpha)$ is

$$-\frac{\ln\left(1-\frac{x}{\alpha}\right)}{1-x} = \sum_{n=1}^{\infty} H_n(\alpha)\, x^n.$$

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In [8], for the generalized harmonic numbers $H_n(\alpha)$, Ömür and Bilgin defined the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ as follows: For $r < 0$ or $n \leq 0$, $H_n^r(\alpha) = 0$ and for $n \geq 1$,

$$H_n^r(\alpha) = \sum_{i=1}^{n} H_i^{r-1}(\alpha) \quad \text{for } r \geq 1,$$

where $H_n^0(\alpha) = \frac{1}{n\alpha^n}$. For $\alpha = 1$, $H_n^r(1) = H_n^r$ are the hyperharmonic numbers of order r. The generating function of $H_n^r(\alpha)$ is

$$-\frac{\ln\left(1 - \frac{x}{\alpha}\right)}{(1-x)^r} = \sum_{n=1}^{\infty} H_n^r(\alpha)\, x^n. \tag{1.1}$$

The Cauchy numbers of order r, C_n^r are defined by the generating function to be

$$\left(\frac{x}{\ln(1+x)}\right)^r = \sum_{n=0}^{\infty} C_n^r \frac{x^n}{n!}. \tag{1.2}$$

The generalized geometric series are given for positive integers a and b by

$$\frac{x^b}{(1-x)^{a+1}} = \sum_{n=b}^{\infty} \binom{n+a-b}{a} x^n. \tag{1.3}$$

The exponential generating function is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{1.4}$$

The derangement numbers d_n are given by the closed form formula

$$d_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

Also, these numbers satisfy the recursive formula given by

$$d_n = (n-1)(d_{n-1} + d_{n-2}) \quad \text{for } n \geq 2$$

with $d_0 = 1, d_1 = 0$ (see sequence A000166 in [9]). The generating function of d_n is given by

$$\frac{1}{1-x} e^{-x} = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!}. \tag{1.5}$$

The generalized derangement numbers $d_{n,m}$ are introduced by Munarini [10] as

$$d_{n,m}=\sum_{k=0}^{n}(-1)^k\binom{m+n-k}{n-k}\frac{n!}{k!},$$

and can be generated by

$$\frac{e^{-x}}{(1-x)^{m+1}}=\sum_{n=0}^{\infty}d_{n,m}\frac{x^n}{n!}.$$

These numbers satisfy the following relation [11]:

$$d_{n,m+1}=d_{n,m}+nd_{n-1,m+1}.$$

In [12], for $0\leq r\leq n$, $D_r(n)$ denotes the number of derangement on $n+r$ elements under the restriction that the first r elements are in disjoint cycles. A closed form formula for $D_r(n)$ is also given by

$$D_r(n)=\sum_{j=r}^{n}\binom{j}{r}\frac{n!}{(n-j)!}(-1)^{n-j}\quad \text{for } n\geq r\geq 0.$$

The r-derangement numbers $D_r(n)$ satisfy the recursive formula

$$D_r(n)=rD_{r-1}(n-1)+(n-1)D_r(n-2)+(n+r-1)D_r(n-1),\ \ n>2, r>0.$$

with initial conditions

$$D_1(n)=d_{n+1},\ \ D_r(r)=r!\ \ (r\geq 1)\ \ \text{and}\ \ D_r(r+1)=r(r+1)!\ \ (r\geq 2).$$

The generating function of $D_r(n)$ is given by

$$\frac{x^r e^{-x}}{(1-x)^{r+1}}=\sum_{n=r}^{\infty}D_r(n)\frac{x^n}{n!}. \tag{1.6}$$

Notice that for $r=0$, $D_0(n)=d_n$. The authors obtained many formulas for the r-derangement numbers. For example, for $r\geq 1, 1\leq s\leq r$ and $s\leq n$,

$$D_r(n)=\sum_{j=s}^{n}\binom{j-1}{s-1}\frac{n!}{(n-j)!}D_{r-s}(n-j).$$

Recently, using generating functions, there are some works including derangement numbers by the authors [13-19]. In [19], Qi and Guo established explicit formulas for derangement numbers and their generating function in terms of Stirling numbers of the second kind. For example, for positive integer n,

$$d_n=\sum_{k=1}^{n}k!\,k^{n-k}\binom{n}{k}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\sum_{i=0}^{n-k}\frac{(-1)^i}{k^i}\binom{n-k}{i}\frac{S(i+l,l)}{\binom{i+l}{l}},$$

where Stirling numbers of the second kind $S(n,k)$ can be defined by

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!} \quad \text{for } k\in\{0\}\cup\mathbb{N}.$$

For $n\geq 2$, the Fibonacci numbers are given by

$$F_n=F_{n-1}+F_{n-2}$$

with $F_0=0, F_1=1$ and the generating function of these numbers is

$$\frac{x}{1-x-x^2}=\sum_{n=0}^{\infty}F_nx^n. \tag{1.7}$$

Let $\mathcal{F}_{\mathrm{n}}$ be the set of all generating functions of the form

$$f_nx^n+f_{n+1}x^{n+1}+f_{n+2}x^{n+2}+\cdots,$$

where $f_n\neq 0$. For $g(x)=\sum_{n\geq 0}g_nx^n\in\mathcal{F}_0$ and $f(x)=\sum_{n\geq 0}f_nx^n\in\mathcal{F}_1$, let $r_{n,k}$ be the coefficient of x^n in gf^k. Riordan array [20] is defined by a couple of analytic functions or formal power series $R=\left(g(x),f(x)\right)=\left(r_{n,k}\right)_{n,k\geq 0}$, such that the generic of R is

$$r_{n,k}=[x^n]g(x)\left(f(x)\right)^k, \tag{1.8}$$

where $[x^n]f(x)$ denotes the coefficient of x^n in $f(x)$. From this definition, $R=\left(g(x),f(x)\right)$ is an infinite, lower triangular array. An important example of Riordan array is the Pascal triangle which can be given with the help of $xg(x)=f(x)=\frac{x}{1-x}$ such that

$$\left(\binom{n}{k}\right)_{n,k\geq 0}=\left(\frac{1}{1-x},\frac{x}{1-x}\right)=\begin{bmatrix}1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}.$$

Basically, the concept of Riordan array is used in a constructive way to find the generating function of many combinatorial identities and sums. For any sequence $\{a_n\}_{n\geq 0}$ generated by $A(x)=\sum_{n\geq 0}a_nx^n$, the summation property for Riordan array $\left(g(x),f(x)\right)=\left(r_{n,k}\right)_{n,k\geq 0}$ [3,20,21] is

$$\sum_{k=0}^{n}r_{n,k}a_k=[x^n]g(x)A\left(f(x)\right). \tag{1.9}$$

In [15], Duran et al. obtained sums including generalized hyperharmonic numbers and special numbers. For example, for any positive integers n,r,

$$\sum_{i=0}^{n}\frac{(-1)^{n-i}}{(n-i)!}H_i^r(\alpha)=\sum_{i=0}^{n}\frac{d_{n-i}}{(n-i)!}H_i^{r-1}(\alpha).$$

For $F(x)=\sum_{n=0}^{\infty}a_nx^n$ and $G(x)=\sum_{n=0}^{\infty}b_nx^n$, the product of these functions is given by

$$F(x)G(x)=\sum_{n=0}^{\infty}a_nx^n\times\sum_{n=0}^{\infty}b_nx^n=\sum_{n=0}^{\infty}c_nx^n, \tag{1.10}$$

where $c_n=\sum_{k=0}^{n}a_kb_{n-k}$.

2. SOME IDENTITIES WITH THE r-DERANGEMENT NUMBERS

In this section, we will give some sums involving the r-derangement numbers, using the generating functions of these numbers.

Theorem 2.1. For any positive integers n and r, then

$$\frac{1}{(n+r)!}\sum_{i=0}^{n}(-1)^ir^i\binom{n+r}{i}D_r(n-i+r)=\sum_{l_1+l_2+\cdots+l_{r+1}=n}\frac{d_{l_1}d_{l_2}\ldots d_{l_{r+1}}}{l_1!\,l_2!\ldots l_{r+1}!}.$$

Proof: Using (1.4) and (1.6), we have

$$x^{-r}e^{-rx}\frac{x^re^{-x}}{(1-x)^{r+1}}=\sum_{n=0}^{\infty}\frac{(-1)^nr^n}{n!}x^n\times\sum_{n=0}^{\infty}\frac{D_r(n+r)}{(n+r)!}x^n,$$

and using (1.10), equals to

$$\sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^i\binom{n+r}{i}\frac{r^i}{(n+r)!}D_r(n-i+r)x^n. \tag{2.1}$$

From (1.5),

$$\begin{aligned}x^{-r}e^{-rx}\frac{x^re^{-x}}{(1-x)^{r+1}}&=\frac{e^{-(r+1)x}}{(1-x)^{r+1}}=\underbrace{\left(\frac{e^{-x}}{1-x}\right)\times\left(\frac{e^{-x}}{1-x}\right)\times\ldots\times\left(\frac{e^{-x}}{1-x}\right)}_{(r+1)-\text{times}}\\&=\sum_{l_1=0}^{\infty}\frac{d_{l_1}}{l_1!}x^{l_1}\times\sum_{l_2=0}^{\infty}\frac{d_{l_2}}{l_2!}x^{l_2}\times\ldots\times\sum_{l_{r+1}=0}^{\infty}\frac{d_{l_{r+1}}}{l_{r+1}!}x^{l_{r+1}}\\&=\sum_{n=0}^{\infty}\sum_{l_1+l_2+\cdots+l_{r+1}=n}\frac{d_{l_1}}{l_1!}\frac{d_{l_2}}{l_2!}\ldots\frac{d_{l_{r+1}}}{l_{r+1}!}x^n.\end{aligned} \tag{2.2}$$

By comparing the coefficients on right sides of (2.1) and (2.2), we have the proof. ■

Theorem 2.2. Let n and r be positive integers such that $n \geq r$. We have

$$\sum_{i=0}^{n-r}(-1)^i\binom{r+1}{i}\frac{D_r(n-i)}{(n-i)!}=\frac{(-1)^{n-r}}{(n-r)!}.$$

Proof: From (1.4), we have

$$\sum_{n=r}^{\infty}\frac{(-1)^{n-r}}{(n-r)!}x^n=x^r\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n=x^re^{-x}=(1-x)^{r+1}\frac{x^re^{-x}}{(1-x)^{r+1}},\tag{2.3}$$

and by (1.6) and Binomial theorem,

$$\begin{aligned}\sum_{n=r}^{\infty}\frac{(-1)^{n-r}}{(n-r)!}x^n&=\sum_{n=0}^{\infty}(-1)^n\binom{r+1}{n}x^n\times\sum_{n=r}^{\infty}\frac{D_r(n)}{n!}x^n\\&=\sum_{n=r}^{\infty}\sum_{i=0}^{n-r}(-1)^i\binom{r+1}{i}\frac{D_r(n-i)}{(n-i)!}x^n\end{aligned}\tag{2.4}$$

and since $D_r(n)=0$ for $r>n$, by comparing the coefficients on right sides of (2.3) and (2.4), we obtain that for $n \geq r$,

$$\sum_{i=0}^{n-r}(-1)^i\binom{r+1}{i}\frac{D_r(n-i)}{(n-i)!}=\frac{(-1)^{n-r}}{(n-r)!},$$

as claimed. ■

Theorem 2.3. Let n and r be positive integers such that $n \geq r$. We have

$$\sum_{i=0}^{n-r}\frac{(-1)^i}{i!}H_{n-r-i}^{r}(\alpha)=\sum_{j=r}^{n}\sum_{i=r}^{j}\frac{(-1)^{n-j}}{i!}\binom{r}{n-j}H_{j-i}^{r-1}(\alpha)D_r(i),$$

and

$$\sum_{i=0}^{n-r}\frac{(-1)^i}{i!}H_{n-r-i}^{2r}(\alpha)=\sum_{i=r}^{n}\frac{1}{i!}D_r(i)H_{n-i}^{r-1}(\alpha).$$

Proof: Firstly, from (1.1) and (1.4), we have

$$\begin{aligned}-e^{-x}x^r\frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^r}&=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n\times\sum_{n=r}^{\infty}H_{n-r}^{r}(\alpha)\,x^n\\&=\sum_{n=r}^{\infty}\sum_{i=0}^{n-r}\frac{(-1)^i}{i!}H_{n-r-i}^{r}(\alpha)x^n,\end{aligned}\tag{2.5}$$

and by (1.6) and Binomial theorem,

$$-e^{-x}x^r \frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^r} = -\frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^{r-1}} \frac{e^{-x}x^r}{(1-x)^{r+1}}(1-x)^r$$
$$= \sum_{n=0}^{\infty} H_n^{r-1}(\alpha)x^n \times \sum_{n=r}^{\infty} \frac{D_r(n)}{n!}x^n \times \sum_{n=0}^{\infty}\binom{r}{n}(-x)^n$$
$$= \sum_{n=r}^{\infty}\sum_{i=r}^{n} \frac{1}{i!}H_{n-i}^{r-1}(\alpha)D_r(i)x^n \times \sum_{n=0}^{\infty}\binom{r}{n}(-1)^n x^n$$
$$= \sum_{n=r}^{\infty}\sum_{j=r}^{n}\sum_{i=r}^{j} \frac{(-1)^{n-j}}{i!}\binom{r}{n-j} H_{j-i}^{r-1}(\alpha)D_r(i)x^n. \quad (2.6)$$

Thus, by comparing the coefficients on right sides of (2.5) and (2.6), we get the desired result. Secondly, by (1.10), we have

$$-e^{-x}x^r \frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^{2r}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n \times \sum_{n=r}^{\infty} H_{n-r}^{2r}(\alpha)\, x^n$$
$$= \sum_{n=r}^{\infty}\sum_{i=0}^{n-r} \frac{(-1)^i}{i!} H_{n-r-i}^{2r}(\alpha)\, x^n, \quad (2.7)$$

and

$$-e^{-x}x^r \frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^{2r}} = \frac{-\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^{r-1}} \frac{-e^{-x}x^r}{(1-x)^{r+1}}$$
$$= \sum_{n=0}^{\infty} H_n^{r-1}(\alpha)\, x^n \times \sum_{n=r}^{\infty} \frac{D_r(n)}{n!}x^n$$
$$= \sum_{n=r}^{\infty}\sum_{i=r}^{n} H_{n-i}^{r-1}(\alpha) \frac{D_r(i)}{i!}x^n. \quad (2.8)$$

Thus, by comparing the coefficients on right sides of (2.7) and (2.8), this completes the proof. ■

Theorem 2.4. Let a, b, n, r be positive integers. For $n \geq r$,

$$\binom{n}{r}n! = \sum_{i=r}^{n}\binom{n}{i}D_r(i),$$

and for $n \geq b + r$,

$$\frac{D_{r+a}(n-b+a)}{(n-b+a)!} = \sum_{i=b}^{n-r}\binom{i-b+a-1}{i-b}\frac{D_r(n-i)}{(n-i)!}.$$

Proof: From (1.3), (1.4) and (1.6), we have

$$\sum_{n=r}^{\infty}\binom{n}{r}x^n = \frac{x^r}{(1-x)^{r+1}} = \frac{x^r e^{-x}}{(1-x)^{r+1}}e^x = \sum_{n=r}^{\infty} D_r(n)\frac{x^n}{n!} \times \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$= \sum_{n=r}^{\infty}\sum_{i=r}^{n}\frac{D_r(i)}{i!\,(n-i)!}x^n = \frac{1}{n!}\sum_{n=r}^{\infty}\sum_{i=r}^{n}\binom{n}{i}D_r(i)\,x^n.$$

Also, using (1.3) and (1.6), we get

$$\begin{aligned}\sum_{n=b+r}^{\infty}\frac{D_{r+a}(n-b+a)}{(n-b+a)!}x^n &= \sum_{n=a+r}^{\infty}D_{r+a}(n)\frac{x^{n+b-a}}{n!} = x^{b-a}\frac{x^{r+a}e^{-x}}{(1-x)^{r+a+1}}\\ &= \frac{x^b}{(1-x)^a}\frac{x^r e^{-x}}{(1-x)^{r+1}}\\ &= \sum_{n=b}^{\infty}\binom{n-b+a-1}{n-b}x^n \times \sum_{n=r}^{\infty}D_r(n)\frac{x^n}{n!}\\ &= \sum_{n=b+r}^{\infty}\sum_{i=b}^{n-r}\binom{i-b+a-1}{i-b}\frac{D_r(n-i)}{(n-i)!}x^n.\end{aligned}$$

From here, by comparing the coefficients on both sides, we have the proof. ■

Theorem 2.5. Let n and r be positive integers such that $n \geq r$. We have

$$\sum_{i=0}^{n}(-1)^i\binom{r}{i}\frac{D_r(n-i)}{(n-i)!} = \frac{d_{n-r}}{(n-r)!}.$$

Proof: Observing that

$$\begin{aligned}\sum_{n=r}^{\infty}d_{n-r}\frac{x^n}{(n-r)!} &= x^r\sum_{n=0}^{\infty}d_n\frac{x^n}{n!}\\ &= x^r\frac{e^{-x}}{1-x} = (1-x)^r\frac{x^r e^{-x}}{(1-x)^{r+1}}\\ &= \sum_{n=0}^{\infty}(-1)^n\binom{r}{n}x^n \times \sum_{n=r}^{\infty}D_r(n)\frac{x^n}{n!}\\ &= \sum_{n=r}^{\infty}\sum_{i=0}^{n-r}(-1)^i\binom{r}{i}\frac{D_r(n-i)}{(n-i)!}x^n.\end{aligned}$$

Since $\binom{n}{k} = 0$ for $k > n$ and $D_r(n) = 0$ for $r > n$, we obtain that for $n \geq r$,

$$\frac{d_{n-r}}{(n-r)!} = \sum_{i=0}^{n}(-1)^i\binom{r}{i}\frac{D_r(n-i)}{(n-i)!},$$

as claimed. ■

Theorem 2.6. Let n and r be positive integers such that $n \geq r$. We have

$$\frac{1}{n!}\sum_{i=0}^{n-r}(-1)^i\binom{n}{i}\frac{D_r(n-i)C_i}{\alpha^i}=\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^j\binom{j}{i}\binom{n-j-i}{r-1}\frac{C_i d_{j-i}}{\alpha^i j!}.$$

Proof: With the help of (1.2) and (1.6), we have

$$\begin{aligned}\frac{x^{r+1}e^{-x}}{(1-x)^{r+1}}\frac{1}{\ln\left(1-\frac{x}{\alpha}\right)}&=-\alpha\frac{x^r e^{-x}}{(1-x)^{r+1}}\frac{-\frac{x}{\alpha}}{\ln\left(1-\frac{x}{\alpha}\right)}\\&=\sum_{n=r}^{\infty}\frac{D_r(n)}{n!}x^n\times\sum_{n=0}^{\infty}(-1)^{n-1}\frac{C_n}{\alpha^{n-1}n!}x^n\\&=\sum_{n=r}^{\infty}\sum_{i=0}^{n-r}\frac{(-1)^{i-1}}{\alpha^{i-1}n!}\binom{n}{i}D_r(n-i)C_i\,x^n,\end{aligned}\tag{2.9}$$

and by (1.2), (1.3) and (1.6), we get

$$\begin{aligned}\frac{x^{r+1}e^{-x}}{(1-x)^{r+1}}\frac{1}{\ln\left(1-\frac{x}{\alpha}\right)}&=-\alpha\frac{e^{-x}}{1-x}\frac{-\frac{x}{\alpha}}{\ln\left(1-\frac{x}{\alpha}\right)}\frac{x^r}{(1-x)^r}\\&=\sum_{n=0}^{\infty}\frac{d_n}{n!}x^n\times\sum_{n=0}^{\infty}\frac{(-1)^{n-1}C_n}{\alpha^{n-1}n!}x^n\times\sum_{n=r}^{\infty}\binom{n-1}{r-1}x^n\\&=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^{i-1}C_i d_{n-i}}{\alpha^{i-1}n!}x^n\times\sum_{n=r}^{\infty}\binom{n-1}{r-1}x^n\\&=\sum_{n=r}^{\infty}\sum_{j=0}^{n-r}\sum_{i=0}^{j}\binom{j}{i}\binom{n-j-1}{r-1}\frac{(-1)^{i-1}C_i d_{j-i}}{\alpha^{i-1}j!}x^n.\end{aligned}\tag{2.10}$$

By comparing the coefficients on right sides of (2.9) and (2.10), we obtain the claimed result. ■

3. SOME SUMS WITH THE HELP OF RIORDAN ARRAYS

In this section, we will give more sums involving the r-derangement numbers with the help of Riordan arrays. From (1.6) and (1.8), we get Riordan arrays as

$$\left(\frac{e^{-x}}{1-x},\frac{x}{1-x}\right)=\left(\frac{D_k(n)}{n!}\right)_{n,k\geq 0},\tag{3.1}$$

$$\left(\frac{e^{x}}{1+x},\frac{-x}{1+x}\right)=\left(\frac{(-1)^n D_k(n)}{n!}\right)_{n,k\geq 0},\tag{3.2}$$

and

$$\left(\frac{e^x}{1+x}, \frac{x}{1+x}\right) = \left(\frac{(-1)^{n+k}D_k(n)}{n!}\right)_{n,k\geq 0}. \tag{3.3}$$

Using these Riordan arrays and some generating functions, we have following theorems:

Theorem 3.1. Let n be non-negative integer. Then we have

$$\sum_{k=0}^{n}(-1)^k D_k(n) = (-1)^n.$$

Proof: Choosing the Riordan array in (3.1) and $A(x) = \frac{1}{1+x}$, by (1.9), we have

$$\sum_{k=0}^{n}\frac{D_k(n)}{n!}(-1)^k = [x^n]\frac{e^{-x}}{1-x}\frac{1}{1+\frac{x}{1-x}} = [x^n]e^{-x} = [x^n]\sum_{k=0}^{n}\frac{(-1)^n}{n!}x^n = \frac{(-1)^n}{n!},$$

as claimed. ∎

Theorem 3.2. Let n and r be non-negative integers. Then we have

$$\binom{n+r}{r}r!\sum_{k=0}^{n}\binom{r}{k}D_k(n) = D_r(n+r).$$

Proof: Choosing the Riordan array in (3.1) and $A(x) = (1+x)^r$, by (1.9), we have

$$\sum_{k=0}^{n}\binom{r}{k}\frac{D_k(n)}{n!} = [x^n]\frac{e^{-x}}{1-x}\left(1+\frac{x}{1-x}\right)^r = [x^n]\frac{e^{-x}}{(1-x)^{r+1}} = \frac{D_r(n+r)}{(n+r)!},$$

as claimed. ∎

When $n = r$ in Theorem 3.2, we have Corollary 3.3.

Corollary 3.3. For non-negative integer n, we have

$$\binom{2n}{n}n!\sum_{k=0}^{n}\binom{n}{k}D_k(n) = D_n(2n).$$

Theorem 3.4. Let n and r be non-negative integers. Then we have

$$\sum_{k=0}^{n}(-1)^{n+k}\binom{k+r}{r}D_k(n) = \sum_{k=0}^{n}k!\binom{n}{k}\binom{r}{k}, \tag{3.4}$$

and

$$\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k}D_k(n) = \sum_{k=1}^{n}(-1)^{n+k}k!\binom{n}{k}H_k. \tag{3.5}$$

Proof: Let us choose the Riordan array in (3.1). Taking $A(x)=\frac{1}{(1+x)^{r+1}}$ for (3.4) and $A(x)=\ln(1+x)$ for (3.5), the proof is similar to the proof of Theorem 3.1. ■

Theorem 3.5. Let n be non-negative integer. Then we have

$$\sum_{k=0}^{n} D_k(n)=\sum_{k=0}^{n} 2^k\, d_k, \tag{3.6}$$

and

$$\sum_{k=0}^{n}(-1)^k D_k(n)F_k = n!\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k!}F_{n-k}. \tag{3.7}$$

Proof: From the Riordan array in (3.2), $A(x)=\frac{1}{1-x}$ for (3.6) and by (1.7), from the Riordan array in (3.3), $A(x)=\frac{1}{1-x-x^2}$ for (3.7), the proof is similar to the proof of Theorem 3.1. ■

CONCLUSION

We would like to study some sums involving the generalized derangement numbers $d_{n,m}$ [9,10], using Riordan arrays.

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