ORIGINAL PAPER

ADAPTIVE ESTIMATORS OF THE GENERAL PARETO DISTRIBUTION PARAMETERS UNDER RANDOM CENSORSHIP AND APPLICATION

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Abstract. *In this article, we introduce adaptive estimators for parameters of the (GPD) Generalized Pareto Distribution under censored data via the KIB-estimator. The KIB-estimator is based on the Maximum Likelihood Estimates (MLE) by the exceedances over the threshold t under random censoring which was developed by [1]. Hence, it was proved that the KIB-estimator is Maximum Likelihood (ML) estimator with the uncensored case. We use the standardized MLE based on the exceedances on the uncensored situation which converge to a centered bivariate normal distribution. Whose found by [2] to standardized our adaptive KIB estimator of the GPD parameters under random censorship. As an application, we establish the asymptotic normality of an estimator of the excess-of- loss reinsurance premium for heavy-tailed distribution, through the adapted KIB estimator of GPD under censored data.*

Keywords: *extreme value index; random censoring; generalized Pareto distributions; KIB estimator; reinsurance premium.*

1. INTRODUCTION

Let $X_1,\ldots,X_n$ be a sequence of independent and identically distribution (i.i.d). random variables from some unknown distribution function(d.f) F. Denote the upper endpoint of F by τ_F where $\tau_F = \sup\{x : F(x) < 1\} \le \infty$ and with $1-F(t)>0$, $t<\tau_F$ and $x>0$, be the conditional d.f. of $X-t$ given $X>t$,

$$F_t(x) = P(X \le t+x \mid X>t) = \frac{F(x+t)-F(t)}{1-F(t)} \tag{1}$$

We denote the order statistics by $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$. The weak convergence of the centered and standardized maxima $X_{n:n}$ implies the existence of sequences of constants $a_n>0$, $b_n>0$ and a d.f $\Phi(x)$ such that:

$$\lim_{n\to+\infty} P\left(\frac{X_{n:n}-b_n}{a_n} \le x\right) = \Phi(x) \tag{2}$$

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for all $x > 0$ where $\Phi(x)$ is continuous. The work of Fisher & Tippett [3], Gnedenko [4]and de Haan [5] answered the question on the possible limits and characterized the classes of distribution functions F having a certain limit in (2). This convergence result is our main assumption. Up to location and scale, the possible limiting dfs $\Phi(x)$ in (2) are given by the so-called extreme value distributions $\Psi_{\gamma_1}(x)$, defined by

$$\Psi_{\gamma_1}(x) = \begin{cases} \exp\left(-\left(1+\gamma_1 x\right)^{-1/\gamma_1}\right), if & \gamma_1 \neq 0 \\ \exp\left(-\exp\left(-x\right)\right), & if \quad \gamma_1 = 0 \end{cases} \tag{3}$$

Then it is well known Balkema et al. [6] and Pickands [7] that up to scale and location transformations the generalized Pareto d.f. given by

$$\varphi_{\gamma_1}(x) = 1 - \left(1+\gamma_1 x\right)^{-1/\gamma_1},$$

with $x > 0$ if $\gamma_1 > 0$ and $0 < x < -1/\gamma_1$ if $\gamma_1 < 0$, and for $\gamma_1 = 0$ read $\varphi_{\gamma_1}(x)$ as

$$\varphi_{\gamma_1}(x) = 1 - \exp(-x)$$

More precisely, it has been proved that there exists a normalizing function $\sigma_1(t) > 0$ such that for all x

$$\lim_{t\to\tau_F} \sup_{0<x<t-\tau_F} \left| F_t(x) - \varphi_{\gamma_1}\left(x/\sigma_1(t)\right)\right| = 0, \tag{4}$$

if and only if $F \in D\left(\Psi_{\gamma_1}\right)$. Recall that in this paper, we consider the estimation (γ_1, σ_1) under random censoring based on the maximum likelihood estimation . To specify it, let $X_1, \ldots, X_n$ be i.i.d random variables with the common distribution F . And $Y_1, \ldots, Y_n$ be i.i.d random variables with the common distribution G. We assume that both of F and G are absolutely continuous:Defined $Z_1, \ldots, Z_n$ be n i.i.d random variables with the common distribution H of $Z_i = \min(X_i, Y_i)$ where $\delta_i = 1$ if $X_i \leq Y_i$. and $\delta_i = 0$ .if $X_i > Y_i$.We will assume that both F and G are the domain of attraction of an extreme value distribution. The extreme value index of d.f. of (Z, δ) exists and is denoted γ where $\gamma = \gamma_1\gamma_2/(\gamma_1+\gamma_2)$ for γ_1 is the extreme value index d.f. of X and γ_2 is the extreme value index d.f. of Y .For τ_F , τ_G and τ_H denote the right endpoint of the support of F , G and H respectively, we assume that the pair (F, G) is in one of the following three cases (we get case 1 and case 2 for $\gamma \neq 0$ and for case 3 for $\gamma = 0$):

$$\begin{cases} case & 1: \gamma_1 > 0, \gamma_2 > 0, \\ case & 2: \gamma_1 < 0, \gamma_2 < 0, \tau_F = \tau_G \\ case & 3: \gamma_1 = \gamma_2 = 0, \tau_F = \tau_G = \infty \end{cases}$$

and the other possibilities are not very interesting.

- **The Adapt of The Profile Log-Likelihood of GPD Under Random Censorship**

We denoting the number of absolute excesses over t by k for rather the largest observations $(Z_{n-k:n},\ldots,Z_{n:n})$. In view of (4) with $\tau_H = \sup\{x : H(x) < 1\} \leq \infty$ denote the right endpoint of the support of H we can expect that observations of $C_i = Z_{n-i+1,n} - Z_{n-k,n}$ for $1 \leq i \leq k$, or, equivalently, on

$$C_0 = Z_{n-k,n}, C_1 = Z_{n-k+1,n} - Z_{n-k,n}, \cdots, C_k = Z_{n,n} - Z_{n-k,n} \tag{5}$$

where in the asymptotic setting $k = k_n$ an intermediate sequence, that is, $k_n \to \infty$ and $k_n / n \to 0$ as $n \to \infty$. And let $\delta_{1,k}, \ldots, \delta_{k,k}$ the δ corresponding to $C_{1,k}, \ldots, C_{k,k}$ respectively, for $C_{1,k} < \cdots < C_{k,k}$ the order statistics of C_i for $i = 1, \ldots, k$.

Hence, in view of convergence (4), the conditional distribution of the $(C_0, C_1, \ldots, C_k)$ given $C_0 = c_0$ can be approximated by the distribution of an ordered sample of k i.i.d. generalized Pareto random variables with d.f. $x \to \varphi_{\gamma_1}(x/\sigma_1)$. This suggests to estimate the unknown parameters γ_1 and σ_1 by a maximum likelihood estimator in the approximating generalized Pareto model under censored data; that is, given the sample $(Z_1, \ldots, Z_n)$ denoting the number of absolute excesses over t by k for rather the largest observations $(Z_{n-k:n}, \ldots, Z_{n:n})$ we can easily adapt the likelihood see Andersen et al.[1], to maximize:

$$\ell\left(C_{i,k}, \delta_{i,k}\right) = \prod_{i=1}^{k} f_{GP}\left(C_{i,k}\right)^{\delta_{i,k}} \left(1 - F_{GP}\left(C_{i,k}\right)\right)^{1-\delta_{i,k}}, \tag{6}$$

where $1 - F_{GP}(x) := \left(1 + (\gamma_1/\sigma_1)x\right)^{-1/\gamma_1}$ for $\gamma_1 \neq 0$ and $1 - F_{GP}(x) := \exp\left(-(x/\sigma_1)\right)$ for $\gamma_1 = 0$. The log-likelihood of (6) is given for $\gamma_1 \neq 0$ and $\gamma_1 = 0$ respectively by:

$$\log\left(\ell\left(C_{i,k}, \delta_{i,k}\right)\right) = \begin{cases} \sum_{i=1}^{k} \left(\delta_{i,k} \log\left(\frac{1}{\sigma_1}\right) - \left(\delta_{i,k} + \frac{1}{\gamma_1}\right) \log\left(1 + \frac{\gamma_1}{\sigma_1} C_{i,k}\right) \right) \\ \sum_{i=1}^{k} \left(\delta_{i,k} \log\left(\frac{1}{\sigma_1}\right) - \frac{1}{\sigma_1} C_{i,k} \right) \end{cases} \tag{7}$$

The likelihood equations from (7) are then given in terms of the partial derivatives

$$\begin{cases} \left.\frac{\partial \log\left(\ell\left(C_{i,k}, \delta_{i,k}\right)\right)}{\partial \gamma_1}\right|_{\gamma_1 \neq 0} = \sum_{i=1}^{k} \left(\left(\frac{1}{\gamma_1}\right)^2 \log\left(1 + \frac{\gamma_1}{\sigma_1} C_{i,k}\right) - \left(\delta_{i,k} + \frac{1}{\gamma_1}\right)\left(\frac{1}{\sigma_1} C_{i,k}\right)\left(1 + \frac{\gamma_1}{\sigma_1} C_{i,k}\right)^{-1} \right) = 0, \\ \left.\frac{\partial \log\left(\ell\left(C_{i,k}, \delta_{i,k}\right)\right)}{\partial \sigma_1}\right|_{\gamma_1 \neq 0} = \sum_{i=1}^{k} \frac{1}{\sigma_1} \left(-\delta_{i,k} + \left(\delta_{i,k} + \frac{1}{\gamma_1}\right)\left(\frac{\gamma_1}{\sigma_1} C_{i,k}\right)\left(1 + \frac{\gamma_1}{\sigma_1} C_{i,k}\right)^{-1} \right) = 0, \\ \left.\frac{\log\left(\ell\left(C_{i,k}, \delta_{i,k}\right)\right)}{\partial \sigma_1}\right|_{\gamma_1 = 0} = \sum_{i=1}^{k} \left(-\delta_{i,k} \frac{1}{\sigma_1} + \frac{1}{\sigma_1^2} C_{i,k} \right) = 0. \end{cases} \tag{8}$$

With third formula on (8), we find that

$$\hat{\sigma}_1 = (k/r)\overline{C} \text{ where } r = \sum_{i=1}^{k} \delta_{i,k} \text{ and } \overline{C} = (1/k) \times \sum_{i=1}^{k} C_{i,k} \tag{8.1}$$

The resulting likelihood equations in terms of $C_{i,k}$ for $\gamma_1 \neq 0$ can be simplified as follows:

$$\frac{1}{r}\sum_{i=1}^{k} \log\left(1+\frac{\gamma_1}{\sigma_1}C_{i,k}\right) = \gamma_1 \tag{8.2}$$

and

$$\frac{1}{k}\sum_{i=1}^{k}\left(\gamma_1\delta_{i,k}+1\right)\left(1+\frac{\gamma_1}{\sigma_1}C_{i,k}\right)^{-1} = 1 \tag{8.3}$$

Denote $\theta_1 = -(\gamma_1/\sigma_1)$ and consider (θ_1, γ_1) to be new parameters of the generalized Pareto distribution. Then we can be easily adapt the likelihood for the sample $(C_1, \dots, C_k)$ under (5), and formula (7) with $\gamma_1 \neq 0$ by

$$\log\left(\ell\left(C_{i,k}, \delta_{i,k}\right)\right) = \sum_{i=1}^{k}\left(-\delta_{i,k}\log\left(-\frac{\gamma_1}{\theta_1}\right) - \left(\delta_{i,k}+\frac{1}{\gamma_1}\right)\log\left(1-\theta_1 C_{i,k}\right)\right) \tag{9}$$

which depend only on θ_1, this is called the profile log-likelihood of θ_1. The value of θ_1 which corresponds to the local maximum of this profile log-likelihood, given by

$$\mathrm{L}\left(\theta_1, \left(C_{i,k}, \delta_{i,k}\right)\right) = \sum_{i=1}^{k}\left(-\frac{r}{k} - \delta_{i,k}\left(\log\left(-\frac{1}{\theta_1}\frac{1}{r}\sum_{i=1}^{k}\log\left(1-\theta_1 C_{i,k}\right)\right)\right) - \left(\delta_{i,k}\log\left(1-\theta_1 C_{i,k}\right)\right)\right) \tag{10}$$

Consider the first derivative of the profile log-likelihood given in (10) by:

$$\frac{\mathrm{L}\left(\theta_1, \left(C_{i,k}, \delta_{i,k}\right)\right)}{\partial\theta_1} = \frac{1}{\theta_1}\sum_{i=1}^{k}\left(\delta_{i,k}\left(\frac{-\sum_{i=1}^{k}\log\left(1-\theta_1 C_{i,k}\right) - \sum_{i=1}^{k}\frac{\theta_1 C_{i,k}}{1-\theta_1 C_{i,k}} + \frac{\theta_1 C_{i,k}}{1-\theta_1 C_{i,k}}\sum_{i=1}^{k}\log\left(1-\theta_1 C_{i,k}\right)}{\sum_{i=1}^{k}\log\left(1-\theta_1 C_{i,k}\right)}\right)\right)$$

Solving $\mathrm{L}\left(\theta_1, \left(C_{i,k}, \delta_{i,k}\right)\right)/\partial\theta_1 = 0$ is equivalent to finding the zeros of the function

$$\psi\left(\theta_1\right) = \left(\frac{1}{r}\sum_{i=1}^{k}\log\left(1-\theta_1 C_{i,k}\right)\right)\left(\frac{1}{k}\sum_{i=1}^{r}\left(1-\theta_1 C_{i,r}\right)^{-1}\right) + \frac{1}{k}\sum_{i=1}^{k}\left(1-\theta_1 C_{i,k}\right)^{-1} - 1 \tag{11}$$

with $\theta_1 = -(\gamma_1/\sigma_1)$ where $1-\theta_1 C_{i,k} > 0$ for all $i = 1, \dots, k$, so must be computed numerically on the space $\mathrm{B} = \{\theta_1 < 1/C_{k,k}, \theta_1 \neq 0\}$. The numerical problem to find a solution of the equation (11) which maximizes the approximate likelihood was discussed in detail by Kouider et al. [1]. Hence, to obtain a finite maximum of the GPD log-likelihood, the constraint $\gamma_1 \geq -1$ must be imposed. Moreover, we look for a maximum of the approximate likelihood function only in the region $]-1/2; +\infty[\times [0; +\infty[$ because the maximum likelihood estimator behaves irregularly if $\gamma_1 \leq (-1/2)$.

Since with uncensored when $k=r$ we estimate the GPD parameters (γ,σ) with $\theta=-(\gamma/\sigma)$ was discussed in detail by Kouider [9]. There More, Kouideret al. [1] was propose a new index estimator for the shape parameters to estimate the extreme value index under censored data by numerical algorithm for solve the function (11) and was developed that estimator is not adaptive estimators note by $\left(\hat{\gamma}_1^{(c,KIB)},\hat{\sigma}_1^{(c,KIB)}\right)$ with $\hat{\theta}_1^{(c,KIB)}$. It was called the KIB estimator. Hence in here numerical application proves that if $\hat{\theta}_1^{(c,KIB)}=\hat{\theta}^{(KIB)}$ the KIB estimator became adaptive. Then, will be the censored ML estimator and the GPD parameters estimation under censored data become the adaptive estimators. The adapted KIB estimator of GPD parameters in censoring defined by multiplying $\hat{\gamma}^{(KIB)}$ and $\hat{\sigma}^{(KIB)}$ estimators not adapted to censoring of γ and σ into $\frac{k}{r}$,

$$\hat{\gamma}_1^{(c.KIB)}=\frac{\hat{\gamma}^{(KIB)}}{\hat{p}},\text{ and }\hat{\sigma}_1^{(c.KIB)}=\frac{\hat{\sigma}^{(KIB)}}{\hat{p}}\text{ where }\hat{p}=\frac{r}{k}\text{ with }r=\sum_{i=1}^{n}\delta_{i,n}1_{Z_{i,n}>t}\tag{12}$$

with $\hat{p}$ is the proportion of non-censored observation in the k largest $Z's$,with $\delta_{1,n},\ldots,\delta_{n,n}$ being the δ's corresponding to $Z_{1,n},\ldots,Z_{n,n}$, respectively. For $Z_{1,n}<\cdots<Z_{n,n}$ are the order statistics of Z_i for $i=1,\ldots,n$. And we note that $\hat{\gamma}^{(KIB)}$ and $\hat{\sigma}^{(KIB)}$ are the same ML estimators.

It's easy that if $\hat{\theta}_1^{(c,KIB)}=\hat{\theta}^{(KIB)}$ we get the adaptive estimators of the GPD parameters with KIB estimator by the Likelihood methodi.e,

$$\hat{\theta}^{(KIB)}=\frac{\hat{\gamma}^{(KIB)}}{\hat{\sigma}^{(KIB)}}=\frac{\dfrac{\hat{\gamma}^{(KIB)}}{\hat{p}}}{\dfrac{\hat{\sigma}^{(KIB)}}{\hat{p}}}=\frac{\hat{\gamma}_1^{(c,KIB)}}{\hat{\sigma}_1^{(c,KIB)}}=\hat{\theta}_1^{(c,KIB)}\text{ with }\hat{p}=\frac{r}{k}$$

The KIB estimator for the GPD parameters under censored data, can be approximate in the following procedure:

- Find the root $\hat{\theta}_1^{(c.KIB)}$ of $\psi(\theta_1)=0$ where:

$$\psi\left(\theta_1\right)=\left(\frac{1}{k}\sum_{i=1}^{k}\log\left(1-\theta_1 C_{i,k}\right)\right)\left(\frac{1}{r}\sum_{i=1}^{r}\left(1-\theta_1 C_{i,r}\right)^{-1}\right)+\frac{1}{k}\sum_{i=1}^{k}\left(1-\theta_1 C_{i,k}\right)^{-1}-1=0$$

- Compute $\hat{\gamma}_1^{(c.KIB)}$ by

$$\frac{1}{r}\sum_{i=1}^{k}\log\left(1-\hat{\theta}_1^{(c.KIB)}C_{i:\mathrm{k}}\right)=\hat{\gamma}_1^{(c.KIB)}$$

- $\hat{\theta}_1^{(c.KIB)}=-\left(\hat{\gamma}_1^{(c.KIB)}/\hat{\sigma}_1^{(c.KIB)}\right)$ then $\hat{\sigma}_1^{(c.KIB)}=-\left(\hat{\gamma}_1^{(c.KIB)}/\hat{\theta}_1^{(c.KIB)}\right)$.

Let us now consider the estimation of an extreme quantile $x_{F,s}=F^{\leftarrow}(1-s)$ where $s=1-F_{GP}\left(x_{F,s}\right)$ under censoring, we can adapt the classical estimators proposed in the literature as follows:

$$\hat{x}_{t,k}^{(c.KIB)} := Z_{n-k,n} + a^{(c.KIB)} \frac{\left(\left(1-\hat{F}_n\left(Z_{n-k,n}\right)\right)/\hat{s}\right)^{\hat{\gamma}_1^{(c.KIB)}} - 1}{\hat{\gamma}_1^{(c.KIB)}} \text{ where } a^{(c.KIB)} = \frac{\hat{\sigma}^{KIB}}{\hat{p}} = \hat{\sigma}_1^{(c.KIB)} \quad (13)$$

Here under random censoring , the empirical estimator of the probability $1-F(t)$ is necessarily given by the Kaplan and Meier [10] product-limit estimator defined as

$$1-\hat{F}_n(x) = \prod_{i=1}^{n}\left[1-\frac{\delta_{i,n} 1_{Z_{i,n}\le x}}{n-i+1}\right] \quad (14)$$

where $Z_{i,n}$ denote the order statistics associated to $Z_1,\dots,Z_n$ and $\delta_{i,n}$ the δ corresponding to $Z_{i,n}$.

In the next part, we present the main result of this work, which is related to the presentation the approximate normality of the adapted KIB estimator of the GPD under censored data. Moreover, we give also the approximate normality of the KIB estimator of the GPD without censored data. These results are based mainly on the findings of each Drees et al. [2] and de Haan and Ferreira [11]. So we adapted their results under censored data. In section (3) is intended for the application of the main result where we derive the approximate normality of the estimator of reinsurance premiums in excess of the loss with censored data.

2. MAIN RESULTS

To specify the asymptotic bias of the adaptive KIB-estimator, we use a second-order condition phrased in terms of function $H^{\leftarrow}$ denotes the generalized inverse of H. We note $\xrightarrow{d}$ convergence in distribution and $\xrightarrow{P}$ convergence in probability. From the theory of generalized regular variation of second-order outlinedin Drees et al. [2]. We assume that there exist measurable, locally bounded functions, a positive function a and a second eventually positive function $A:[0;1]\to[0;+\infty[$, such that the limit

$$\lim_{t\to 0}\frac{1}{A(t)}\left\{H^{\leftarrow}(1-tx)-H^{\leftarrow}(1-t)-a(t)\frac{x^{-\gamma}-1}{\gamma}\right\}=\tilde{\Psi}(x) \quad (15)$$

exist for some $\gamma\ge-(1/2)$, for all $t\in[0;1]$ and $x>0$ with $A(t)\to 0$ as $t\to 0$. Moreover, according to de Haan and Ferreira [11], there exist function a_0 positive and $A_0(t)$ positive or negative function, such that

$$\lim_{t\to 0}\frac{1}{A_0(t)}\left\{H^{\leftarrow}(1-tx)-H^{\leftarrow}(1-tx)-a_0(t)\frac{x^{-\gamma}-1}{\gamma}\right\}=\tilde{\Psi}(x) \quad (16)$$

where

$$a_0(t)=a(t)\left[\left(1-\frac{1}{\rho}A(t)\right)1_{\rho<0}+\left(1-\frac{1}{\gamma}A(t)\right)1_{\rho=0\neq\gamma}+1_{\rho=0=\gamma}\right]$$

with

$$\lim_{t\to 0}\frac{1}{A(t)}\left(\frac{a_0(t)}{a(t)}-1\right)=\left[\left(-\frac{1}{\rho}\right)1_{\rho<0}+\left(-\frac{1}{\gamma}\right)1_{\rho=0\neq\gamma}+1_{\rho=0=\gamma}\right]:=\Gamma$$

and

$$A_0(t)=\frac{1}{\rho}A(t)1_{\rho<0}+A(t)1_{\rho=0}$$

where A not changing sign eventually with $\lim_{t\to 0}A(t)=0$ Then, according to de Haan and Stadtmüller [12], for some $\rho\leq 0$, $|A|$ is regularly varying with index $-\rho$ for , that is,

$$\lim_{t\to 0}\frac{A(tx)}{A(t)}=x^{-\rho}$$

It follows that there exists a $c\neq 0$ and a second-order parameter $\rho\leq 0$ for which the function a satisfies

$$\lim_{t\to 0}\frac{1}{A(t)}\left\{\frac{a(tx)}{a(x)}-x^{-\gamma}\right\}=cx^{-\gamma}\frac{x^{-\rho}-1}{\rho}$$

For an appropriate choice of the function a, the function $\tilde{\Psi}$ that appears in (15) and (16) admits the representation

$$\tilde{\psi}(x)=\tilde{\alpha}\Psi(x)\text{ with }\tilde{\alpha}=\left(\frac{1}{\rho}1_{\rho<0}+1_{\rho=0}\right):=\frac{A_0(t)}{A(t)}$$

where $x\to\Psi(x)/\left(x^{-\gamma}-1\right)$, is not constant . Then, according to de Haan and Ferreira [11], for all $x>0$ we have,

$$\Psi(x)=\begin{cases}\left(x^{-(\rho+\gamma)}-1\right)/(\rho+\gamma), & \rho<0\\ -x^{-\gamma}\log(x)/\gamma, & \gamma\neq\rho=0\\ \log^2(x), & \gamma=\rho=0\end{cases}$$

provided that the normalizing function a and the function A are chosen suitably.

This leads to the following Proposition, the proof of which is rather straightforward. For the KIB estimator (the uncensored ML estimator), the asymptotic bias-term follows easily from direct computations, it follows from the expressions for the asymptotic bias terms (de Haan and Ferreira [11]) of the corresponding “uncensored” estimators, see Drees et al. [2]. We assume throughout that $k=k_n$ is an integer sequence satisfying, $1<k<n, k\to\infty$ and $k/n\to 0$ as $n\to\infty$.

Proposition 2.1. Assume conditions (16) for some $\gamma\geq-(1/2)$ and that the intermediate sequence $k=k_n$ satisfies $A(k/n)=O\left(\sqrt{k}\right)$ and if

$$\sqrt{k}\lim_{n\to\infty}A\left(\frac{k}{n}\right)\to\lambda\in\mathbb{R}\text{ with }\left(\frac{1}{\rho}1_{\rho<0}+1_{\rho=0}\right):=\tilde{\alpha}\qquad(17)$$

For uncensored case (k=r) the system of likelihood equations(8.2)-(8.3) has a sequence of solutions $(\hat{\gamma},\hat{\sigma})$ that verifies

$$\sqrt{k}\left(\hat{\gamma}^{KIB}-\gamma\right)-\lambda\tilde{\alpha}\frac{(1+\gamma)^2}{\gamma}\int_0^1\left(t^{\gamma}-(2\gamma+1)t^{2\gamma}\right)\Psi(t)\,dt$$
$$\xrightarrow{d}\frac{(1+\gamma)^2}{\gamma}\left(t^{\gamma}-(2\gamma+1)t^{2\gamma}\right)\left(W(1)-t^{-\gamma-1}W(t)\right)dt$$

and

$$\sqrt{k}\left(\frac{\hat{\sigma}^{KIB}}{a(k/n)}-1\right)-\left(\lambda\tilde{\alpha}\frac{(\gamma+1)}{\gamma}\int_0^1\left((\gamma+1)(2\gamma+1)t^{2\gamma}-t^{\gamma}\right)\Psi(t)\,dt+\lambda\Gamma\right)$$
$$\xrightarrow{d}\frac{(\gamma+1)}{\gamma}\int_0^1\left((\gamma+1)(2\gamma+1)t^{2\gamma}-t^{\gamma}\right)\left(W(1)-t^{-\gamma-1}W(t)\right)dt$$

where W is a standard Brownian motion.

Corollary 2.1.Assume condition of proposition(2.1).We have

$$\sqrt{k}\begin{bmatrix}\hat{\gamma}^{KIB}-\gamma\\ \dfrac{\hat{\sigma}^{KIB}}{a(k/n)}-1\end{bmatrix}\xrightarrow{d}N(\lambda\mu,\Sigma)$$

where N denotes the bivariate normal distribution, μ equals

$$\mu=\begin{cases}\left[\dfrac{(\gamma+1)}{(1-\rho)(\gamma-\rho+1)},\dfrac{-\rho}{(1-\rho)(\gamma-\rho+1)}\right]^T & if \quad \rho<0\\ [1,0]^T & if \quad \rho=0\neq\gamma\\ [2,1]^T & if \quad \rho=0=\gamma\end{cases}$$

and

$$\Sigma=\begin{pmatrix}(1+\gamma)^2 & -(1+\gamma)\\ -(1+\gamma) & (1+\gamma)^2+1\end{pmatrix}$$

Since, if $\lambda=0$ we was shown that the standardized KIB's based on the exceedances Z_i-t converge to a centered bivariate normal distribution with covariance matrix Σ with no asymptotic bias, $\mu=[0,0]^T$. Drees et al. [2] were setting one has to condition at the event $Z_{n-k,n}=t$ with

$$\sqrt{k}\left(\frac{\hat{\sigma}^{KIB}}{a(k/n)}-1\right)=\frac{\sigma}{a(k/n)}T_n+\sqrt{k}\left(\frac{\sigma}{a(k/n)}-1\right)$$

where T_n is a normal random variable with variance $2(1+\gamma)$, so that the first term tends to a normal random variable with variance $2(1+\gamma)$ and unconditionally the second term converges to $\gamma\times W(1)$ with variance γ^2, leading to the variance given in corollary (2.1).

Moreover for $\gamma \neq 0$, $\rho < 0$ and $\lambda = 0$ setting considered here, $a(k/n) = \gamma Z_{n-k,n}$ for $\gamma > 0$ and $a(k/n) = |\gamma|\left(Z_{n,n} - Z_{n-k,n}\right)$ for $\gamma < 0$ with $\sigma / a(k/n) \to 1$ in probability, so that the first term tends to a normal random variable with variance $2(1+\gamma)$ (i.e $\hat{\sigma}^{KIB} = \hat{\sigma}\left(1 + T_n / \sqrt{k}\right)$). Then the standardized $\sqrt{k}\left(\hat{\gamma}^{KIB} - \gamma, \hat{\sigma}^{KIB} / a(k/n) - 1\right)$ converge to a centered bivariate normal distributionwith covariance matrix

$$\begin{pmatrix} (1+\gamma)^2 & -(1+\gamma) \\ -(1+\gamma) & 2(1+\gamma) \end{pmatrix}$$

which is Smith's results [13-14] examined a slightly different version of the maximum likelihood estimator that is based on the excesses over a deterministicthreshold $Z_{n-k,n} = t$.

Now, let define

$$p(z) = \frac{d\overline{H}^1(z)}{d\overline{H}(z)}$$

where $\overline{H}(z) = \overline{F}(z)\overline{G}(z)$ and $\overline{H}^1(z) = -\int_z^{\tau_H} \overline{G}(x) d\overline{F}(x)$ whose introduced by Einmahl et al. [15], they noted that, $\lim_{z \to \tau_H} p(z) = \frac{\gamma}{\gamma_1} := p \in [0;1]$ for $\gamma = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2}$. Then, with $k = k_n$ is an integer sequence satisfying , $1 < k < n, k \to \infty$ and $k/n \to 0$ as $n \to \infty$ we have:

$$\hat{p} = \frac{1}{k} \sum_{i=1}^{k} \delta_{i,k} = \frac{r}{k}$$

Theorem 1. Under the assumptions corollary (2.1) and for $n \to \infty$

$$\frac{1}{\sqrt{k}} \sum_{i=1}^{k} \left[p\left(U_H\left(\frac{i}{n}\right)\right) - p \right] \to \alpha_1 \in \mathbb{R}$$

and

$$\sqrt{k} \sup_{\left\{1-k/n \le t < 1; |t-s| \le C/\sqrt{k}/n; s < 1\right\}} S\left| p\left(U_H(t)\right) - p\left(U_H(s)\right) \right| \to 0 \text{ for all } C > 0 \qquad (18)$$

we have, for the adaptive KIB-estimator of the GPD parameters under censoring, that $\gamma_1 > (-1/2)$

$$\sqrt{k}\left(\hat{\gamma}_1^{(c,KIB)} - \gamma_1\right) \xrightarrow{d} N\left(\frac{1}{p}\left(\lambda\mu_{11} - \gamma_1\alpha_1\right), \frac{1}{p}\left(\frac{\Sigma_{11}}{p} + \gamma_1^2(1-p)\right)\right)$$

where $\lambda\mu_{11}$ denotes the bias and Σ_{11} denotes the variance of $\sqrt{k}\left(\hat{\gamma}^{(KIB)}, \gamma\right)$, and

$$\sqrt{k}\left(\frac{\hat{\sigma}_1^{(c,KIB)}}{a^{(C)}(k/n)} - \frac{1}{p}\right) \xrightarrow{d} N\left(\lambda\mu_{12} - \frac{\alpha_1}{p^2}, \Sigma_{11} + \frac{(1-p)}{p^3}\right)$$

where $\lambda\mu_{12}$ denotes the bias and Σ_{22} denotes the variance of $\sqrt{k}\left(\hat{\sigma}^{KIB}/a(k/n)-1\right)$ and $a^{(c)}(k/n)=a(k/n)/p$.

Proof 2.1: Proof of Theorem (1)**,** it is clear that for $\gamma_1=\gamma/p$ and $\sigma_1=\sigma/p$ we have the following decomposition

$$\sqrt{k}\left(\hat{\gamma}_1^{(c,KIB)}-\gamma_1\right)=\frac{1}{\hat{p}}\sqrt{k}\left(\hat{\gamma}^{(KIB)}-\gamma\right)-\sqrt{k}\,\frac{\gamma}{\hat{p}}\left(\frac{\hat{p}}{p}-1\right) \tag{19}$$

Next for $\sigma/a(k/n)\to 1$ based on the sample $C_i=Z_{n-i+1,n}-Z_{n-k,n}$ for $1\le i\le k$ and with (13) we get

$$\frac{a^{(C)}(k/n)}{a(k/n)}\xrightarrow{P}\frac{1}{p} \tag{20}$$

as similar with $\sigma_1/a(k/n)\xrightarrow{p}\frac{1}{p}$ we have,

$$\sqrt{k}\left(\frac{\hat{\sigma}_1^{(c,KIB)}}{a^{(C)}(k/n)}-\frac{1}{p}\right)=\frac{p}{\hat{p}}\sqrt{k}\left(\frac{\hat{\sigma}^{(KIB)}}{a(k/n)}-1\right)-\sqrt{k}\,\frac{1}{\hat{p}p}(\hat{p}-p) \tag{21}$$

where k denote the (random) number of exceedances over the threshold t and $\hat{p}=(r/k)$It is worth mentioning that $\hat{p}$ is a consistent estimator for p, that is $\hat{p}\to p$ (in probability) as $n\to\infty$. Then the terms (19) and (21) becomes

$$\begin{cases}\sqrt{k}\left(\hat{\gamma}_1^{(c,KIB)}-\gamma_1\right)=\frac{1}{\hat{p}}\sqrt{k}\left(\hat{\gamma}^{(c,KIB)}-\gamma\right)-\sqrt{k}\,\frac{\gamma_1}{p}(\hat{p}-p)\\[2ex] \sqrt{k}\left(\frac{\hat{\sigma}_1^{(c,KIB)}}{a^{(C)}(k/n)}-\frac{1}{p}\right)=\sqrt{k}\left(\frac{\hat{\sigma}^{(KIB)}}{a(k/n)}-1\right)-\sqrt{k}\,\frac{1}{p^2}(\hat{p}-p)\end{cases} \tag{22}$$

Under Einmahl et al. [15], we have

$$\sqrt{k}(\hat{p}-p)=\sqrt{k}(\tilde{p}-p)+\sqrt{k}(\hat{p}-\tilde{p}) \tag{23}$$

They get that for $U_{1,n},U_{2,n},\dots,U_{n,n}$ are i.i.d. and independent of the Z-sample with the assumptions that

$$\sqrt{k}(\hat{p}-\tilde{p})\xrightarrow{P}\frac{1}{\sqrt{k}}\sum_{i=1}^{k}\left[p\left(U_H\left(\frac{i}{n}\right)\right)-p\right] \tag{24}$$

where $U_H(i/n)=H^{\leftarrow}(1-(i/n))$ then for $n\to\infty$, we have

$$\sqrt{k}\left(\hat{p}-p\right)=\sqrt{k}\left(\tilde{p}-p\right)+\alpha_1-o_p(1) \tag{25}$$

which α_1 turns out to be a bias term. Since $\tilde{p}=\frac{1}{k}\sum_{i=1}^{k}1_{U_{i,k}<p}$, we have

$$\sqrt{k}\left(\tilde{p}-p\right)\xrightarrow{d}N\left(0;p(1-p)\right)$$

Finally, combining (22) and (23) with (25) yields

$$\sqrt{k}\left(\hat{\gamma}_1^{(c,KIB)}-\gamma_1\right)=\frac{1}{p}\left(\sqrt{k}\left(\hat{\gamma}^{(c,KIB)}-\gamma\right)-\sqrt{k}\gamma_1\left(\tilde{p}-p\right)\right)-\frac{\gamma_1}{p}\alpha_1+o_p(1)$$

$$\sqrt{k}\left(\frac{\hat{\sigma}_1^{(c,KIB)}}{a^{(C)}(k/n)}-\frac{1}{p}\right)=\sqrt{k}\left(\frac{\hat{\sigma}^{(KIB)}}{a(k/n)}-1\right)-\sqrt{k}\,\frac{1}{p^2}\left(\tilde{p}-p\right)-\frac{\alpha_1}{p^2}+o_p(1)$$

with the two terms within the brackets independent since the first is based on the Z-sample and the second on the U -sample. Therefore, under the assumptions (17) and (18), we get

$$\sqrt{k}\left(\hat{\gamma}_1^{(c,KIB)}-\gamma_1\right)\xrightarrow{d}N\left(\frac{1}{p}\left(\lambda\mu_{11}-\gamma_1\alpha_1\right);\frac{1}{p}\left(\frac{\Sigma_{11}}{p}+\gamma_1^2(1-p)\right)\right),$$

where $\lambda\mu_{11}$ denotes the bias and Σ_{11} denotes the variance of $\sqrt{k}\left(\hat{\gamma}^{(KIB)},\gamma\right)$. And

$$\sqrt{k}\left(\frac{\hat{\sigma}_1^{(c,KIB)}}{a^{(C)}(k/n)}-\frac{1}{p}\right)\xrightarrow{d}N\left(\lambda\mu_{12}-\frac{\alpha_1}{p^2};\Sigma_{11}+\frac{(1-p)}{p^3}\right)$$

With

$$Cov\left(\sqrt{k}\left(\hat{\gamma}_1^{(c,KIB)}-\gamma_1\right);\sqrt{k}\left(\frac{\hat{\sigma}_1^{(c,KIB)}}{a^{(C)}(k/n)}-\frac{1}{p}\right)\right)=\frac{1}{p}Cov\left(\sqrt{k}\left(\hat{\gamma}^{(KIB)}-\gamma\right);\sqrt{k}\left(\frac{\hat{\sigma}^{(KIB)}}{a(k/n)}-1\right)\right)$$
$$+\frac{\gamma_1}{p^2}Var\left(\sqrt{k}\left(\hat{p}-p\right)\right).$$

The following corollary, represent the expressions for the asymptotic bias terms of the corresponding adaptive KIB –estimator.

Corollary 2.2. Under the assumptions of Theorem (1), we have

$$\sqrt{k}\begin{bmatrix}\hat{\gamma}_1^{(c,KIB)}-\gamma_1\\ \dfrac{\hat{\sigma}_1^{(c,KIB)}}{a^{(C)}(k/n)}-\dfrac{1}{p}\end{bmatrix}\xrightarrow{d}N\left(\tilde{\lambda}\mu^{(C)},\Sigma^{(C)}\right)$$

where N denotes the bivariate normal distribution, $\mu^{(c)}$ equals

$$\mu^{(C)}=\begin{cases}\left[\tilde{\lambda}\left(\frac{(\gamma+1)}{(1-\rho)(\gamma-\rho+1)}\right)-\gamma_1\frac{\alpha_1}{p},\tilde{\lambda}\left(\frac{-\rho}{(1-\rho)(\gamma-\rho+1)}\right)-\frac{\alpha_1}{p^2}\right]^T & if \quad \rho<0\\ \left[\tilde{\lambda}-\frac{\gamma_1}{p}\alpha_1,-\frac{\alpha_1}{p^2}\right]^T & if \quad \rho=0\neq\gamma\\ \left[2\tilde{\lambda}-\frac{\gamma_1}{p}\alpha_1,p\tilde{\lambda}-\frac{\alpha_1}{p^2}\right]^T & if \quad \rho=0=\gamma\end{cases}$$

where $\tilde{\lambda}=\frac{\lambda}{p}$. And $\Sigma^{(c)}$ equals

$$\Sigma^{(C)}=\begin{pmatrix}\frac{1}{p}\left(\left(1+\gamma_1\right)^2+\frac{1-p}{p}\right) & \frac{1}{p}\left(\gamma_1(1-2p)-1\right)\\ \frac{1}{p}\left(\gamma_1(1-2p)-1\right) & \left(1+p\gamma_1\right)^2+\frac{1}{p}\end{pmatrix}$$

Since if $\tilde{\lambda}=0$ we get that the standardized adaptive KIB's $\sqrt{k}\left(\hat{\gamma}_1^{(c,KIB)}-\gamma_1\right)$ and $\sqrt{k}\left(\frac{\hat{\sigma}_1^{(c,KIB)}}{a^{(c)}(k/n)}-\frac{1}{p}\right)$ based on the exceedances Z_i-t for $Z_{n-k,n}-t$ under censoring converge to bivariate normal distribution with covariance matrix $\Sigma^{(c)}$ with asymptotic bias, $\mu^{(C)}=\left[-\left(\gamma_1/p\right)\alpha_1,-\left(1/p^2\right)\alpha_1\right]^T$. There more, we setting

$$\sqrt{k}\left(\frac{\hat{\sigma}_1^{(c,KIB)}}{a^{(C)}(k/n)}-\frac{1}{p}\right)=\frac{\sigma}{a(k/n)}T_n+\sqrt{k}\left(\frac{\sigma}{a(k/n)}-1\right)-\sqrt{k}\,\frac{1}{p^2}(\hat{p}-p)$$

So that $T_n\times\sigma/\left(a(k/n)\right)$ tends to a normal random variable with variance $2(1+\gamma)$ unconditionally $\sigma/\left(a(k/n)\right)\to 1$ and the second term converges to $\gamma\times W(1)$ with variance γ^2 and third term converges to $N\left(\alpha_1/p^2;(1-p)/p^3\right)$. Since asymptotically $Z_{n-k,n}=t$ and the excesses $Z_{n-i+1,n}-Z_{n-k,n}$ for $1\le i\le k$ are independent, so are T_n and $\gamma W(1)$ and the third term based on the U-sample so the asymptotic variance for the scale estimator under censoring is $(1+\gamma)^2+(1/p)$.
Since if

$$\sqrt{k}\left(\frac{\hat{\sigma}_1^{(c,KIB)}}{a^{(C)}(k/n)}-\frac{1}{p}\right)=\frac{\sigma}{a(k/n)}T_n-\sqrt{k}\,\frac{1}{p^2}(\hat{p}-p)$$

the asymptotic variance for the scale estimator under censoring is $2(1+\gamma)+(1-p)/p^3$

3. RESULTS AND APPLICATION

Our work will be of great interest in establishing the limit distributions of many statistics in extreme value theory under random censoring such as the estimators of tail indices and actuarial risk premiums for heavy-tailed distribution. As an application, we propose an estimator for the excess-of-loss reinsurance premium and establish its asymptotic normality.

Let $X_1,\ldots,X_n$ $(n \geq 1)$ be n individual claim amounts of an insured heavy-tailed loss X with finite mean. Note that a Pareto-like distribution, with tail index greater than or equal to 1**,** do not have finite mean. Hence, assuming that $E[X]$ exists necessarily implies that $\gamma_1 \in]-1;1[$ In practice, each claim will have a policy limit Y equal to the maximal amount(specific to each contract) that the company can insure. When the amount of the claim exceeds the policy limit (i.e when $X \geq Y$), the loss variable is right censored. for a discussion on this issue.

Let $X_1,\ldots,X_n$ be n independent copies $(n \geq 1)$ of a non-negative random variable (rv) X defined over some probability space $(\Omega;A;P)$, with cumulative distribution function (cdf) F. We assume that the distribution tail $\overline{F} = 1-F$ is regularly varying at infinity, with index $(-1/\gamma_1)$, notation: $\overline{F} \in RV_{(-1/\gamma_1)}$. That is

$$\lim_{t\to\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} := x^{-1/\gamma_1}, \text{ for any } x > 0 \tag{26}$$

We notice that the asymptotic normality of extreme value theory based estimators is achieved in the second-order framework (see de Haan and Stadtmüller [12]). Thus, it seems quite natural to suppose that cdf's *F*, *G* and *H* satisfy the well-known second-order condition of regular variation. That is, we assume that there exist a constant $\tau_j < 0$ and a function A_j, $j = 1,2,3$ tending to zero and not changing sign near infinity, such that for any $x > 0$

$$\begin{aligned}
&\lim_{t\to\infty} \frac{\overline{F}(tx)/\overline{F}(t) - x^{-1/\gamma_1}}{A_1(t)} := x^{-1/\gamma_1}\,\frac{x^{\tau_1/\gamma_1}-1}{\tau_1\gamma_1},\\
&\lim_{t\to\infty} \frac{\overline{G}(tx)/\overline{G}(t) - x^{-1/\gamma_2}}{A_2(t)} := x^{-1/\gamma_2}\,\frac{x^{\tau_2/\gamma_2}-1}{\tau_2\gamma_2},\\
&\lim_{t\to\infty} \frac{\overline{H}(tx)/\overline{H}(t) - x^{-1/\gamma}}{A_3(t)} := x^{-1/\gamma}\,\frac{x^{\tau_3/\gamma_3}-1}{\tau_3\gamma_3},
\end{aligned} \tag{27}$$

Let $X_1,\ldots,X_n$ be n independent copies $(n \geq 1)$ of a non-negative random variable (rv) X defined over some probability space $(\Omega;A;P)$, for distributed risks with cumulative distribution function (cdf) *F* such as

$$\overline{F}(x) := cx^{-1/\gamma_1}\left(1 + x^{-\alpha}L(x)\right) as\, x \to \infty \tag{28}$$

for $\gamma_1 \in [0;1]$**,** $\alpha > 0$ a constant c and $L(x)$ a function with slow variation. We note that $\overline{F} \in RV_{(-1/\gamma_1)}$ it's mean the same distribution with (26), (see de Haan and Ferreira [11] and de Haan and Stadtmüller [12]).

Let $F_t(x) := P(X \leq x + t \mid X > t)$ be the distribution of excesses over the threshold *t*. It follows with similar from(1) that

$$\overline{F}t(x) := \frac{\overline{F}(x+t)}{\overline{F}(t)} = \left(1 + \frac{x}{t}\right)^{-1/\gamma_1} \frac{1 + (x+t)^{-\alpha}\, L(x+t)}{1 + t^{-\alpha} L(t)}, \tag{29}$$

Under the convergence (4), the law of excesses beyond of a threshold t is asymptotically a law of GPD. Where the function of distribution GPD is:

$$F_{GP}(x) := \begin{cases} 1-\left(1+\frac{\gamma_1}{\sigma_1}x\right)^{-1/\gamma_1} & if \quad \gamma_1 \neq 0 \\ 1-\exp\left(-\frac{x}{\sigma_1}\right) & if \quad \gamma_1 = 0 \end{cases}, \quad (30)$$

with $x \in [0; \tau_F - t]$ is $\gamma_1 > 0$, $x \in [0; \min(-\sigma_1/\gamma_1; \tau_F - t)]$ and if $\gamma_1 < 0$. And if $\gamma_1 \to 0$, $F_{GP}(x)$ is approximated by the exponential distribution with σ_1 corresponds to the mean.

For the large value of t if $\sigma_1(t) = t\gamma_1$ the function $\bar{F}_t(x)$ is perturbed generalized Pareto distribution (GPD), where is defines in (30). For large value of t (i.e. close to the right endpoint of the support of *F;* $\tau_F = \sup\{x : F(x) < 1\}$, use that

$$\bar{F}t(x) \approx \bar{F}GP(x) \text{ if } \sigma_1(t) = t\gamma_1, \quad (31)$$

In the excess-of-loss reinsurance treaty, the ceding company covers claims that do not exceed a (high) number $t \geq 0$, called retention level, while the reinsurer pays the part $(X_i - t)_+ = \max(0; X_i - t)$ of each claim exceeding t. the net premium for the layer from t to infinity is defined as follow:

$$\Pi(t) = E[(X-t)_+] = \int_t^{\infty} \bar{F}(x)dx, \quad (32)$$

By definition $\bar{F}(x) = P(X > x - t)$ and with decaled with (1) and that $\bar{F}(x) = \bar{F}_t(x).\bar{F}(t)$. We can deduce

$$\Pi(t) = \int_t^{\infty} \bar{F}t(x).\bar{F}(t)dx$$

Therefore, the estimator of the net premium will be written:

$$\hat{\Pi}(t) = \int_t^{\infty} \hat{\bar{F}}t(x).\hat{\bar{F}}(t)dx$$

A natural estimator of $\hat{\bar{F}}(t)$ is given according to the empirirque distribution which he is the estimator of Kaplan-Meier [10] or other studies [16-18], given by (33). Note that the distribution tail estimator is of the form

$$\hat{\bar{F}}(t) := \bar{F}n(t) = \prod_{i=1}^{n}\left(1 - \frac{\delta_{i,n}1_{Z_{i,n}\leq t}}{n-i+1}\right), \quad (33)$$

In the other dimensioned, the theorem of Pickands [7], watch which the tail of excesses is approximated by the GPD

$$\hat{\bar{F}}t(x) \approx \hat{\bar{F}}GP(x)$$

We can write this estimator under censored data in the form

$$\hat{\Pi}(t)=\overline{F}n(t)\int_0^\infty\left(1+\frac{\hat{\gamma}_1^{(c,KIB)}}{\hat{\sigma}_1^{(c,KIB)}}x\right)^{-1/\hat{\gamma}_1^{(c,KIB)}}dx$$

Under the condition that $\hat{\gamma}_1^{(c,KIB)}\in[0;1]$ the estimator of the net premium under random censoring can determine by

$$\hat{\Pi}(t)=\overline{F}n(t)\frac{\hat{\sigma}_1^{(c,KIB)}}{1-\hat{\gamma}_1^{(c,KIB)}},\tag{34}$$

And we can transformed that for $\hat{\sigma}_1^{(c,KIB)}(t)=t\,\hat{\gamma}_1^{(c,KIB)}$ this estimator (34) by

$$\hat{\Pi}(t)=\frac{\hat{\gamma}_1^{(c,KIB)}}{1-\hat{\gamma}_1^{(c,KIB)}}t\overline{F}n(t),\tag{35}$$

Theorem 2. Assume that all corollary (2.2) hold. Then, for $0<\gamma_1<1$, $\overline{H}(t):=\overline{F}(t)\overline{G}(t)$ and $\overline{H}^1(s):=-\int_s^{\tau_H}\overline{G}(t)d\overline{F}(t)$ we have as $n\to\infty$,

$$\frac{\sqrt{k}\left(\hat{\Pi}(t)-\Pi(t)\right)}{t\overline{F}(t)}\xrightarrow{d}N\left(\mu^{(C)},\sigma^2\right),\tag{36}$$

where $\mu^{(c)}$ is given in corollary (2.2) and

$$\sigma^2=\left(\frac{1}{1-\gamma_1}\right)^2\left(\gamma_1^2\int_t^0\frac{d\overline{F}(s)}{\overline{F}^2(s)\overline{G}(s)}+\frac{1}{\left(1-\gamma_1\right)^2}\frac{1}{p}\left(\left(1+\gamma_1\right)^2+\frac{1-p}{p}\right)\right).$$

Corollary 3.1. Under the assumptions (27), with no common discontinuity, we have

$$\sqrt{k}\left(\frac{\overline{F}n(t)}{\overline{F}(t)}-1\right)\xrightarrow{d}N\left(0,\frac{1}{p}\frac{1}{\gamma_1^2}t^{\frac{1}{p\gamma_1}}\right)$$

where N denotes the bivariate normal distribution. Then (36) became

$$\frac{\sqrt{k}\left(\hat{\Pi}(t)-\Pi(t)\right)}{t\overline{F}(t)}\xrightarrow{d}N\left(\mu^{(C)},\sigma^2\right)$$

where $\mu^{(c)}$ is given in corollary (2.2) and

$$\sigma^2=\left(\frac{1}{1-\gamma_1}\right)^2\left(\frac{1}{p}\left(t^{\frac{1}{p\gamma_1}}+\frac{1}{\left(1-\gamma_1\right)^2}\left(\left(1+\gamma_1\right)^2+\frac{1-p}{p}\right)\right)\right)$$

Proof of corollary 3.1. First, we provide the estimator of the net premium given in (34). Consider the transformation

$$h=1+\frac{\gamma_1}{\sigma_1}x$$

thus we find that $x=(h-1)(\sigma_1/\gamma_1)$ and $dx=dh\,(\sigma_1/\gamma_1)$ therefor $x\to 0$ implies $h\to 1$. The tail index is $(1/\gamma_1)$ and the case where $\gamma_1\in[0;1]$ is will be considered in this work. This is the range of $(1/\gamma_1)$ which is often of interest in financial applications, thus with possibly an infinite second moment.

$$\hat{\Pi}(t)=\overline{F}n(t)\int_0^\infty\left(1+\frac{\hat{\gamma}_1}{\hat{\sigma}_1}x\right)^{-1/\hat{\gamma}_1}dx=\frac{\hat{\sigma}_1}{\hat{\gamma}_1}\overline{F}n(t)\int_1^\infty h^{-1/\hat{\gamma}_1}dh$$
$$=\frac{\hat{\sigma}_1}{\hat{\gamma}_1}\overline{F}n(t)\frac{1}{-\frac{1}{\hat{\gamma}_1}+1}\left[h^{-1/\hat{\gamma}_1+1}\right]_1^\infty=\overline{F}n(t)\frac{\hat{\sigma}_1}{1-\hat{\gamma}_1}$$

We consider the following decomposition

$$\frac{\sqrt{k}\left(\hat{\Pi}(t)-\Pi(t)\right)}{t\overline{F}(t)}=\frac{\gamma_1}{1-\gamma_1}\sqrt{k}\left(\frac{\overline{F}n(t)}{\overline{F}(t)}-1\right)-\frac{1}{\left(1-\gamma_1\right)^2}\sqrt{k}\left(\hat{\gamma}_1^{(c,KIB)}-\gamma_1\right),$$

Under the asymptotic normality of the estimator of Kaplan-Meier [10], we deduce that

$$\sqrt{k}\left(\frac{\overline{F}n(t)}{\overline{F}(t)}-1\right)\xrightarrow{d}N\left(0,\sigma_a^2\right)\text{ where }\sigma_a^2=-\int_0^t\frac{d\overline{F}(s)}{\overline{F}^2(s)\overline{G}(s)}=\int_0^t\frac{d\overline{H}^1(s)}{\overline{H}^2(s)}. \quad (37)$$

Then with the term (37) and corollary (2.2), we get

$$\frac{\sqrt{k}\left(\hat{\Pi}(t)-\Pi(t)\right)}{t\overline{F}(t)}\xrightarrow{d}N\left(\mu^{(c)},\left(\frac{1}{1-\gamma_1}\right)^2\left(\gamma_1^2\sigma_a^2+\frac{1}{\left(1-\gamma_1\right)^2}\frac{1}{p}\left(\left(1+\gamma_1\right)^2+\frac{1-p}{p}\right)\right)\right)$$

Since $\overline{F}\in RV_{(-1/\gamma_1)}$, $\overline{G}\in RV_{(-1/\gamma_2)}$ and $\overline{H}\in RV_{(-1/\gamma)}$ where $\overline{H}(t):=\overline{F}(t)\overline{G}(t)$. It's easy to chekd that

$$\sqrt{k}\left(\frac{\overline{F}n(t)}{\overline{F}(t)}-1\right)\xrightarrow{d}N\left(0,\frac{1}{p}\frac{1}{\gamma_1^2}t^{\frac{1}{p\gamma_1}}\right)$$

and so the corollary (3.1) it has been proven.

4. CONCLUSIONS

In this paper, we examine the properties of the adaptive KIB estimator for the tail behavior under random censorship. Whose we illustrate earlier as the same the adaptive ML estimator of extreme value index under censored data. By studying the adaptive estimator of the shape parameter and the scale parameter of the GPD . We also adapted the theoretical results that he mentioned under censoring with asymptotic bias terms of the corresponding adaptive KIB estimators of the shape and the scale parameters of the GPD under random censoring.

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Appendix A:

The first derivative of the profile log-likelihood under censoring for $\hat{\theta}$ is given by

$$\frac{L\left(\theta_1,\left(C_{i,k},\delta_{i,k}\right)\right)}{\partial\theta_1}=\sum_{i=1}^{k}\left(\delta_{i,k}\left(\frac{-\left(\frac{1}{\theta_1}\right)^2\frac{1}{r}\sum_{i=1}^{k}\log\left(1-\theta_1C_{i,k}\right)-\frac{1}{\theta_1}\frac{1}{r}\sum_{i=1}^{k}\frac{C_{i,k}}{1-\theta_1C_{i,k}}}{\frac{1}{\theta_1}\frac{1}{r}\sum_{i=1}^{k}\log\left(1-\theta_1C_{i,k}\right)}\right)+\delta_{i,k}\frac{C_{i,k}}{1-\theta_1C_{i,k}}\right)$$

$$=\sum_{i=1}^{k}\left(\delta_{i,k}\left(\frac{-\frac{1}{\theta_1}\sum_{i=1}^{k}\log\left(1-\theta_1C_{i,k}\right)-\frac{1}{\theta_1}\sum_{i=1}^{k}\frac{\theta_1C_{i,k}}{1-\theta_1C_{i,k}}}{\sum_{i=1}^{k}\log\left(1-\theta_1C_{i,k}\right)}\right)+\delta_{i,k}\frac{1}{\theta_1}\frac{\theta_1C_{i,k}}{1-\theta_1C_{i,k}}\right)$$

$$=\frac{1}{\theta_1}\sum_{i=1}^{k}\left(\delta_{i,k}\left(\frac{-\sum_{i=1}^{k}\log\left(1-\theta_1C_{i,k}\right)-\sum_{i=1}^{k}\frac{\theta_1C_{i,k}}{1-\theta_1C_{i,k}}}{\sum_{i=1}^{k}\log\left(1-\theta_1C_{i,k}\right)}\right)+\delta_{i,k}\frac{\theta_1C_{i,k}}{1-\theta_1C_{i,k}}\right)$$

$$=\frac{1}{\theta_1}\sum_{i=1}^{k}\left(\delta_{i,k}\left(\frac{-\sum_{i=1}^{k}\log\left(1-\theta_1C_{i,k}\right)-\sum_{i=1}^{k}\frac{\theta_1C_{i,k}}{1-\theta_1C_{i,k}}+\frac{\theta_1C_{i,k}}{1-\theta_1C_{i,k}}\sum_{i=1}^{k}\log\left(1-\theta_1C_{i,k}\right)}{\sum_{i=1}^{k}\log\left(1-\theta_1C_{i,k}\right)}\right)\right)$$

$$=\frac{1}{\theta_1}\sum_{i=1}^{k}\left(\delta_{i,k}\left(\frac{-\sum_{i=1}^{k}\log\left(1-\theta_1C_{i,k}\right)-\sum_{i=1}^{k}\frac{\theta_1C_{i,k}}{1-\theta_1C_{i,k}}+\frac{\theta_1C_{i,k}}{1-\theta_1C_{i,k}}\sum_{i=1}^{k}\log\left(1-\theta_1C_{i,k}\right)}{\sum_{i=1}^{k}\log\left(1-\theta_1C_{i,k}\right)}\right)\right)$$