ORIGINAL PAPER

WEAKLY BINARY $g\alpha$ – CLOSED SETS AND WEAKLY BINARY αg – CLOSED SETS IN BINARY TOPOLOGICAL SPACE

DURAISINGAM ABINAYA[1], MICHAEL GILBERT RANI[2], RAJENDRAN PREMKUMAR[2]

Manuscript received: 21.01.2023; Accepted paper: 30.04.2023; Published online: 30.06.2023.

Abstract. *In this paper, we will define some new class of generalized closed sets called weakly binary generalized α-closed sets, weakly binary α generalized closed sets, binary generalized $^\star\alpha$-closed sets and weakly binary generalized $^\star\alpha$-closed sets in binary topological spaces and study some of their characterizations and properties.*

Keywords: *$wbg\alpha$-closed sets; $wb\alpha g$-closed sets; $bg^\star\alpha$-closed sets and $wbg^\star\alpha$-closed sets.*

1. INTRODUCTION AND PRELIMINARIES

In 1970 Levine [1] gives the concept and properties of generalized closed (briefly g-closed) sets and the complement of g-closed set is said to be g-open set. Njasted [2] introduced and studied the concept of α-sets. Later these sets are called as α-open sets in 1983. Mashhours et.al [3] introduced and studied the concept of α-closed sets, α-closure of set, α-continuous functions, α-open functions and α-closed functions in topological spaces. Maki et.al [4, 5] introduced and studied generalized α-closed sets and α-generalized closed sets. In 2011, S.Nithyanantha Jothi and P.Thangavelu [6] introduced topology between two sets and also studied some of their properties. Topology between two sets is the binary structure from X to Y which is defined to be the ordered pairs (A, B) where $A \subseteq X$ and $B \subseteq Y$. In this paper, we will define some new class of generalized closed sets called weakly binary generalized α-closed sets, weakly binary α generalized closed sets, binary generalized $^\star\alpha$-closed sets and weakly binary generalized $^\star\alpha$-closed sets in binary topological spaces and study some of their characterizations and properties.

Let X and Y be any two nonempty sets. A binary topology [6] from X to Y is a binary structure $\mathcal{M} \subseteq \mathbb{P}(X) \times \mathbb{P}(Y)$ that satisfies the axioms namely

1. (ϕ, ϕ) and $(X, Y) \in \mathcal{M}$,
2. $(A_1 \cap A_2, B_1 \cap B_2) \in \mathcal{M}$ whenever $(A_1, B_1) \in \mathcal{M}$ and $(A_2, B_2) \in \mathcal{M}$, and
3. If $\{(A_\alpha, B_\alpha): \alpha \in \delta\}$ is a family of members of $\mathcal{M}$, then $(\bigcup_{\alpha\in\delta} A_\alpha, \bigcup_{\alpha\in\delta} B_\alpha) \in \mathcal{M}$.

If $\mathcal{M}$ is a binary topology from X to Y then the triplet $(X, Y, \mathcal{M})$ is called a binary topological space and the members of $\mathcal{M}$ are called the binary open subsets of the binary topological space $(X, Y, \mathcal{M})$. The elements of $X \times Y$ are called the binary points of the binary topological space $(X, Y, \mathcal{M})$. If $Y = X$ then $\mathcal{M}$ is called a binary topology on X in which case we write $(X, \mathcal{M})$ as a binary topological space.

[1] Madurai Kamaraj University, Department of Mathematics, Madurai District, Tamil Nadu, India.
E-mail: abimat@annejac.ac.in.
[2] Arul Anandar College, Department of Mathematics, Karumathur, Madurai District, Tamil Nadu, India.
E-mail: gilmathaac@gmail.com; prem.rpk27@gmail.com.

Definition 1.1. [6] Let X and Y be any two nonempty sets and let (A, B) and $(C, D) \in \mathbb{P}(X) \times \mathbb{P}(Y)$. We say that $(A, B) \subseteq (C, D)$ if $A \subseteq C$ and $B \subseteq D$.

Definition 1.2. [6] Let $(X, Y, \mathcal{M})$ be a binary topological space and $A \subseteq X$, $B \subseteq Y$. Then (A, B) is called binary closed in $(X, Y, \mathcal{M})$ if $(X \backslash A, Y \backslash B) \in \mathcal{M}$.

Proposition 1.3. [6] Let $(X, Y, \mathcal{M})$ be a binary topological space and $(A, B) \subseteq (X, Y)$. Let $(A, B)^{1*} = \cap \{A_\alpha : (A_\alpha, B_\alpha)$ is binary closed and $(A, B) \subseteq (A_\alpha, B_\alpha)\}$ and $(A, B)^{2*} = \cap \{B_\alpha : (A_\alpha, B_\alpha)$ is binary closed and $(A, B) \subseteq (A_\alpha, B_\alpha)\}$. Then $((A, B)^{1*}, (A, B)^{2*})$ is binary closed and $(A, B) \subseteq ((A, B)^{1*}, (A, B)^{2*})$.

Proposition 1.4. [6] Let $(X, Y, \mathcal{M})$ be a binary topological space and $(A, B) \subseteq (X, Y)$. Let $(A, B)^{1*} = \cup \{A_\alpha : (A_\alpha, B_\alpha)$ is binary open and $(A_\alpha, B_\alpha) \subseteq (A, B)\}$ and $(A, B)^{2*} = \cup \{B_\alpha : (A_\alpha, B_\alpha)$ is binary open and $(A_\alpha, B_\alpha) \subseteq (A, B)\}$.

Definition 1.5. [6] The ordered pair $((A, B)^{1*}, (A, B)^{2*})$ is called the binary closure of (A, B), denoted by b-cl(A, B) in the binary space $(X, Y, \mathcal{M})$ where $(A, B) \subseteq (X, Y)$.

Definition 1.6. [6] The ordered pair $((A, B)^{1*}, (A, B)^{2*})$ defined in proposition 1.4 is called the binary interior of of (A, B), denoted by b-int(A, B). Here $((A, B)^{1*}, (A, B)^{2*})$ is binary open and $((A, B)^{1*}, (A, B)^{2*}) \subseteq (A, B)$.

Definition 1.7. [6] Let $(X, Y, \mathcal{M})$ be a binary topological space and let $(x, y) \subseteq (X, Y)$. The binary open set (A, B) is said to be a binary neighbourhood of (x, y) if $x \in A$ and $y \in B$.

Proposition 1.8. [6] Let $(A, B) \subseteq (C, D) \subseteq (X, Y)$ and $(X, Y, \mathcal{M})$ be a binary topological space. Then, the following statements hold:

1. b-int$(A, B) \subseteq (A, B)$.
2. If (A, B) is binary open, then b-int$(A, B) = (A, B)$.
3. b-int$(A, B) \subseteq$ b-int(C, D).
4. b-int(b-int(A, B)) = b-int(A, B).
5. $(A, B) \subseteq$ b-cl(A, B).
6. If (A, B) is binary closed, then b-cl$(A, B) = (A, B)$.
7. b-cl$(A, B) \subseteq$ b-cl(C, D).
8. b-cl(b-cl(A, B)) = b-cl(A, B).

Definition 1.9. A subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is called

1. a binary semi open set [7] if $(A, B) \subseteq$ b-cl(b-int(A, B)).
2. a binary pre open set [8] if $(A, B) \subseteq$ b-int(b-cl(A, B)),
3. a binary regular open set [9] if $(A, B) =$ b-int(b-cl(A, B)).

Definition 1.10. A subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is called

1. a binary g-closed set [10] if b-cl$(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary open.
2. binary $g^\star$-closed set [11] if b-cl$(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary g-open in (X, Y).
3. a binary gs-closed set [12] if b-scl$(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary open.
4. a binary sg-closed set [12] if b-scl$(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary semi open.

5. a binary gr-closed set [9] if b-rcl$(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary open.

6. a binary gsp-closed set [13] if b-βcl$(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary open.

7. a binary gp-closed set [13] if b-pcl$(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary open.

8. a binary gpr-closed set [9] if b-pcl$(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary regular open.

Definition 1.11. A subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is called

1. a binary α-open [14] if $(A, B) \subseteq$ b-int(b-cl(b-int(A, B))).
2. a binary β-open [8] if $(A, B) \subseteq$ b-cl(b-int(b-cl(A, B))).

Definition 1.12. [15] A subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is called

1. a binary gα-closed if b-cl$(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary α-open.
2. a binary αg-closed if b-αcl$(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary open.

2. WEAKLY BINARY gα-CLOSED SETS AND WEAKLY BINARY αg-CLOSED SETS

Definition 2.1. A subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is said to be

1. weakly binary generalized α-closed (briefly wbgα-closed) set if b-αcl(b-int(A, B)) $\subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary α-open in $(X, Y, \mathcal{M})$.
2. weakly binary α-generalized closed (briefly wbαg-closed) set if b-αcl(b-int(A, B)) $\subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary open in $(X, Y, \mathcal{M})$.
3. binary generalized*α-closed (briefly bg*α-closed) set if b-αcl$(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary gα-open in $(X, Y, \mathcal{M})$.
4. weakly binary generalized*α-closed (briefly wbg*α-closed) set if b-αcl(b-int(A, B)) $\subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary gα-open in $(X, Y, \mathcal{M})$.

The collection of all wbgα-closed (resp. wbαg-closed, bg*α-closed and wbg*α-closed) sets of $(X, Y, \mathcal{M})$ is denoted by WBGαC(X, Y) (resp. WBαGC(X, Y), BG*αC(X, Y) and WBG*αC(X, Y)).

Proposition 2.2. Every binary closed set in a topological space $(X, Y, \mathcal{M})$ is bg*α-closed.

Proof: Assume that a subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is binary closed. Let (U, V) be a binary gα-open set containing (A, B). Then $(U, V) \supseteq (A, B) =$ b-cl(A, B), as (A, B) is binary closed. Also, $(U, V) \supseteq$ b-cl$(A, B) \supseteq$ b-αcl(A, B). We have $(U, V) \supseteq$ b-αcl(A, B). Hence (A, B) is a bg*α-closed set in $(X, Y, \mathcal{M})$. The converse of Proposition 2.2 need not be true as seen from the following example.

Example 2.3. Let $X = \{a, b\}$, $Y = \{1,2\}$ and $\mathcal{M} = \{(\phi, \phi), (\{a\}, \{1\}), (\phi, \{1\}), (X, Y)\}$. Then the sets in $\{(\phi, \phi), (\{b\}, \{2\}), (X, \{2\}), (X, Y)\}$ are binary closed. Then the subset $(\{b\}, \{1\})$ is bg*α-closed but not binary closed in $(X, Y, \mathcal{M})$.

Corollary 2.4. Every binary regular closed set in $(X, Y, \mathcal{M})$ is bg*α-closed.

Proof: We know that every binary regular closed set is binary closed and by Proposition 2.2, every binary closed set is $bg^\star\alpha$-closed. Hence every binary regular closed set is $bg^\star\alpha$-closed.

Proposition 2.5. Every binary α-closed set in $(X, Y, \mathcal{M})$ is $bg^\star\alpha$-closed.

Proof: Assume that a subset (A, B) is binary α-closed in $(X, Y, \mathcal{M})$. Let (U, V) be a binary gα-open set containing (A, B). Then $(U, V) \supseteq (A, B) = \text{b-}\alpha\text{cl}(A, B)$, as (A, B) is α-closed. Thus $(U, V) \supseteq \text{b-}\alpha\text{cl}(A, B)$. Hence (A, B) is $bg^\star\alpha$-closed in $(X, Y, \mathcal{M})$. The converse of Proposition 2.5 need not be true as seen from the following example.

Example 2.6. In Example 2.3, then the subset $(\{a\}, Y)$ is $bg^\star\alpha$-closed but not binary α-closed in $(X, Y, \mathcal{M})$.

Proposition 2.7. Every binary gα-closed set in a binary topological space $(X, Y, \mathcal{M})$ is wbgα-closed.

Proof: Let (A, B) be a subset of $(X, Y, \mathcal{M})$ which is binary gα-closed and let (U, V) be an binary α-open set containing (A, B). Since (A, B) is binary gα-closed, $(U, V) \supseteq \text{b-}\alpha\text{cl}(A, B)$. Then $(U, V) \supseteq \text{b-}\alpha\text{cl}(A, B) \supseteq \text{b-}\alpha\text{cl}(\text{b-int}(A, B))$. i.e., $(U, V) \supseteq \text{b-}\alpha\text{cl}(\text{b-int}(A, B))$. Hence (A, B) is wbgα-closed in $(X, Y, \mathcal{M})$. The converse of Proposition 2.7 need not be true as seen from the following example.

Example 2.8. Let $X = \{a, b\}$, $Y = \{1,2\}$ and $\mathcal{M} = \{(\phi, \phi), (\phi, \{2\}), (\{a\}, \{1\}), (X, \{1\}), (\{a\}, Y), (X, Y)\}$. Then the subset $(\phi, \{1\})$ is wbgα-closed but not binary gα-closed in $(X, Y, \mathcal{M})$.

Proposition 2.9. If a subset (A, B) of $(X, Y, \mathcal{M})$ is both binary open and wbgα-closed, then it is binary gα-closed.

Proof: Let (A, B) be a subset of $(X, Y, \mathcal{M})$ which is both binary open and wbgα-closed. Then $(A, B) \supseteq \text{b-}\alpha\text{cl}(\text{b-int}(A, B)) \supseteq \text{b-}\alpha\text{cl}(A, B)$. i.e., $(A, B) \supseteq \text{b-}\alpha\text{cl}(A, B)$. Hence (A, B) is binary gα-closed in $(X, Y, \mathcal{M})$.

Proposition 2.10. Every binary αg-closed set in $(X, Y, \mathcal{M})$ is wbαg-closed.

Proof: Assume that a subset (A, B) of $(X, Y, \mathcal{M})$ is binary αg-closed. Let (U, V) be an binary open set containing (A, B), Then $(U, V) \supseteq \text{b-}\alpha\text{cl}(A, B)$, as (A, B) is binary αg-closed. Thus $(U, V) \supseteq \text{b-}\alpha\text{cl}(A, B) \supseteq \text{b-}\alpha\text{cl}(\text{b-int}(A, B))$. i.e., $(U, V) \supseteq \text{b-}\alpha\text{cl}(\text{b-int}(A, B))$. Hence (A, B) is wbαg-closed. The converse of Proposition 2.10 need not be true as seen from the following example.

Example 2.11. In Example 2.8, then the subset $(\{a\}, \{2\})$ is wbαg-closed but not binary αg-closed in $(X, Y, \mathcal{M})$.

Proposition 2.12. If a subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is both binary open and wbαg-closed, then it is binary αg-closed.

Proof: Assume that a subset (A, B) of $(X, Y, \mathcal{M})$ is both binary open and wbαg-closed. Then $(A, B) \supseteq \text{b-}\alpha\text{cl}(\text{b-int}(A, B)) \supseteq \text{b-}\alpha\text{cl}(A, B)$. i.e., $(A, B) \supseteq \text{b-}\alpha\text{cl}(A, B)$. Hence (A, B) is binary αg-closed in $(X, Y, \mathcal{M})$.

Proposition 2.13. Every $bg^{\star}\alpha$-closed set in $(X, Y, \mathcal{M})$ is

1. binary gα-closed,
2. wbgα-closed,
3. binary αg-closed and
4. wbαg-closed.

Proof: Assume that (A, B) is a $bg^{\star}\alpha$-closed set in $(X, Y, \mathcal{M})$.

1. Let (U, V) be an binary α-open set containing (A, B). Then (U, V) is a binary gα-open set, as every binary α-open set is binary gα-open. Since (A, B) is a $bg^{\star}\alpha$-closed set, $(U, V) \supseteq (A, B)$ and $(U, V) \supseteq$ b-αcl(A, B). Therefore (A, B) is binary gα-closed in $(X, Y, \mathcal{M})$.
2. Follows from (1) and from Proposition 2.7.
3. Let (U, V) be an binary open set containing (A, B). Then (U, V) is a binary gα-open set containing (A, B). Thus $(U, V) \supseteq$ b-αcl(A, B), as (A, B) is a $bg^{\star}\alpha$-closed set. Therefore (A, B) is binary αg-closed in $(X, Y, \mathcal{M})$.
4. Follows from (3) and from Proposition 2.10.

The converse of Proposition 2.13 need not be true as seen from the following example.

Example 2.14. Let $X = \{a, b, c\}$, $Y = \{1,2\}$ and $\mathcal{M} = \{(\phi, \phi), (\{a\}, \{1\}), (\{b\}, \phi), (\{b\}, \{2\}), (\{a, b\}, \{1\}), (\{a, b\}, Y), (X, Y)\}$. Then the subset $(\{a\}, \{2\})$ is binary gα-closed, wbgα-closed, binary αg-closed and wbαg-closed but not $bg^{\star}\alpha$-closed in $(X, Y, \mathcal{M})$.

Remark 2.15. The following examples show that the concept of binary semi-closed and wbgα-closed sets are independent.

Example 2.16. In Example 2.14, then the subset $(\{c\}, \{1\})$ is wbgα-closed but not binary semi-closed in $(X, Y, \mathcal{M})$ and then the subset $(\{a\}, \{1\})$ is binary semi-closed but not wbgα-closed in $(X, Y, \mathcal{M})$.

Proposition 2.17. Every wbgα-closed set in $(X, Y, \mathcal{M})$ is wbαg-closed.

Proof: Let (A, B) be a wbgα-closed set in $(X, Y, \mathcal{M})$ and let (U, V) be an binary open set containing (A, B). Since every binary open set is binary α-open, (U, V) is an binary α-open set containing (A, B). Since (A, B) is wbgα-closed, $(U, V) \supseteq$ b-αcl(b-int(A, B)). Hence (A, B) is wbαg-closed in $(X, Y, \mathcal{M})$. The converse of Proposition 2.17 need not be true as seen from the following example.

Example 2.18. Let $X = \{a, b\}$, $Y = \{1,2\}$ and $\mathcal{M} = \{(\phi, \phi), (\phi, \{1\}), (\{a\}, \{1\}), (\{b\}, \{1\}), (X, \{1\}), (X, Y)\}$. Then the subset $(\{b\}, Y)$ is wbαg-closed but not wbgα-closed in $(X, Y, \mathcal{M})$.

Corollary 2.19. Every binary closed set is wbαg-closed.

Proof: By Proposition 2.2, every binary closed set is $bg^{\star}\alpha$-closed and by Proposition 2.13, every $bg^{\star}\alpha$-closed set is wbαg-closed. Hence every binary closed set is wbαg-closed.

Remark 2.20. The following example show that the concept of binary g-closed and wbgα-closed sets are independent.

Example 2.21. In Example 2.18 Then the subset $(\{a\}, \phi)$ is wbgα-closed but not binary g-closed in $(X, Y, \mathcal{M})$ and the subset $(\{b\}, Y)$ is binary g-closed but not wbgα-closed in $(X, Y, \mathcal{M})$.

Proposition 2.22. Every bg*α-closed set is wbg*α-closed.

Proof: Assume that a subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is bg*α-closed. Let (U, V) be a binary gα-open set containing (A, B). Then $(U, V) \supseteq$ b-αcl(A, B), as (A, B) is bg*α-closed. Thus $(U, V) \supseteq$ b-αcl$(A, B) \supseteq$ b-αcl(b-int(A, B)). So, $(U, V) \subseteq$ b-αcl(b-int(A, B)). Hence (A, B) is wbg*α-closed in $(X, Y, \mathcal{M})$. The converse of Proposition 2.22 need not be true as seen from the following example.

Example 2.23. In Example 2.14, then the subset $(\{a\}, \{2\})$ is wbg*α-closed but not bg*α-closed in $(X, Y, \mathcal{M})$.

Proposition 2.24. If a subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is both binary open and wbg*α-closed, then it is bg*α-closed.

Proof: Assume that a subset (A, B) in $(X, Y, \mathcal{M})$ is both binary open and wbg*α-closed. Let (U, V) be a binary gα-open set containing (A, B). Since (A, B) is wbg*α-closed, $(U, V) \supseteq$ b-αcl(b-int(A, B)). Thus $(U, V) \supseteq$ b-αcl(b-int(A, B)) $\supseteq$ b-αcl(A, B), as (A, B) is binary open. Hence (A, B) is bg*α-closed.

Remark 2.25. The following examples show that the concept of binary sg-closed and wbgα-closed sets are independent.

Example 2.26. In Example 2.14, then the subset $(\{b\}, \{2\})$ is binary sg-closed but not wbgα-closed in $(X, Y, \mathcal{M})$ and then the subset $(\{b, c\}, \{2\})$ is wbgα-closed but not binary sg-closed in $(X, Y, \mathcal{M})$.

Proposition 2.27. If a subset A of a binary topological space $(X, Y, \mathcal{M})$ is binary nowhere dense, then it is wbgα-closed.

Proof: Let (A, B) be a binary nowhere dense set. Then b-int(b-cl(A, B)) $= (\phi, \phi)$. It is obvious that $(A, B) \subseteq$ b-αcl(A, B) and also b-int$(A, B) \subseteq$ b-int(b-αcl(A, B)) $\subseteq$ b-int(b-cl(A, B)). Hence b-int$(A, B) = (\phi, \phi)$ which implies b-αcl(b-int(A, B)) $= (\phi, \phi)$. Thus (A, B) is wbgα-closed in $(X, Y, \mathcal{M})$. The converse of Proposition 2.27 need not be true as seen from the following example.

Example 2.28. In Example 2.14, then the subset $(\{a, c\}, \{1\})$ is wbgα-closed but not binary nowhere dense.

Proposition 2.29. If a subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is binary nowhere dense, then it is wbαg-closed.

Proof: The proof follows from Propositions 2.17 and 2.27.

Proposition 2.30. Every wbg*α-closed set in $(X, Y, \mathcal{M})$ is wbαg-closed.

Proof: Let (A, B) be a wbg*α-closed set in $(X, Y, \mathcal{M})$ and let (U, V) be an binary open set containing (A, B). Since every binary open set is binary gα-open, (U, V) is a binary gα-open set containing (A, B). Since (A, B) is wbg*α-closed, $(U, V) \supseteq$ b-αcl(b-int(A, B)). Hence (A, B) is wbαg-closed. The converse of Proposition 2.30 need not be true as seen from the following example.

Example 2.31. In Example 2.8, then the subset $(\{a\}, \{2\})$ is wbαg-closed but not wbg⋆α-closed in $(X, Y, \mathcal{M})$.

Remark 2.32. The following examples show that the concept of binary β-closed and wbαg-closed sets are independent.

Example 2.33. In Example 2.3, then the subset $(\phi, \{1\})$ is binary β-closed but not wbαg-closed in $(X, Y, \mathcal{M})$ and then the subset $(\{b\}, \{2\})$ is wbαg-closed but not binary β-closed in $(X, Y, \mathcal{M})$.

Remark 2.34. The following examples show that the concept of binary sg-closed and wbαg-closed sets are independent.

Example 2.35. In Example 2.14, then the subset $(\{a\}, Y)$ is binary sg-closed but not wbαg-closed in $(X, Y, \mathcal{M})$ and then the subset $(\{b, c\}, \{2\})$ is wbαg-closed but not binary sg-closed in $(X, Y, \mathcal{M})$.

Proposition 2.36. Every binary g-closed set in $(X, Y, \mathcal{M})$ is wbαg-closed.

Proof: Assume that a subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is binary g-closed. Let (U, V) be an binary open set containing (A, B). Then $(U, V) \supseteq$ b-cl(A, B), as (A, B) is binary g-closed. Thus $(U, V) \supseteq$ b-cl$(A, B) \supseteq$ b-αcl$(A, B) \supseteq$ b-αcl(b-int(A, B)). i.e., $(U, V) \supseteq$ b-αcl(b-int(A, B)). Hence (A, B) is wbαg-closed in $(X, Y, \mathcal{M})$. The converse of Proposition 2.36 need not be true as seen from the following example.

Example 2.37. In Example 2.8, then the subset $(\phi, \{1\})$ is wbαg-closed but not binary g-closed in $(X, Y, \mathcal{M})$.

Proposition 2.38. Every bg⋆α-closed set in $(X, Y, \mathcal{M})$ is binary gs-closed.

Proof: Let a subset (A, B) be bg⋆α-closed and let (U, V) be an binary open set containing (A, B). Then (U, V) is a binary gα-open set containing (A, B). Since (A, B) is bg⋆α-closed, $(U, V) \supseteq$ b-αcl(A, B). Thus $(U, V) \supseteq$ b-αcl$(A, B) \supseteq$ b-scl(A, B). i.e., $(U, V) \supseteq$ b-scl(A, B). Hence (A, B) is binary gs-closed in $(X, Y, \mathcal{M})$. The following example shows that the converse of Proposition 2.38 need not be true:

Example 2.39. In Example 2.14, then the subset $(\{a\}, \{2\})$ is binary gs-closed but not bg⋆α-closed in $(X, Y, \mathcal{M})$.

Remark 2.40. The following examples show that the concepts of binary semi-closed and bg⋆α-closed sets are independent.

Example 2.41. In Example 2.14, then the subset $(\{b, c\}, \phi)$ is bg⋆α-closed but not binary semi-closed in $(X, Y, \mathcal{M})$ and then the subset $(\{a\}, \{1\})$ is binary semi-closed but not bg⋆α-closed in $(X, Y, \mathcal{M})$.

Remark 2.42. The following examples show that the concepts of binary semi-closed and wbg⋆α-closed sets are independent.

Example 2.43. In Example 2.14, then the subset $(\{a\}, Y)$ is binary semi-closed but not wbg⋆α-closed in $(X, Y, \mathcal{M})$ and then the subset $(\{b, c\}, \{2\})$ is wbg⋆α-closed but not binary semi-closed in $(X, Y, \mathcal{M})$.

Proposition 2.44. If a subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is binary nowhere dense, then it is wbg*α-closed in $(X, Y, \mathcal{M})$.

Proof: Assume that (A, B) is binary nowhere dense in (X, Y), then $\text{b-int}(\text{b-cl}(A, B)) = (\phi, \phi)$. It is obvious that $(A, B) \subseteq \text{b-}\alpha\text{cl}(A, B)$ and also $\text{b-int}(A, B) \subseteq \text{b-int}(\text{b-}\alpha\text{cl}(A, B)) \subseteq \text{b-int}(\text{b-cl}(A, B))$. Thus $\text{b-int}(A, B) \subseteq \text{b-int}(\text{b-cl}(A, B))$. Since $\text{b-int}(\text{b-cl}(A, B)) = (\phi, \phi)$, we get $\text{b-int}(A, B) = (\phi, \phi)$. This implies that $\text{b-cl}(\text{b-int}(A, B)) = (\phi, \phi)$. Hence $(\phi, \phi) = \text{b-cl}(\text{b-int}(A, B)) \supseteq \text{b-}\alpha\text{cl}(\text{b-int}(A, B))$. So, $\text{b-}\alpha\text{cl}(\text{b-int}(A, B)) = (\phi, \phi)$. Therefore, (A, B) is wbg*α-closed in $(X, Y, \mathcal{M})$. The converse of Proposition 2.44 need not be true as seen from the following example.

Example 2.45. In Example 2.14, then the subset $(\{a, c\}, \{1\})$ is wbg*α-closed but not binary nowhere dense in $(X, Y, \mathcal{M})$.

Proposition 2.46. Every wbg*α-closed set is binary gsp-closed.

Proof: Let (A, B) be a wbg*α-closed set and let (U, V) be an binary open set containing (A, B). Then (U, V) is a binary gα-open set containing (A, B). Since (A, B) is wbg*α-closed, $(U, V) \supseteq \text{b-}\alpha\text{cl}(\text{b-int}(A, B))$. Now, $(U, V) \supseteq \text{b-}\alpha\text{cl}(\text{b-int}(A, B)) \supseteq \text{b-int}(\text{b-}\alpha\text{cl}(\text{b-int}(A, B)))$. This implies $(A, B) \cup (U, V) \supseteq (A, B) \cup \text{b-int}(\text{b-}\alpha\text{cl}(\text{b-int}(A, B)))$. i.e., $(U, V) \supseteq \text{b-spcl}(A, B)$. Hence (A, B) is binary gsp-closed in $(X, Y, \mathcal{M})$. The converse of Proposition 2.46 need not be true as seen from the following example.

Example 2.47. In Example 2.8, then the subset $(\{a\}, \{2\})$ is binary gsp-closed but not wbg*α-closed in $(X, Y, \mathcal{M})$.

Theorem 2.48. A set (A, B) is bg*α-closed if and only if $\text{b-}\alpha\text{cl}(A, B) - (A, B)$ contains no non-empty binary gα-closed set.

Proof:

<u>Necessity:</u> Assume that (A, B) is bg*α-closed. Let (U, V) be a binary gα-closed set such that $(U, V) \subseteq \text{b-}\alpha\text{cl}(A, B) - (A, B)$. Then $(U, V)^c$ is a binary gα-open set containing (A, B). From the definition of bg*α-closed, $\text{b-}\alpha\text{cl}(A, B) \subseteq (U, V)^c$. i.e., $(U, V) \subseteq (\text{b-}\alpha\text{cl}(A, B))^c$. This implies that $(U, V) \subseteq \text{b-}\alpha\text{cl}(A, B) \cap (\text{b-}\alpha\text{cl}(A, B))^c = (\phi, \phi)$. i.e., $\text{b-}\alpha\text{cl}(A, B) - (A, B)$ contains no non-empty binary gα-closed set.

<u>Sufficiency:</u> Let us assume that $\text{b-}\alpha\text{cl}(A, B) - (A, B)$ contains no non-empty binary gα-closed set. Let $(A, B) \subseteq (U, V)$ where (U, V) is a binary gα-open subset of $(X, Y, \mathcal{M})$. If $\text{b-}\alpha\text{cl}(A, B)$ is not contained in (U, V), then $\text{b-}\alpha\text{cl}(A, B) \cap (U, V)^c$ is a non-empty binary gα-closed set of $\text{b-}\alpha\text{cl}(A, B) - (A, B)$. Thus we obtain a contradiction. Therefore $\text{b-}\alpha\text{cl}(A, B) \subseteq (U, V)$ and hence (A, B) is bg*α-closed.

Corollary 2.49. Let (A, B) be a bg*α-closed set. Then (A, B) is binary α-closed if and only if $\text{b-}\alpha\text{cl}(A, B) - (A, B)$ is binary gα-closed.

Proof:

<u>Necessity:</u> Assume that a bg*α-closed set (A, B) is binary α-closed. i.e., $\text{b-}\alpha\text{cl}(A, B) = (A, B)$. Then $\text{b-}\alpha\text{cl}(A, B) — (A, B) = (\phi, \phi)$ is binary α-closed and hence binary gα-closed.

<u>Sufficiency:</u> Let $\text{b-}\alpha\text{cl}(A, B) - (A, B)$ be binary gα-closed. By Theorem 2.48, $\text{b-}\alpha\text{cl}(A, B) - (A, B)$ contains no non-empty binary gα-closed set. i.e., $\text{b-}\alpha\text{cl}(A, B) - (A, B) = (\phi, \phi)$. Therefore $\text{b-}\alpha\text{cl}(A, B) = (A, B)$. Hence (A, B) is binary α-closed.

Proposition 2.50. Let $(X, Y, \mathcal{M})$ be a binary topological space and let $(C, D) \subseteq (A, B) \subseteq (X, Y)$. If (C, D) is a bg⋆α-closed set relative to (A, B) and (A, B) is a bg⋆α-closed subset of $(X, Y, \mathcal{M})$. Then (C, D) is a bg⋆α-closed set relative to $(X, Y, \mathcal{M})$.

Proof: Let $(C, D) \subseteq (U, V)$ and (U, V) be a binary gα-open set in $(X, Y, \mathcal{M})$. Then $(C, D) \subseteq (A, B) \cap (U, V)$. Since (C, D) is bg⋆α-closed relative to (A, B), b-αcl$(C, D) \subseteq (A, B) \cap (U, V)$. i.e., $(A, B) \cap$ b-αcl$(C, D) \subseteq (A, B) \cap (U, V)$. We have $(A, B) \cap$ αb-cl$(C, D) \subseteq (U, V)$ and hence $(A, B) \cap$ b-αcl$(C, D) \cup$ (b-αcl$(C, D))^c \subseteq (U, V) \cup$ (b-αcl$(C, D))^c$. Since (A, B) is bg⋆α-closed in $(X, Y, \mathcal{M})$, we have b-αcl$(A, B) \subseteq (U, V) \cup$ (b-αcl$(C, D))^c$. Also $(C, D) \subseteq (A, B)$ implies b-αcl$(C, D) \subseteq$ b-αcl(A, B), thus b-αcl$(C, D) \subseteq$ b-αcl$(A, B) \subseteq (U, V) \cup$ (b-αcl$(C, D))^c$. Therefore b-αcl$(C, D) \subseteq (U, V)$, since b-αcl(C, D) is not contained in (b-αcl$(C, D))^c$. Thus (C, D) is a bg⋆α-closed set relative to $(X, Y, \mathcal{M})$.

Proposition 2.51. If (A, B) is bg⋆α-closed and (E, F) is closed in a binary topological space $(X, Y, \mathcal{M})$, then $(A, B) \cap (E, F)$ is bg⋆α-closed in $(X, Y, \mathcal{M})$.

Proof: Clearly, $(A, B) \cap (E, F)$ is binary closed in (A, B). Therefore b-cl$((A, B) \cap (E, F)) = (A, B) \cap (E, F)$ in (A, B). Let $(A, B) \cap (E, F) \subseteq (U, V)$, where (U, V) is binary gα-open in (A, B). Then b-αcl$((A, B) \cap (E, F)) \subseteq$ b-cl$((A, B) \cap (E, F)) = (A, B) \cap (E, F) \subseteq (U, V)$. Thus b-αcl$((A, B) \cap (E, F)) \subseteq (U, V)$ and hence $(A, B) \cap (E, F)$ is bg⋆α-closed in (A, B). By Proposition 2.50, $(A, B) \cap (E, F)$ is bg⋆α-closed in $(X, Y, \mathcal{M})$.

Proposition 2.52. If (A, B) is a bg⋆α-closed set in $(X, Y, \mathcal{M})$ and $(A, B) \subseteq (C, D) \subseteq$ B-αcl(A, B), then (C, D) is a bg⋆α-closed set in $(X, Y, \mathcal{M})$.

Proof: Let (G, H) be a binary gα-open set in $(X, Y, \mathcal{M})$ such that $(C, D) \subseteq (G, H)$ and hence $(A, B) \subseteq (G, H)$. Since (A, B) is bg⋆α-closed, b-αcl$(A, B) \subseteq (G, H)$. Since $(C, D) \subseteq$ b-αcl(A, B) we have, b-αcl$(C, D) \subseteq$ b-αcl$(A, B)) =$ b-αcl$(A, B) \subseteq (G, H)$. Hence b-αcl$(C, D) \subseteq (G, H)$ which implies that (C, D) is bg⋆α-closed in $(X, Y, \mathcal{M})$.

Proposition 2.53. Let $(X, Y, \mathcal{M})$ be a binary topological space and $(A, B) \subseteq (E, F) \subseteq (X, Y)$, If (A, B) is a bg⋆α-closed set in (X, Y), then (A, B) is bg⋆α-closed relative to (E, F).

Proof: Let (A, B) be a bg⋆α-closed set in (X, Y) and let $(A, B) \subseteq (E, F) \cap (U, V)$, where (U, V) is binary gα-open in $(X, Y, \mathcal{M})$. Since (A, B) is bg⋆α-closed in $(X, Y, \mathcal{M})$ and $(A, B) \subseteq (U, V)$ implies b-αcl$(A, B) \subseteq (U, V)$. That is $(E, F) \cap$ b-αcl$(A, B) \subseteq (E, F) \cap (U, V)$ where $(E, F) \cap$ b-αcl(A, B) is binary α-closure of (A, B) in (E, F). Thus, (A, B) is ng⋆α-closed relative to (E, F).

Result 2.54. For a binary topological space $(X, Y, \mathcal{M})$ the following results hold:

1. Let $(C, D) \subseteq (A, B) \subseteq (X, Y)$. If (C, D) is a wbg⋆α-closed set relative to (A, B) and (A, B) is a wbg⋆α-closed subset of (X, Y). Then (C, D) is a wbg⋆α-closed set relative to (X, Y).
2. If (A, B) is wbg⋆α-closed and (G, H) is binary closed in $(X, Y, \mathcal{M})$, then $(A, B) \cap (G, H)$ is wbg⋆α-closed in $(X, Y, \mathcal{M})$.
3. If (A, B) is wbg⋆α-closed and $(A, B) \subseteq (C, D) \subseteq$ b-αcl(b-int(A, B)), then (C, D) is wbg⋆α-closed.
4. Let $(A, B) \subseteq (E, F) \subseteq (X, Y)$. If (A, B) is wbg⋆α-closed in (X, Y), then (A, B) is wbg⋆α-closed relative to (E, F).

Proposition 2.55. If (A, B) is binary gα-open and bg⋆α-closed in $(X, Y, \mathcal{M})$, then (A, B) is binary α-closed in $(X, Y, \mathcal{M})$.

Proof: Let (A, B) be binary gα-open and bg⋆α-closed in $(X, Y, \mathcal{M})$. Then b-αcl$(A, B) \subseteq (A, B)$. Also we know that $(A, B) \subseteq$ b-αcl(A, B) for every subset (A, B) of (X, Y). Hence b-αcl$(A, B) = (A, B)$. This implies that (A, B) is binary α-closed in $(X, Y, \mathcal{M})$.

Proposition 2.56. For each $\{i, j\} \in (X, Y)$, either $\{i, j\} \notin BG\alpha C(X, Y)$ or its complement $(X, Y) - \{i, j\} \in BG^{\star}\alpha C(X, Y)$.

Proof: Assume that $\{i, j\} \notin BG\alpha C(X, Y)$, for each $(i, j) \in (X, Y)$. Therefore $(X, Y) - \{i, j\} \notin BG\alpha O(X, Y)$. Then (X, Y) is the only binary gα-open set containing $(X, Y) - \{i, j\}$. Hence b-αcl$((X, Y) - \{i, j\}) \subseteq (X, Y)$ which implies that $(X, Y) - \{i, j\}$ is bg⋆α-closed in $(X, Y, \mathcal{M})$. Hence $(X, Y) - \{i, j\} \in BG^{\star}\alpha C(X, Y)$.

CONCLUSION

In this work weakly binary *gα*-closed sets and weakly binary *αg*-closed sets in binary topological space were introduced and the propertires of the sets were investigated. The continuation of this work will deal with some special functions on these topological spaces.

REFERENCES

[1] Levine, N., *Rendiconti del Circolo Matematico di Palermo*, **19**(2), 89, 1970.
[2] Njastad, O., *Pacific Journal of Mathematics*, **15**(3), 961, 1965.
[3] Mashhour, A.S., Abd El-Monsef, M.E., EL-Deeb, S.N., *Acta Mathematica Hungarica*, **41**, 213, 1983.
[4] Maki, H., Devi, R., Balachandran, K., *Bulletin of Fukuoka University of Education*, **42**, 13, 1993.
[5] Maki, H., Devi, R., Balachandran, K., *Memoirs of the Faculty of Science. Series A. Mathematics*, **15**, 51, 1994.
[6] Nithyanantha Jothi, S., Thangavelu, P., *Journal of Mathematical Sciences & Computer Applications*, **1**(3), 95, 2011.
[7] Nithyanantha Jothi, S., *International journal of Mathematical Archieve*, **7**(9), 73, 2016.
[8] Jayalakshmi, S., Manonmani, A., *International Journal of Analytical and Experimental Modal Analysis*, **12**(4), 494, 2020.
[9] Nithyanantha Jothi, S., Thangavelu, P., *IRA-International Journal of Applied Sciences*, **4**(2), 259, 2016.
[10] Nithyanantha Jothi, S., Thangavelu, P., *Ultra Scientist*, **26**(1), 25, 2014.
[11] Gnana Arockiam, A., Gilber Rani, M., Premkumar, R., *Indian Journal of Natural Science*, **13**(76), 52299, 2023.
[12] Santhini, C., Dhivya, T., *International Journal of Mathematical Archive*, **9**(10), 1, 2018.
[13] Jayalakshmi, S., Manonmani, A., *International Journal of Mathematics Trends and Technology*, **66**(7), 18, 2020.
[14] Granados, C., *South Asian Journal of Mathematics*, **11**(1), 1, 2021.
[15] Abinaya, D., Gilbert Rani, M. and Premkumar, R., *Indian Journal of Natural Science*, **14**(77), 54089, 2023.
[16] Gilber Rani, M., Premkumar, R., *Journal of Education: Rabindra Bharati University*, **XXIV**(1)(XII), 164, 2022.
[17] Nithyanantha Jothi, S., Thangavelu, P., *Acta Ciencia Indica*, **XLIM**(3), 241, 2015.