ORIGINAL PAPER

INVOLUTE-EVOLUTE CURVES ACCORDING TO MODIFIED ORTHOGONAL FRAME

AYŞE ZEYNEP AZAK[1]

Manuscript received: 02.11.2020; Accepted paper: 23.04.2021;
Published online: 30.06.2021.

Abstract. *In this paper, the involute-evolute curve concept has been defined according to two type modified orthogonal frames at non-zero points of curvature and torsion in the Euclidean space E^3, respectively. Later, the characteristic theorems related to the distance between the corresponding points of these curves have been given. Besides, the relations have been found between the curvatures and also torsions of the two type the involute-evolute modified orthogonal pairs.*

Keywords: *modified orthogonal frames; involute-evolute curves; Euclidean 3-space.*

1. INTRODUCTION

The specific curve pairs are important and interesting topics of the curve theory. Especially, obtaining the relations between the curvatures of these curve pairs is important for their classification. One of these special curve pairs is the involute-evolute curve pair which was discovered by Huygens [1]. The involute curve of any curve is defined as its tangent line remains perpendicular to the tangent line of the original curve. Thus, the original curve is called an involute curve. In the three dimensional Euclidean space, many studies have been conducted with the involute-evolute curve pair [2-4]. Recently, these studies focused on the examination of their characteristic properties according to different frames [5-10]. Several researchers studied the problem of analytic space curves which their curvatures have discrete zero points. Some indefiniteness that arises on the choice of the normal vectors. Namely, they encountered some difficulties in deciding on the choice of the sign of curvature and torsion. Sasai created a length-varying frame to overcome this problem such that the normal directions of this new frame coincide with those of the classical frame at non-zero points of the first curvature κ [11]. After this study, the theorems have been proved about the radii and the centers of the osculating spheres of any spherical curve with modified orthogonal frame by Bükcü and Karacan [12]. Then, they have obtained the counterpart of the modified orthogonal frame with a curvature which is defined by Sasai for three dimensional Minkowski space. Also, they have introduced a new version of modified orthogonal frame with torsion. Moreover, they have given the equivalent of this frame in three dimensional Minkowski space [13]. Baş and Körpınar have introduced the rollar coaster surface for the modified orthogonal frame in the Euclidean 3-space. Additionally, some characteristic features of this surfaces have been obtained and visual applications related to modified orthogonal rollar coaster surface and its parallel surface have been given [14]. Lone et al. have defined Mannheim pairs, Frenet-Mannheim curves and weakened Mannheim curves according to modified orthogonal frame in the Euclidean 3-space and obtained some characterizations of these

[1] Sakarya University, Faculty of Education, Mathematics and Science Education Department, 54300 Sakarya, Turkey. E-mail: apirdal@sakarya.edu.tr.

curves [15]. Later, Lone et al. have studied helices and the Bertrand curves according to modified orthogonal frame in the Euclidean 3-space. They have investigated some theorems and results concerning this frame and given examples [16].

The paper is arranged as follows. In section 2, some fundamental concepts regarding modified orthogonal frames with non-zero curvature and torsion are given. Involute-evolute modified orthogonal pair is defined for modified orthogonal frames at non-zero points of curvature. The relationship at the corresponding points is found. Later, the relation is obtained between the curvatures and also torsions of the involute-evolute modified orthogonal pair in Section 3. In section 4, the involute-evolute modified orthogonal pair is defined for modified orthogonal frames at non-zero points of torsion. Finally, the theorems concerning involute-evolute modified orthogonal pair which is mentioned before in section 3 are presented for the modified orthogonal frames at non-zero points of torsion.

2. MATHERIALS AND METHODS

In this section, we will give a brief summary regarding modified orthogonal frames with non-zero curvature and torsion and their properties, respectively.

Let $\alpha(s)$ be a C^3 space curve in the Euclidean 3-space with parametrized by arc-length s. In this case, it is assumed that the curvature $\kappa(s)$ of the given curve does not vanish anywhere. Thus, an orthonormal frame $\{\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}\}$ can be constituted such that satisfies the Frenet-Serret equation as follows

$$\begin{bmatrix}\boldsymbol{t}'\\ \boldsymbol{n}'\\ \boldsymbol{b}'\end{bmatrix}=\begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{bmatrix} \tag{1}$$

where $\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}$ are tangent, principal normal, binormal vectors and κ,τ denotes the curvature and torsion of the curve $\alpha(s)$, respectively.

Now let $\alpha(s)$ be an analytical curve. So, it can be reparametrized by its arc-length s. For this curve, we will assume that the curvature function $\kappa(s)$ is not identically zero or has discrete zero points. Thus, an orthogonal frame $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$ can be defined as follows:

$$\boldsymbol{T}=\frac{d\alpha}{ds},\qquad \boldsymbol{N}=\frac{d\boldsymbol{T}}{ds},\qquad \boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}. \tag{2}$$

Considering the above equations and the Serret-Frenet equations, the relations between the Serret-Frenet frame $\{\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}\}$ and this new frame at the non-zero points of κ is obtained as

$$\boldsymbol{T}=\boldsymbol{t},\qquad \boldsymbol{N}=\kappa\boldsymbol{n},\qquad \boldsymbol{B}=\kappa\boldsymbol{b}. \tag{3}$$

In this case, $\boldsymbol{N}(s_0)=\boldsymbol{B}(s_0)=0$ when $\kappa(s_0)=0$ and the squares of the length of $\boldsymbol{N}$ and $\boldsymbol{B}$ vary analytically in s. Differentiating the Eq. (3) with respect to s and by using the Eq. (1), the derivative equations are found as

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}' \\ \boldsymbol{B}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \dfrac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \dfrac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix} \tag{4}$$

The torsion of the curve α is given by

$$\tau(s)=\frac{\det(\alpha',\alpha'',\alpha''')}{\kappa^2} \tag{5}$$

where a dash denotes the differentiation with respect to arc-length s. If we consider the Frenet-Serret equation, it is obvious that any point where $\kappa^2=0$ is a removable singularity of τ.

Let $\langle , \rangle$ be the standard inner product of the Euclidean space E^3, then the following equalities for $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$ hold:

$$\begin{aligned} &\langle \boldsymbol{T},\boldsymbol{T}\rangle = 1, \qquad \langle \boldsymbol{N},\boldsymbol{N}\rangle = \langle \boldsymbol{B},\boldsymbol{B}\rangle = \kappa^2 \\ &\langle \boldsymbol{T},\boldsymbol{N}\rangle = \langle \boldsymbol{T},\boldsymbol{B}\rangle = \langle \boldsymbol{N},\boldsymbol{B}\rangle = 0. \end{aligned} \tag{6}$$

Thus, the orthogonal frame $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$ is called the modified orthogonal frame at non-zero points of the curvature κ [11]. It is easily seen that for $\kappa=1$, Serret-Frenet frame coincides with the modified orthogonal frame.

Let $\tilde{\alpha}(s)$ be a C^3 space curve in the Euclidean 3-space with parametrized by arc-length s. Then, the orthonormal Serret-Frenet frame $\{\tilde{\boldsymbol{t}},\tilde{\boldsymbol{n}},\tilde{\boldsymbol{b}}\}$ of the curve $\tilde{\alpha}(s)$ is given by

$$\begin{bmatrix} \tilde{\boldsymbol{t}}' \\ \tilde{\boldsymbol{n}}' \\ \tilde{\boldsymbol{b}}' \end{bmatrix} = \begin{bmatrix} 0 & \tilde{\kappa} & 0 \\ -\tilde{\kappa} & 0 & \tilde{\tau} \\ 0 & -\tilde{\tau} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{t}} \\ \tilde{\boldsymbol{n}} \\ \tilde{\boldsymbol{b}} \end{bmatrix} \tag{7}$$

where $\tilde{\boldsymbol{t}},\tilde{\boldsymbol{n}},\tilde{\boldsymbol{b}}$ are tangent, principal normal, binormal vectors and $\tilde{\kappa},\tilde{\tau}$ denote the curvature and torsion of the curve $\tilde{\alpha}(s)$, respectively.

Let $\tilde{\alpha}(s)$ be an analytical curve. Then, this curve can be reparametrized by its arc-length s. For this curve, we will assume that the curvature function $\tilde{\tau}(s)$ is not identically zero. Thus, an orthogonal frame $\{\tilde{\boldsymbol{T}},\tilde{\boldsymbol{N}},\tilde{\boldsymbol{B}}\}$ can be defined as follows:

$$\begin{bmatrix} \tilde{\boldsymbol{T}}' \\ \tilde{\boldsymbol{N}}' \\ \tilde{\boldsymbol{B}}' \end{bmatrix} = \begin{bmatrix} 0 & \frac{\tilde{\kappa}}{\tilde{\tau}} & 0 \\ -\tilde{\kappa}\tilde{\tau} & \frac{\tilde{\tau}'}{\tilde{\tau}} & \tilde{\tau} \\ 0 & -\tilde{\tau} & \frac{\tilde{\tau}'}{\tilde{\tau}} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{T}} \\ \tilde{\boldsymbol{N}} \\ \tilde{\boldsymbol{B}} \end{bmatrix} \tag{8}$$

and

$$\tilde{\boldsymbol{T}} = \tilde{\boldsymbol{t}}, \qquad \tilde{\boldsymbol{N}} = \tilde{\tau}\tilde{\boldsymbol{n}}, \qquad \tilde{\boldsymbol{B}} = \tilde{\tau}\tilde{\boldsymbol{b}}. \tag{9}$$

Let $\langle \, , \rangle$ be the standard inner product of the Euclidean space E^3, then the following equalities are given

$$\begin{aligned} &\langle \tilde{\boldsymbol{T}}, \tilde{\boldsymbol{T}} \rangle = 1, \qquad \langle \tilde{\boldsymbol{N}}, \tilde{\boldsymbol{N}} \rangle = \langle \tilde{\boldsymbol{B}}, \tilde{\boldsymbol{B}} \rangle = \tilde{\tau}^2 \\ &\langle \tilde{\boldsymbol{T}}, \tilde{\boldsymbol{N}} \rangle = \langle \tilde{\boldsymbol{T}}, \tilde{\boldsymbol{B}} \rangle = \langle \tilde{\boldsymbol{N}}, \tilde{\boldsymbol{B}} \rangle = 0. \end{aligned} \tag{10}$$

Moreover, the torsion of the curve $\tilde{\alpha}$ is given by

$$\tilde{\tau}(s) = \frac{\det(\tilde{\alpha}', \tilde{\alpha}'', \tilde{\alpha}''')}{\tilde{\kappa}^2} \tag{11}$$

where a dash denotes the differentiation with respect to arc-length s.
Thus, the orthogonal frame $\{\tilde{\boldsymbol{T}}, \tilde{\boldsymbol{N}}, \tilde{\boldsymbol{B}}\}$ is called the modified orthogonal frame with non-zero torsion [13].

3. INVOLUTE-EVOLUTE CURVE PAIR ACCORDING TO MODIFIED ORTHOGONAL FRAME WITH CURVATURE

In this section, the involute-evolute curve pair will be defined according to the modified orthogonal frame at non-zero points of the curvature. After, some characteristic theorems are proven.

Definition 1. Let α and β be the arc-length parameter curves in E^3. Denote the modified orthogonal frames of the curves $\alpha(s)$ and $\beta(s)$ by $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ and $\{\boldsymbol{T}^*, \boldsymbol{N}^*, \boldsymbol{B}^*\}$, respectively. If the tangent vector $\boldsymbol{T}$ of the curve α becomes orthogonal to the tangent vector $\boldsymbol{T}^*$ of the curve β at the corresponding points of the curves, then α is called the evolute curve of the curve β with modified orthogonal frame and β is the involute curve of the curve α with modified orthogonal frame. Then, the pair $\{\alpha, \beta\}$ is called involute-evolute modified orthogonal pair.

Theorem 1. Let α and β be two curves with arc-length parameters s and s^* according to modified orthogonal frame in the Euclidean 3-space E^3, respectively. If the pair $\{\alpha, \beta\}$ is involute-evolute modified orthogonal pair, then the relationship at the corresponding points is given by

$$\beta(s)=\alpha(s)+(c-s)\boldsymbol{T}(s).$$

Proof: Let $\alpha(s)$ and $\beta(s^*)$ be two curves with modified orthogonal frame. By the Definition 1, we have

$$\beta(s)=\alpha(s)+\lambda(s)\boldsymbol{T}(s). \tag{12}$$

If we differentiate both sides of the Eq. (12) with respect to s and considering the Eq. (4), the following equality is obtained

$$\boldsymbol{T}^*(s^*)\frac{ds^*}{ds}=\boldsymbol{T}(s)+\lambda'(s)\boldsymbol{T}(s)+\lambda(s)\boldsymbol{N}(s).$$

Since $\langle \boldsymbol{T},\boldsymbol{T}^*\rangle=0$, we get

$$\lambda'(s)=-1.$$

Thus, the above equality yields to

$$\lambda(s)=c-s \tag{13}$$

where c is constant.

Finally, the Eq. (12) can be written as follows:

$$\beta(s)=\alpha(s)+(c-s)\boldsymbol{T}(s). \tag{14}$$

Theorem 2. Let α and β be two arc-length parameter curves with modified orthogonal frames $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\},\{\boldsymbol{T}^*,\boldsymbol{N}^*,\boldsymbol{B}^*\}$, the curvatures κ,κ^*, torsions τ,τ^*, respectively. If the pair $\{\alpha,\beta\}$ be involute-evolute modified orthogonal pair. Then, there exists a relation between the curvature κ^* of the curve β and the torsion τ, the curvature κ of the curve α as follows:

$$\kappa^*=\frac{\sqrt{\kappa^2+\tau^2}}{(c-s)\kappa}.$$

Proof: Let $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$ and $\{\boldsymbol{T}^*,\boldsymbol{N}^*,\boldsymbol{B}^*\}$ be modified orthogonal frames at the points $\alpha(s)$ and $\beta(s)$, respectively. Then, taking the derivative of Eq. (14) and using Eq. (4), the following equality is obtained

$$\boldsymbol{T}^*\frac{ds^*}{ds}=(c-s)\boldsymbol{N}. \tag{15}$$

By taking the norm of the Eq. (15), we get

$$\frac{ds^*}{ds} = (c-s)\kappa. \tag{16}$$

Substituting Eq. (16) into the Eq. (15), we have

$$\boldsymbol{T}^* = \frac{1}{\kappa}\boldsymbol{N}. \tag{17}$$

By taking the derivative of Eq. (17) with respect to s and using Eq. (16) and Eq. (4), the following equation is written

$$\boldsymbol{N}^* = \frac{-\kappa\boldsymbol{T} + (\tau/\kappa)\boldsymbol{B}}{(c-s)\kappa}. \tag{18}$$

By taking the norm of the Eq. (18), the desired relation is obtained as follows

$$\kappa^* = \frac{\sqrt{\kappa^2+\tau^2}}{(c-s)\kappa}.$$

Theorem 3. Let the pair $\{\alpha,\beta\}$ be involute-evolute modified orthogonal pair. Then, the curvature τ^* of the curve β satisfies the condition

$$\tau^* = \frac{(c-s)^5\kappa^5\left(\frac{\tau}{\kappa}\right)'}{\left(\left(\frac{\tau}{\kappa}\right)^2+1\right)}$$

Proof: Taking the first, second and third-order derivatives of the relation in Theorem 1 with respect to s, then substituting Eq. (4) into the results give

$$\begin{aligned}
\beta' &= (c-s)\boldsymbol{N}\\
\beta'' &= -(c-s)\kappa^2\boldsymbol{T} + \left((c-s)\frac{\kappa'}{\kappa}-1\right)\boldsymbol{N} + (c-s)\tau\boldsymbol{B}\\
\beta''' &= \left(2\kappa^2 - 3(c-s)\kappa\kappa'\right)\boldsymbol{T} + \left(-(c-s)\kappa^2-(c-s)\tau^2-2\frac{\kappa'}{\kappa}+(c-s)\frac{\kappa''}{\kappa}\right)\boldsymbol{N}\\
&\quad+\left(2(c-s)\frac{\kappa'}{\kappa}\tau - 2\tau + (c-s)\tau'\right)\boldsymbol{B}.
\end{aligned}$$

The determinant of β',β'',β'' is calculated by

$$\det(\beta',\beta'',\beta''') = (c-s)^3\kappa^5\left(\frac{\tau}{\kappa}\right)'. \tag{19}$$

If we consider Eq. (5), the torsion of the curve β is given by

$$\tau^{*}=\frac{\det\left(\beta',\beta'',\beta'''\right)}{\kappa^{*}} \tag{20}$$

Putting Eq. (19) and the curvature of the curve β in Theorem 2 into Eq. (20), we have

$$\tau^{*}=\frac{\left(c-s\right)^{5}\kappa^{5}\left(\dfrac{\tau}{\kappa}\right)'}{\left(\left(\dfrac{\tau}{\kappa}\right)^{2}+1\right)}.$$

4. INVOLUTE-EVOLUTE CURVE PAIR ACCORDING TO MODIFIED ORTHOGONAL FRAME WITH TORSION

In this section, firstly the involute-evolute curve pair will be defined according to the modified orthogonal frame at non-zero points of the torsion. After, some characteristic theorems regarding this curve pair are concerned.

Definition 2. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be the arc-length parameter curves in E^3. Denote the modified orthogonal frames of the curves $\tilde{\alpha}(s)$ and $\tilde{\beta}(s)$ by $\{\tilde{\boldsymbol{T}},\tilde{\boldsymbol{N}},\tilde{\boldsymbol{B}}\}$ and $\{\tilde{\boldsymbol{T}}^{*},\tilde{\boldsymbol{N}}^{*},\tilde{\boldsymbol{B}}^{*}\}$, respectively. If the tangent vector $\tilde{\boldsymbol{T}}$ of the curve $\tilde{\alpha}$ becomes orthogonal to the tangent vector $\tilde{\boldsymbol{T}}^{*}$ of the curve $\tilde{\beta}$ at the corresponding points of the curves, then $\tilde{\alpha}$ is called the evolute curve of the curve $\tilde{\beta}$ and $\tilde{\beta}$ is called the involute curve of the curve $\tilde{\alpha}$ with modified orthogonal frame at non-zero points of the torsion. Then, the pair $\{\tilde{\alpha},\tilde{\beta}\}$ is called involute-evolute modified orthogonal pair.

Theorem 4. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two curves with arc-length parameters s and s^{*} according to modified orthogonal frame in the Euclidean 3-space E^3, respectively. If the pair $\{\tilde{\alpha},\tilde{\beta}\}$ is involute-evolute modified orthogonal pair, then the relationship at the corresponding points is given by

$$\tilde{\beta}(s)=\tilde{\alpha}(s)+(\tilde{c}-s)\tilde{\boldsymbol{T}}(s).$$

Proof: If the pair $\{\tilde{\alpha},\tilde{\beta}\}$ is involute-evolute modified orthogonal pair, then from Definition 2 we have

$$\tilde{\beta}(s)=\tilde{\alpha}(s)+\tilde{\lambda}(s)\tilde{\boldsymbol{T}}(s). \tag{21}$$

By taking the derivative of both sides of the Eq. (21) with respect to s and substituting the Eq. (8) into the result, we get

$$\tilde{\boldsymbol{T}}^*\left(s^*\right)\frac{ds^*}{ds}=\tilde{\boldsymbol{T}}\left(s\right)+\tilde{\lambda}'\left(s\right)\tilde{\boldsymbol{T}}\left(s\right)+\tilde{\lambda}\left(s\right)\frac{\tilde{\kappa}}{\tilde{\tau}}\tilde{\boldsymbol{N}}\left(s\right). \tag{22}$$

Since $\left\langle \tilde{\boldsymbol{T}},\tilde{\boldsymbol{T}}^*\right\rangle=0$, Eq. (22) becomes

$$\tilde{\lambda}'\left(s\right)=-1.$$

Then, we have

$$\tilde{\lambda}\left(s\right)=\tilde{c}-s \tag{23}$$

where $\tilde{c}$ is constant.

From Eq. (23), the Eq. (21) can be expressed as

$$\beta\left(s\right)=\alpha\left(s\right)+\left(\tilde{c}-s\right)\boldsymbol{T}\left(s\right). \tag{24}$$

Theorem 5. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two arc-length parameter curves with modified orthogonal frames $\left\{\tilde{\boldsymbol{T}},\tilde{\boldsymbol{N}},\tilde{\boldsymbol{B}}\right\},\left\{\tilde{\boldsymbol{T}}^*,\tilde{\boldsymbol{N}}^*,\tilde{\boldsymbol{B}}^*\right\}$, the curvatures $\tilde{\kappa},\tilde{\kappa}^*$, torsions $\tilde{\tau},\tilde{\tau}^*$, respectively. If the pair $\left\{\tilde{\alpha},\tilde{\beta}\right\}$ be involute-evolute modified orthogonal pair. Then, the curvature $\tilde{\kappa}^*$ of the curve $\tilde{\beta}$ is obtained with respect to the torsion $\tilde{\tau}$ and the curvature $\tilde{\kappa}$ of the curve $\tilde{\alpha}$ as follows:

$$\tilde{\kappa}^*=\frac{\sqrt{\tilde{\kappa}^2+\tilde{\tau}^2}}{\left(\tilde{c}-s\right)\tilde{\kappa}}.$$

Proof: Let $\left\{\tilde{\boldsymbol{T}},\tilde{\boldsymbol{N}},\tilde{\boldsymbol{B}}\right\}$ and $\left\{\tilde{\boldsymbol{T}}^*,\tilde{\boldsymbol{N}}^*,\tilde{\boldsymbol{B}}^*\right\}$ be modified orthogonal frames at the points $\tilde{\alpha}\left(s\right)$ and $\tilde{\beta}\left(s\right)$, respectively. When we take the derivative of the equality given in Theorem 4 with respect to s and use Eq. (8), then the following equality is found

$$\tilde{\boldsymbol{T}}^*\frac{ds^*}{ds}=\left(\tilde{c}-s\right)\frac{\tilde{\kappa}}{\tilde{\tau}}\tilde{\boldsymbol{N}}. \tag{25}$$

By taking the norm of the Eq. (25), we have

$$\frac{ds^*}{ds}=\left(\tilde{c}-s\right)\tilde{\kappa}. \tag{26}$$

Substituting Eq. (26) into the Eq. (25), we get

$$\tilde{\boldsymbol{T}}^*=\frac{1}{\tilde{\tau}}\tilde{\boldsymbol{N}}. \tag{27}$$

By taking the derivative of Eq. (27) with respect to s and considering Eq. (8) and Eq. (26), the following equation is written

$$\frac{\tilde{\kappa}^*}{\tilde{\tau}^*}\left(\tilde{c}-s\right)\tilde{\kappa}\,\tilde{\boldsymbol{N}}^* = -\tilde{\kappa}\tilde{\boldsymbol{T}} + \tilde{\boldsymbol{B}}. \tag{28}$$

By taking the norm of the Eq. (28), the curvature $\tilde{\kappa}^*$ is obtained by

$$\tilde{\kappa}^* = \frac{\sqrt{\tilde{\kappa}^2 + \tilde{\tau}^2}}{\left(\tilde{c}-s\right)\tilde{\kappa}}.$$

Theorem 6. Let the pair $\left\{\tilde{\alpha},\tilde{\beta}\right\}$ be involute-evolute modified orthogonal pair. Then, the torsion $\tilde{\tau}^*$ of the curve $\tilde{\beta}$ is given by

$$\tilde{\tau}^* = \frac{\left(\tilde{c}-s\right)^5 \tilde{\kappa}^5 \left(\frac{\tilde{\tau}}{\tilde{\kappa}}\right)'}{\left(\left(\frac{\tilde{\tau}}{\tilde{\kappa}}\right)^2 + 1\right)}.$$

Proof: Taking the first, second and third-order derivatives of the relation in Theorem 4 with respect to s, then substituting Eq. (8) into the results give

$$\begin{aligned}
\tilde{\beta}' &= \left(\tilde{c}-s\right)\frac{\tilde{\kappa}}{\tilde{\tau}}\tilde{\boldsymbol{N}},\\
\tilde{\beta}'' &= -\left(\tilde{c}-s\right)\tilde{\kappa}^2\tilde{\boldsymbol{T}} + \left(-\frac{\tilde{\kappa}}{\tilde{\tau}} + \left(\tilde{c}-s\right)\frac{\tilde{\kappa}'\tilde{\tau}}{\tilde{\tau}^2} + 2\left(\tilde{c}-s\right)\frac{\tilde{\kappa}\tilde{\tau}'}{\tilde{\tau}^2}\right)\tilde{\boldsymbol{N}} + \left(\tilde{c}-s\right)\tilde{\kappa}\tilde{\boldsymbol{B}},\\
\tilde{\beta}''' &= \left(2\tilde{\kappa}^2 - 3\left(\tilde{c}-s\right)\tilde{\kappa}\,\tilde{\kappa}' - 2\left(\tilde{c}-s\right)\frac{\tilde{\kappa}^2\tilde{\tau}'}{\tilde{\tau}}\right)\tilde{\boldsymbol{T}}\\
&\quad + \left(-\left(\tilde{c}-s\right)\frac{\tilde{\kappa}^3}{\tilde{\tau}} + \left(-\frac{\tilde{\kappa}'}{\tilde{\tau}}\right)' - \frac{\tilde{\kappa}'}{\tilde{\tau}} + \left(\tilde{c}-s\right)\left(\frac{\tilde{\kappa}'}{\tilde{\tau}}\right)' - 3\frac{\tilde{\kappa}\tilde{\tau}'}{\tilde{\tau}^2}\right.\\
&\qquad \left. + 2\left(\tilde{c}-s\right)\left(\frac{\tilde{\kappa}\tilde{\tau}'}{\tilde{\tau}^2}\right)' + \left(\tilde{c}-s\right)\frac{\tilde{\kappa}'\tilde{\tau}'}{\tilde{\tau}^2} + 2\left(\tilde{c}-s\right)\frac{\tilde{\kappa}\tilde{\tau}'^2}{\tilde{\tau}^3} + \left(\tilde{c}-s\right)\tilde{\kappa}\tilde{\tau}\right)\tilde{\boldsymbol{N}}\\
&\quad + \left(-2\tilde{\kappa} + 2\left(\tilde{c}-s\right)\tilde{\kappa}' + 3\left(\tilde{c}-s\right)\frac{\tilde{\kappa}\tilde{\tau}'}{\tilde{\tau}}\right)\tilde{\boldsymbol{B}}.
\end{aligned}$$

Then, the determinant of $\tilde{\beta}',\tilde{\beta}'',\tilde{\beta}'''$ is calculated by

$$\det\left(\tilde{\beta}',\tilde{\beta}'',\tilde{\beta}'''\right) = \left(\tilde{c}-s\right)^3 \tilde{\kappa}^5 \left(\frac{\tilde{\tau}}{\tilde{\kappa}}\right)'. \tag{29}$$

If we consider Eq. (11), the torsion of the curve $\tilde{\beta}$ is found as

$$\tilde{\tau}^* = \frac{\det\left(\tilde{\beta}', \tilde{\beta}'', \tilde{\beta}'''\right)}{\tilde{\kappa}^*} \tag{30}$$

Putting Eq. (29) and the curvature of the curve $\tilde{\beta}$ in Theorem 5 into Eq. (30), we get

$$\tilde{\tau}^* = \frac{\left(\tilde{c}-s\right)^5 \tilde{\kappa}^5 \left(\dfrac{\tilde{\tau}}{\tilde{\kappa}}\right)'}{\left(\left(\dfrac{\tilde{\tau}}{\tilde{\kappa}}\right)^2 + 1\right)}.$$

5. CONCLUSION

Many researchers have made a major contribution to the field of the involute-evolute curve theory. Besides being an important subject in terms of curve theory, the applications appear in the design of gears, optics, winding clocks and toys. In this study, we have defined the involute-evolute curve couple by using the two type modified orthogonal frames in the Euclidean 3-space. Then, some well-known theorems are provided for these new involute-evolute curves. The most important result of our study is that there are similar relationships between torsions and curvatures of involute-evolute curves compared to both modified orthogonal frames. At the same time, it is thought that this study will contribute to the studies that will be done concerning the involute-evolute curve pair according to modified orthogonal frame in Minkowski space.

REFERENCES

[1] Boyer, C., *A history of mathematics*, Wiley, New York, 1968.
[2] Çalışkan, M., Bilici, M., *Bulletin of Pure and Applied Sciences*, **21E**(2), 289, 2002.
[3] Erişir, T., Kardağ, N. C., *Fundamental Journal of Mathematics and Applications*, **2**(2), 148, 2019.
[4] Fuchs, D., *American Mathematical Monthly*, **120**(3), 217, 2013.
[5] As, E., Sarıoğlugil, A., *International Journal of Physical Sciences*, **9**(7), 140, 2014.
[6] Bektaş, O., Yüce, S., *European Journal of Pure and Applied Mathematics*, **6**(1), 20, 2013.
[7] Körpınar T., Asil V, Körpınar Z., *Journal of Science and Arts*, **20**(1), 43, 2020.
[8] Körpınar T., Kaymanlı G. U., *Journal of Science and Arts*, **20**(1), 55, 2020.
[9] Sarıaydın M. T., Körpınar T., *Journal of Science and Arts*, **19**(2), 281, 2019.
[10] Şenyurt, S., Cevahir, C., Altun, Y., Kocayiğit, H., *Thermal Science*, **23**(1), 413, 2019.
[11] Sasai, T., *Tohoku Mathematical Journal*, **36**, 17, 1984.
[12] Bükcü, B., Karacan, M. K., *Journal of New Results in Science*, **5**(10), 60, 2016.
[13] Bükcü, B., Karacan, M. K., *Mathematical Sciences and Applications E-Notes*, **4**(1), 184, 2016.
[14] Baş S., Körpınar T., *Mathematics*, **7**(2), 1, 2019.
[15] Lone, M. S., Es, H., Karacan, M. K., Bükcü, B., *Turkish Journal of Mathematics*, **43**(2), 648, 2019.
[16] Lone, M. S., Es, H., Karacan, M. K., Bükcü, B., *Iranian Journal of Science and Technology, Transactions A: Science*, **43**(4), 1905, 2019.