ORIGINAL PAPER

DARBOUX ASSOCIATE CURVES OF SPACELIKE CURVES IN $\mathbb{E}_1^3$

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Abstract. *In this paper, we introduce a new type of curve called the k-directional Darboux curve. These curves are generated by vector fields that are constructed using the Darboux frame of a given spacelike curve α lying on a timelike surface in Minkowski 3-space* $\mathbb{E}_1^3$. *We give the relationships between these curves and their curvatures. In particular, we show how k-directional Darboux curves can be used to classify some special curves, such as helices and slant helices.*

Keywords: *Direction curves; general helix; slant helix; Darboux frame; Minkowski space.*

1. INTRODUCTION

In the theory of curves in differential geometry, characterizing a regular curve and giving general information about its structure is one of the interesting problems. By considering the curvatures and relationships between Frenet vectors, mathematicians can better understand these important mathematical objects. Namely, the curvatures κ and τ of regular curves have an effective role in the solution of the problem since the shape and size of a regular curve can be determined by using its curvatures.

A *general helix* is a type of curve in three-dimensional space that has a constant slope. This means that the tangent vector to the curve at any point makes a constant angle with a fixed direction. Lancret's theorem also states that a regular curve with non-zero first κ and second curvature τ is a general helix if and only if it has constant conical curvature (the ratio of the second curvature to the first curvature). The theorem is attributed to French mathematician Jean-Marie Lancret [1]. The concept of helices is an important result in the theory of curves and their significance in various fields. For example, helices have practical applications in engineering and physics, such as describing the shape of screw threads, springs, and DNA molecules [2-4].

The *slant helix* is introduced as a curve with non-vanishing curvature whose principal normal vector N makes a constant angle with a fixed direction, as defined by Izumiya and Takeuchi in [5]. There is a relationship between slant helices and general helices, with slant helices being the successor curves of general helices, as demonstrated by Menninger in [6]. The Frenet equations, which describe the motion of a particle along a curve in space, can be expressed differently depending on the causal character of the Frenet trihedron in the Minkowski 3-space $\mathbb{E}_1^3$. Ali and Lucas, however, were able to define the slant helix in

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Minkowski space in a similar way to the previously introduced definition and characterized these curves in [7].

General and slant helices also appear on the surfaces. These curves can be seen as generalizations of slants and general helices in Euclidean space, where the fixed axis is replaced by a fixed direction. The study of these curves on surfaces in Minkowski space is an interesting topic in differential geometry [8-11].

In addition to using curvatures, another approach to understanding regular curves is to consider the relationship between the corresponding Frenet vectors of two curves. This approach can lead to the discovery of interesting curves, such as involute-evolute curves, Bertrand curves, and Mannheim curves. Associated curves generated from a given curve are another example of curves that can be studied through this approach. Examples of associated curves include the normal curve, osculating circle, and pedal curve of a given curve.

In Minkowski $3-$space $\mathbb{E}_1^3$, Choi *et al.* introduced the notion of the principal (binormal)-directional curve and the principal (binormal)-donor curve of the Frenet curve [12]. They also introduced a natural representation of general helices and slant helices from the natural representation of plane curves and general helices, respectively. Many mathematicians have studied associated curves in different ambient spaces and obtained interesting results [13-24].

This new approach provides a powerful tool for understanding the geometry of these curves and their properties. Therefore, this paper provides a novel and insightful approach to studying curves on timelike surfaces in Minkowski 3-space using the Darboux frame describing the geometry of the curve.

In this paper, we define k-directional Darboux curves of a given spacelike curve α lying on a timelike surface in Minkowski 3-space $\mathbb{E}_1^3$. In Section 2, some fundamental facts of spacelike curves and the Darboux frame of α are reviewed. In Section 3, we give some relationships between the curvatures of the spacelike curve α and each k-directional Darboux curve. We obtain the necessary and sufficient conditions for $k-$directional Darboux curves to be a helix and a slant helix.

2. PRELIMINARIES

Minkowski space $\mathbb{E}_1^3$ is the real vector space $\mathbb{E}^3$ endowed with an indefinite flat metric $\langle\cdot,\cdot\rangle$ given by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3,$$

for any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{E}_1^3$. An arbitrary vector $x \in \mathbb{E}_1^3$ can be *spacelike*, *timelike*, or *null* (*lightlike*), if $\langle x, x\rangle > 0$,$\langle x, x\rangle < 0$, or $\langle x, x\rangle = 0$ and $x \neq 0$ respectively. In particular, the vector $x = 0$ is said to be spacelike. The *norm* (length) of a non-null vector $x \in \mathbb{E}_1^3$ is given by $\|x\| = \sqrt{|\langle x, x\rangle|}$. If $\|x\| = 1$, the vector x is called *unit*. The vector x and y are said to be orthogonal, if $\langle x, y\rangle = 0$.

An arbitrary curve $\alpha: I \to \mathbb{E}_1^3$ can be the *spacelike*, the *timelike* or the *null (lightlike)*, if all of its velocity vectors $\alpha'(s) = \frac{d\alpha}{ds}$ are spacelike, timelike, or null, respectively [25, 26]. In purticular, if $\alpha'(s)$ is spacelike and the acceleration vector field $\alpha''(s) = \frac{d^2\alpha}{ds^2}$ is null, then the curve α is called *pseudo null curve*.

The Frenet formulae of a unit speed spacelike curve α and $\langle \alpha''(s), \alpha''(s) \rangle \neq 0$ in $\mathbb{E}_1^3$ is given by [25]

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \epsilon\kappa(s) & 0 \\ -\kappa(s) & 0 & -\epsilon\tau(s) \\ 0 & -\epsilon\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{1}$$

where $\kappa(s)$ and $\tau(s)$ are the *first curvature* and the *second curvature* of α, respectively and it holds

$$\langle T, T \rangle = 1, \quad \langle N, N \rangle = \epsilon = \pm 1, \quad \langle B, B \rangle = -\epsilon, \tag{2}$$

$$f : \mathbb{R} \to \mathbb{R}, \ f(x) = x^3 - 3x^2 + 2x + 10 \tag{3}$$

The Frenet formulae of a unit speed pseudo null curve α in $\mathbb{E}_1^3$ have the form ([27])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & \tau & 0 \\ -\kappa & 0 & -\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{4}$$

where the first curvature $\kappa(s) = 1$. The second curvature (torsion) $\tau(s)$ is an arbitrary function of the arclength parameter s of α. The Frenet's frame vectors of α satisfy the equations

$$\begin{aligned} &\langle T, T \rangle = 1, \langle N, N \rangle = \langle B, B \rangle = 0, \\ &\langle T, N \rangle = \langle T, B \rangle = 0, \langle N, B \rangle = 1, \end{aligned} \tag{5}$$

and

$$T \times N = N, N \times B = T, B \times T = B. \tag{6}$$

Definition 2.1. A surface in Minkowski space $\mathbb{E}_1^3$ is called timelike, spacelike or lightlike if the normal vector field on the surface is timelike, spacelike or lightlike, respectively (see [26] for details).

Spacelike or timelike surface in Minkowski 3-space is also called a *non-degenerate surface*. Let M be a non-degenerate surface in Minkowski space $\mathbb{E}_1^3$ with parametrization

$$X : U \subseteq \mathbb{E}^2 \longrightarrow \mathbb{E}^3, X(u,t) = \big(x_1(u,t), x_2(u,t), x_3(u,t)\big). \tag{7}$$

Denote by

$$n(u,t) = \frac{X_u \times X_t}{||X_u \times X_t||} \tag{1}$$

a unit non-null normal vector field of M. Consider a spacelike curve $\alpha : I \subset \mathbb{R} \to M$ lying on the surface M. The *Darboux frame* of α is positively oriented a orthonormal frame $\{T, \zeta, \eta\}$, consisting of the tangential vector field T, the unit spacelike normal vector field $\eta = n|_\alpha$ and a unit timelike vector field $\zeta = T \times \eta$. The Darboux's frame equations of α read

$$\begin{bmatrix} T'(s) \\ \zeta'(s) \\ \eta'(s) \end{bmatrix} = \begin{bmatrix} 0 & -k_g(s) & k_n(s) \\ -k_g(s) & 0 & \tau_g(s) \\ -k_n(s) & \tau_g(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ \zeta(s) \\ \eta(s) \end{bmatrix}, \tag{9}$$

where $k_n(s)$, $k_g(s)$ and $\tau_g(s)$ are called the *normal curvature*, the *geodesic curvature* and the *geodesic torsion* of α, respectively. These curvature functions are defined by

$$k_g(s) = \langle T'(s), \zeta(s)\rangle, \quad k_n(s) = \langle T'(s), \eta(s)\rangle, \quad \tau_g(s) = \langle \zeta'(s), \eta(s)\rangle. \tag{10}$$

The Darboux frame of α satisfies the relations

$$\langle T, T\rangle = 1, \quad \langle \zeta, \zeta\rangle = -1, \quad \langle \eta, \eta\rangle = 1, \tag{11}$$

and

$$T \times \zeta = \eta, \quad \zeta \times \eta = T, \quad \eta \times T = -\zeta. \tag{12}$$

Lemma 2.2. Let M be a surface in $\mathbb{E}_1^3$ and $\alpha: \mathrm{I} \to \mathbb{E}_1^3$ be an arbitrary curve lying on M with the geodesic curvature $\mathrm{k_g}$, normal curvature $\mathrm{k_n}$ and geodesic torsion τ_g. Then the following statements hold:
(i) α is a geodesic curve on M if and only if $\mathrm{k_g} = 0$;
(ii) α is an asymptotic curve on M if and only if $\mathrm{k_n} = 0$;
(iii) α is a line of principal curvature on M if and only if $\tau_\mathrm{g} = 0$.

Also, the Frenet frame and Darboux frame are related to the Euclidean rotation in the timelike normal plane $T^\perp$ of α, given as follows:
(i) If α is a spacelike curve with the timelike principal normal N, then the relations (1) and (9) yield

$$k_g = \kappa\cosh\theta, \quad k_n = \kappa\sinh\theta, \quad \tau_g = \tau + \theta'. \tag{13}$$

where $\theta(s) = \angle(\eta, N)$ is an angle between a timelike and a spacelike vector.
(ii) If α is a spacelike curve with the spacelike principal normal N, then the relations (1) and (9) imply

$$k_g = \kappa\sinh\theta, \quad k_n = \kappa\cosh\theta, \quad \tau_g = \tau + \theta'. \tag{14}$$

where $\theta(s) = \angle(\eta, N)$ is an angle between two spacelike vectors.
(iii) If α is a pseudo null curve, then from the relations (4) and (9) we get

$$k_g = -k_n, \quad \tau_g = \frac{k_g'}{k_g} + \tau. \tag{15}$$

Definition 2.3. Isophote curve in Minkowski space $\mathbb{E}_1^3$ is spacelike, timelike or lightlike curve having a property that the scalar product of the surface's normal along that curve and a constant vector spanning its axis is constant.

The special isophote curve along which the surface's normal is orthogonal to its axis, is called *silhouette* curve.

3. DARBOUX ASSOCIATED CURVES OF A SPACELIKE CURVE

In this section, we introduce a k-directional Darboux curve of a spacelike curve lying on a timelike surface in Minkowski 3-space. We give the necessary and sufficient conditions in terms of the geodesic curvature, normal curvature and geodesic torsion of spacelike curves lying on the timelike surface, to be a relatively normal-slant helix and an isophote curve. Throughout this section, all curves and surfaces under consideration are smooth and regular unless otherwise stated.

Let M be a timelike surface in Minkowski space $\mathbb{E}_1^3$, and let α be a unit spacelike curve lying on M with the Darboux frame $\{T, \zeta, \eta\}$ in Minkowski 3-space. Suppose k is a vector field along the curve α. In this case, there exists a nice relation between k and the Darboux frame $\{T, \zeta, \eta\}$ of α. Specifically, k can be represented in the form

$$k(s) = k_1(s)T(s) + k_2(s)\zeta(s) + k_3(s)\eta(s), \tag{16}$$

where $k_1(s)$, $k_2(s)$, and $k_3(s)$ are scalar functions in the arclength parameter s of α, and $\epsilon_k \in -1,0,1$ is such that

$$\langle k, k\rangle = k_1^2 - k_2^2 + k_3^2 = \epsilon_k. \tag{17}$$

Using α, we can define a curve $\beta = \beta(s)$, with the same parameter as the parameter of the curve α, such that β is an integral curve of k, i.e., $\beta'(s) = k$. Based on this, we can give the following definition.

Definition 3.1. Let α be a unit spacelike curve lying on a timelike surface M in $\mathbb{E}_1^3$ and k be a vector field given by (16). A curve β is called a k-directional Darboux curve of α if β is an integral curve of k (i. e. $\beta' = k$).

Remark 3.2. If $k_1(s) = 1$ and $k_2(s) = k_3(s) = 0$ for all s in (17), then $k(s) = T(s)$ for all s in (16), and so β is called a $T-$directional Darboux curve of α. It is easy to see that the $T-$direction curve β of the curve α is trivially α itself.

Remark 3.3. If $k_1(s) = k_3(s) = 0$ and $k_2(s) = 1$ for all s in (17), then $k(s) = \zeta(s)$ for all s in (16). In this case, β is called a ζ-directional Darboux curve of α. Furthermore, if α is a relatively normal-slant helix, then β is a general helix, and vice versa. Thus, the notion of a general helix corresponds to the notion of a relatively normal-slant helix as defined in [10].

Remark 3.4. If $k_1(s) = k_2(s) = 0$ and $k_3(s) = 1$ for all s in (17), then $k(s) = \eta(s)$ for all s in (16), and so β is called an η −-directional Darboux curve of α. This means that β is a curve in which the tangent vector at each point is parallel to the vector field η along α. Furthermore, the notion of the general helix β corresponds to the notion of an isophote curve α defined in [8, 11]. In other words, if α is an isophote curve, then β is a general helix, and vice versa.

3.1. ζ-DIRECTION DARBOUX CURVE

Let α be a unit spacelike curve lying on a timelike surface M with the Darboux frame $\{T, \zeta, \eta\}$ and non-zero curvatures k_g, k_n, and τ_g. Let β be a ζ-direction Darboux curve of α. From the Definition 3.1, we have

$$\beta' = T_\beta = \zeta, \tag{18}$$

and so, β is a timelike curve. Differentiating (18) with respect to s and using (9), we obtain

$$T_\beta' = \zeta' = -k_g T + \tau_g \eta,$$

with

$$\langle T_\beta', T_\beta'\rangle = k_g^2 + \tau_g^2.$$

Using (11) and (12), the Frenet vectors along β are given by

$$T_\beta = \zeta,$$
$$N_\beta = \frac{1}{\sqrt{k_g^2+\tau_g^2}}\left(-k_g T + \tau_g \eta\right),$$
$$B_\beta = T_\beta \times N_\beta = \frac{1}{\sqrt{k_g^2+\tau_g^2}}\left(\tau_g T + k_g \eta\right).$$

Moreover, the curvature κ_β and torsion τ_β of β are given by

$$\kappa_\beta = \langle T_\beta', N_\beta\rangle = \sqrt{k_g^2+\tau_g^2}, \quad \tau_\beta = -\langle B_\beta', N_\beta\rangle = -k_n + \frac{\tau_g' k_g - k_g' \tau_g}{k_g^2+\tau_g^2}.$$

Theorem 3.5. Let α be a unit spacelike curve lying on a timelike surface M with the Darboux frame $\{T, \zeta, \eta\}$ and non-zero curvatures k_g, k_n, and τ_g. Suppose β be a ζ-direction Darboux curve of α. Then β is a timelike curve and its Frenet vector fields $\{T_\beta, N_\beta, B_\beta\}$ are given by:

$$T_\beta = \zeta,$$
$$N_\beta = \frac{1}{\sqrt{k_g^2+\tau_g^2}}\left(-k_g T + \tau_g \eta\right),$$
$$B_\beta = T_\beta \times N_\beta = \frac{1}{\sqrt{k_g^2+\tau_g^2}}\left(\tau_g T + k_g \eta\right),$$

and the curvature κ_β and torsion τ_β of β are given by:

$$\kappa_\beta = \sqrt{k_g^2+\tau_g^2}, \quad \tau_\beta = -k_n + \frac{\tau_g' k_g - k_g' \tau_g}{k_g^2+\tau_g^2}.$$

Corollary 3.6. Let α be a spacelike geodesic curve lying on a timelike surface M with the Darboux frame $\{T, \zeta, \eta\}$ in $\mathbb{E}_1^3$. Then, we have the following cases:
(i) if N is timelike, then α and β are a straight line;
(ii) if N is spacelike, then β is a general helix if and only if α is a general helix.

Proof:
(i) if N is timelike, then the relation (13) yields $\kappa(s) = 0$ and $k_n(s) = 0$ for all s. Hence, α is a straight line. Also, from Theorem 3.5 $\kappa_\beta = \tau_\beta = 0$,and so β is a straight line
(ii) if N is spacelike, then from (14) we have $\kappa(s) \neq 0$ and $\theta(s) = 0$ for all s. Moreover, we have $k_n = \kappa$ and $\tau_g = \tau$. Using the Theorem 3.5 the curvatures of β are given by $\kappa_\beta = \tau$ and $\tau_\beta = -\kappa$. Now, let β be a general helix, i.e. $\frac{\tau_\beta}{\kappa_\beta} = -\frac{\kappa}{\tau} = -\frac{1}{c}$, where $c \in \mathbb{R}_0$. Then we have $\frac{\tau}{\kappa} = \pm c = constant$ and it means that α is a general helix.

Conversely, assume that α is a general helix. In this case, we have $\frac{\tau}{\kappa} = constant$, that is, $\frac{\tau_\beta}{\kappa_\beta} = constant$. This means that β is a general helix. □

In particular, if N is null, then the relations (4) and (9) imply

$$N = -k_g \zeta + k_n \eta,$$

and since $k_g = -k_n$, N is zero vector which is coincide. Thus, there is not any geodesic pseudo null curve lying on a timelike surface M.

Corollary 3.7. Let α be a spacelike asymptotic curve on a timelike surface M in $\mathbb{E}_1^3$. Then,
(i) if N is timelike, then β is a general helix if and only if α is a slant helix;
(ii) if N is spacelike, then α and β are a straight line.

Proof:
(i) If N is timelike, then the relation (13) yields $\kappa(s) \neq 0$ and $\theta(s) = 0$ for all s. Also, we get $k_g = \kappa$ and $\tau_g = \tau$. From the Theorem 3.5 the curvatures of β are given by $\kappa_\beta = \sqrt{k_g^2 + \tau_g^2}$ and $\tau_\beta = \frac{\tau_g' k_g - k_g' \tau_g}{k_g^2 + \tau_g^2}$. Now, we assume that β is a general helix. Then we have

$$\frac{\tau_\beta}{\kappa_\beta} = -\frac{k_g^2}{\left(k_g^2 + \tau_g^2\right)^{\frac{3}{2}}} \left(\frac{\tau_g}{k_g}\right)' = c,$$

or

$$\frac{\tau_\beta}{\kappa_\beta} = \frac{\kappa^2}{(\kappa^2 + \tau^2)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)' = c$$

where $c \in \mathbb{R}_0$. This means that α is a slant helix. Conversely, assume that α is a slant helix. In this case,

$$\frac{\kappa^2}{(\kappa^2 + \tau^2)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)' = c,$$

is constant. That is, $\frac{\tau_\beta}{\kappa_\beta} = constant$. Namely, β is a general helix.
(ii) If N is spacelike, then from (14) we get $\kappa(s) = 0$ and $k_g(s) = 0$ for all s. Thus, α is a straight line. From Theorem 3.5 $\kappa_\beta = \tau_\beta = 0$,and so β is a straight line. □

Here, if N is null, then the relations (4) and (9) imply

$$N = -k_g \zeta + k_n \eta$$

and since $k_g = -k_n$, N is zero vector which is coincide. Thus, there is not any geodesic pseudo null curve lying on a timelike surface M.

Corollary 3.8. Let α be a spacelike line of principal curvature on a timelike surface M in $\mathbb{E}_1^3$. Then,
(i) if N is timelike or spacelike, then β is a general helix if and only if α is a plane curve;
(ii) if N is null, then β is a general helix.

Proof:
(i) If N is timelike, then the relation (13) yields $\theta' = \tau$, $k_g = \kappa\cosh\theta$, and $k_n = \kappa\sinh\theta$. Using the Theorem 3.5 the curvatures κ_β and τ_β of β are given by $\kappa_\beta = k_g$ and $\tau_\beta = -k_n$. Let β be a general helix, i.e. $\frac{\tau_\beta}{\kappa_\beta} = -\tanh\theta = constant$. Then, we have $\theta = constant$. It means that $\theta' = 0$, and so $\tau = 0$. Therefore, α is a plane curve.

Conversely, assume that α is a plane curve. In this case, we have $\tau = 0$, that is, $\frac{\tau_\beta}{\kappa_\beta} = constant$. This means that β is a general helix. The other case can be proved analogously, so we omit its proof.
(ii) If N is null, then from (15) we get $k_g = -k_n = e^{-\int \tau ds}$. From the Theorem 3.5 the curvatures of β are given by $\kappa_\beta = k_g$ and $\tau_\beta = -k_n$. Then, we have $\frac{\tau_\beta}{\kappa_\beta} = -\frac{k_n}{k_g} = -1$ and so β is a general helix. □

Theorem 3.9. Let α be a unit spacelike curve lying on a timelike surface M with the Darboux frame $\{T, \zeta, \eta\}$ and non-zero curvatures k_g, k_n, and τ_g, and let β be a ζ-direction Darboux curve of α. If β is lying on M, then the geodesic curvature $\left(k_g^\beta\right)$, the normal curvature $\left(k_n^\beta\right)$, and the geodesic torsion $\left(\tau_g^\beta\right)$, of β are given by

$$\begin{aligned} k_g^\beta &= -k_g \cos\psi + \tau_g \sin\psi, \\ k_n^\beta &= -k_g \sin\psi - \tau_g \cos\psi, \\ \tau_g^\beta &= -\psi' - k_n, \end{aligned}$$

where $\psi(s)$ is an angle between the tanget vector T of α and the Darboux vector field ζ_β of β.

Proof: Let $\{T_\beta, \zeta_\beta, \eta_\beta\}$ denote the Darboux frame along the curve β. Here, T_β is the tangent vector field of β, $\eta_\beta = n|_\beta$ is the unit spacelike normal vector field of M along β, and ζ_β is a unit spacelike vector field such that $\zeta_\beta = \eta \times T$. Since β is the ζ −direction Darboux curve of α, from the Definition 3.1, we have

$$\beta' = T_\beta = \zeta,$$

and so, β is a timelike curve. Also, since β is perpenticular to T and η, the Darboux frames are related by the hyperbolic rotation in spacelike normal plane $T_\beta^\perp$ of β, given by

$$\begin{bmatrix} T_\beta \\ \zeta_\beta \\ \eta_\beta \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \cos\psi & 0 & \sin\psi \\ \sin\psi & 0 & -\cos\psi \end{bmatrix} \begin{bmatrix} T \\ \zeta \\ \eta \end{bmatrix} \tag{19}$$

where $\psi(s) = \angle(\zeta_\beta, T)$ is an angle between two spacelike vectors. By using the relations (9), and (19), we get

$$\begin{aligned} k_g^\beta &= \langle T_\beta', \zeta_\beta \rangle = -k_g \cos\psi + \tau_g \sin\psi, \\ k_n^\beta &= \langle T_\beta', \eta_\beta \rangle = -k_g \sin\psi - \tau_g \cos\psi, \\ \tau_g^\beta &= \langle \zeta_\beta', \eta_\beta \rangle = -\psi' - k_n. \ \square \end{aligned}$$

Using the Lemma 2.2 and the Theorem 3.9 we can give the following corollary.

Corollary 3.10. Let α be a spacelike curve lying on a timelike surface M and β be a ζ-direction Darboux curve of α. Then,
i) β is a geodesic curve on M if and only if α is a geodesic curve and $\psi = 0$;
ii) β is an asmyptotic curve on M if and only if α is a line of principal curvature and $\psi = 0$;
iii) β is a line of principal curvature on M if and only if α is an asmyptotic curve and $\psi = \text{constant}$.

3.2. η-DIRECTION DARBOUX CURVE

The following theorem and corollaries can be proved analogously as the ζ-direction Darboux curve, so we omit their proofs.

Theorem 3.11. Let α be a unit spacelike curve lying on a timelike surface M with the Darboux frame $\{T,\zeta,\eta\}$ and the non-zero curvatures k_g, k_n, and τ_g and β be a η-direction Darboux curve of α. Then
(i) if $k_n^2-\tau_g^2>0$ for all s, then β is a spacelike curve with the spacelike principal normal and its curvatures κ_β and τ_β are given by

$$\kappa_\beta=\sqrt{k_n^2-\tau_g^2},\quad \tau_\beta=k_g+\frac{k_n\tau_g'-\tau_g k_n'}{k_n^2-\tau_g^2},$$

(ii) if $k_n^2-\tau_g^2<0$ for all s, then β is a spacelike curve with the timelike principal normal and its curvatures κ_β and τ_β are given by

$$\kappa_\beta=\sqrt{\tau_g^2-k_n^2},\quad \tau_\beta=-k_g+\frac{k_n\tau_g'-\tau_g k_n'}{\tau_g^2-k_n^2},$$

(iii) if $k_n=\tau_g$ for all s, then β is a pseudo curve whose curvatures κ_β and τ_β are given by

$$\kappa_\beta=1,\quad \tau_\beta=\frac{k_n'}{k_n}+k_g.$$

Corollary 3.12. If a unit spacelike curve α lies on a timelike surface M with the Darboux frame $\{T,\zeta,\eta\}$ and non-zero curvatures k_g, k_n, and τ_g, then there does not exist any η −direction Darboux curve of α if $k_n=-\tau_g$ for all s.

Corollary 3.13. Let α be a spacelike geodesic curve lying on a timelike surface M with the curvatures k_g, k_n, and τ_g in $\mathbb{E}_1^3$. Then, its principal normal N is a spacelike vector and
(i) if $k_n^2-\tau_g^2>0$, or $k_n^2-\tau_g^2<0$ for all s, then then α is a slant helix with the non-zero curvature κ if and only if any η-direction Darboux curve of α is a general helix;
(ii) if $k_n=\tau_g$ for all s, then α is a general helix whose curvatures satisfy the relation

$$\kappa=\tau=Ae^s,$$

where $A\in\mathbb{R}^+$, if and only if any η-direction Darboux curve of α is a general helix.

Corollary 3.14. Let α be a spacelike asymptotic curve on a timelike surface M with the curvatures k_g, k_n, and τ_g in $\mathbb{E}_1^3$. Then, the principal normal N of α is timelike and the following holds:
(i) if $k_n^2-\tau_g^2>0$ for all s, then there is no η-direction Darboux curve of α;
(ii) if $k_n^2-\tau_g^2<0$ for all s, then β is a general helix if and only if α is a general helix;
(ii) if $k_n=\tau_g$ for all s, then α is a plane curve and the curvatures κ_β and τ_β of β are given by

$$\kappa_\beta=1,\quad \tau_\beta=\kappa.$$

Corollary 3.15. Let α be a spacelike line of principal curvature on a timelike surface M with the non-null principal normal N in $\mathbb{E}_1^3$. Then,
(i) if $k_n^2 - \tau_g^2 < 0$ for all s, then there is no η −direction Darboux curve of α;
(ii) if $k_n^2 - \tau_g^2 > 0$ for all s, then β is a general helix if and only if α is a plane curve;
(ii) if $k_n = \tau_g$ for all s, then α is a plane curve and the curvatures κ_β and τ_β of β are given by

$$\kappa_\beta = 1, \quad \tau_\beta = \kappa.$$

Corollary 3.16. Let α be a pseudo null line of principal curvature on a timelike surface M in $\mathbb{E}_1^3$. Then,
(i) if $k_n^2 - \tau_g^2 < 0$, or $k_n = \tau_g$ for all s, then there is no η −direction Darboux curve of α;
(ii) if $k_n^2 - \tau_g^2 > 0$ for all s, then β is a general helix.

Theorem 3.17. Let α be a unit spacelike curve lying on a timelike surface M with the Darboux frame $\{T, \zeta, \eta\}$ and non-zero curvatures k_g, k_n, and τ_g. Let β be an η −direction Darboux curve of α. Then, the geodesic curvature $\left(\overline{k}_g^\beta\right)$, the normal curvature $\left(\overline{k}_n^\beta\right)$, and the geodesic torsion $\left(\overline{\tau}_g^\beta\right)$ of β are given by

$$\overline{k}_g^\beta = -k_n sinh\overline{\psi} - \tau_g cosh\overline{\psi},$$
$$\overline{k}_n^\beta = k_n cosh\overline{\psi} + \tau_g sinh\overline{\psi},$$
$$\overline{\tau}_g^\beta = k_g - \overline{\psi}',$$

where $\overline{\psi}(s) = \angle(\eta_\beta, T)$ is an angle between the tanget vector T of α and the Darboux vector field η_β of β.

Corollary 3.18. Let α be a spacelike curve lying on a timelike surface M, and β be a η −direction Darboux curve of α. Then
i) β is a geodesic curve on M if and only if α is a line of principal curvature and $\overline{\psi} = 0$;
ii) β is an asmyptotic curve on M if and only if α is an asmyptotic curve and $\psi = 0$;
iii) β is a line of principal curvature on M if and only if α is a geodesic curve and $\psi =$ constant.

4. CONCLUSIONS

In this paper, we have introduced and studied k-directional Darboux curves of a spacelike curve lying on a timelike surface in Minkowski 3-space $\mathbb{E}_1^3$. These curves are generated by vector fields constructed using the Darboux frame of the spacelike curve. We have established relationships between the curvatures of the spacelike curve and its k-directional Darboux curves, characterizing conditions for these curves to be helices and slant helices.

Our study contributes to the understanding of curves on timelike surfaces in Minkowski 3-space and provides a novel approach to studying the geometry of such curves. Future research could explore further properties and applications of k-directional Darboux curves, potentially leading to new insights into the theory of curves in Minkowski spaces.

In conclusion, the investigation of k-directional Darboux curves opens up avenues for exploring the rich geometry of curves on timelike surfaces, adding to the broader understanding of differential geometry in non-Euclidean spaces.

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