ORIGINAL PAPER

APPLICATIONS OF COMPLEX SEE INTEGRAL TRANSFORM

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Manuscript received: 14.08.2023; Accepted paper: 16.02.2024;
Published online: 30.03.2024.

Abstract. *This paper refers to the study of applications of complex SEE (Sadaq-Emad-Emann) integral transform on various functions. In this paper, we have applied complex SEE transform on some special functions including the Mittag-Leffler function, Generalized function, piecewise continuous function, and integrals. Additionally, we apply the complex SEE integral transform to determine the transfer function of mechanical systems and control engineering. As well as, present and discuss the Beam problem by utilizing complex SEE transform.*

Keywords: *complex SEE integral transform; Mittag-Leffler function; special G function; piecewise continuous function; Heaviside shifting theorem.*

1. INTRODUCTION

The complex SEE (complex Sadaq-Emad-Emann) integral transformation is a new complex transformation that was introduced by Eman A. Mansour, et. all. [1] It makes it more straightforward to tackle the issue of designing applications and makes differential equations easy to solve. The ideas of complex SEE transformation are applied in the areas of science and innovation like Electric circuit examination, Correspondence designing, Control designing also, Atomic material science, and so forth. The present paper deals with the evaluation of complex SEE transforms of the Mittag-Leffler function, generalized function, and piecewise continuous function.

Definition 1.1. Complex SEE integral transform: Eman, Sadiq, and Emad defined the following integral transform named as complex SEE transform in 2021. [1] We dissect functions in set E characterized by a complex integral transform for exponential order functions.

$$E = \{f(t): \exists M, l_1, l_2 > 0. |f(t)| < Me^{-il_j|t|}, if\ t \in (-1)^j \times [0, \infty]\} \tag{1.1}$$

where i is a complex number, and i^2=-1.

The consistent M should be a finite value for a specific function in the arrangement of A, while l_1, l_2 may be infinite or finite. Operator $S^C[.]$ is the notation used for complex SEE integral transform. This transform's formula can be found in:

$$S^c[f(t)] = T(iv) = \frac{1}{v^n}\int_0^\infty f(t)e^{-ivt}\,dt\,, t \geq 0\,, l_1 \leq v \leq l_2\,, n\epsilon\ Z \tag{1.2}$$

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https://doi.org/10.46939/J.Sci.Arts-24.1-a15

The function's argument, f (t) contains, the variable t is factored by using the variable v in this complex transformation.

Definition 1.2. The Mittag-Leffler function: The Mittag-Leffler function E (z), was invented by Swedish mathematician G.N. with α > 0 at the start of this century in a series of five notes, giving it its name. He gave his definition in the form of a series,

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{1.3}$$

Mittag-Leffler admitted that for α; ℜ (α)>0 by himself, the series in equation (1.3) converges in the entire complex plane (and is, thus, an entire function of complex variable z). The generalization of Mittag-Leffler function $E_{\alpha}(Z)$ was introduced by Wiman (1905) [5] is defined as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{1.4}$$

For each α, β ϵ ℂ, ℜ (α) >0; $E_{\alpha,\beta}(Z)$ is an entire function. If take β=1, the function$E_{\alpha,\beta}(Z)$ reduces to classical Mittag-Leffler function. For detailed study of this function you can see H.J. Haubold et. all [4]

Definition 1.3. Special G-function: The following special G-function $G_{\rho,\eta,r}[a,z]$ also named as generalized function, was defined by Lorenzo and Hartley as [2]:

$$G_{\rho,\eta,r}[a,z] = z^{r\rho-\eta-1} \sum_{n=0}^{\infty} \frac{(r)_n (az^{\rho})^n}{\Gamma(n\rho+\rho r-\eta) n!}, \text{ Re } (\boldsymbol{\rho} r\text{-}\boldsymbol{\eta}) > 0 \tag{1.5}$$

Details related to the special G-function $G_{\rho,\eta,r}[a,z]$ can be seen in Lorenzo and Hartley [2] and H. Nagar and Menaria [3].

Definition 1.4. Piecewise continuous function: A function that is consistent on each point, besides at a limited number of points in its space is known as piecewise continuous function. If a function f is characterized and nonstop in the span I = [a, b] with the exception of, potentially, a limited number of points, t_1, t_2, ..., t_k, at every one of which the left hand and right hand limits of this function exist (i.e., every one of the discontinuities of the principal type, or "jump" discontinuities).

2. COMPLEX SEE INTEGRAL TRANSFORM OF SPECIAL FUNCTIONS

We try to demonstrate that any function f (t) meets the integral of condition (1.2). Eman et al., had evaluated various important results after calculating the complex SEE integral transform of some famous functions like as constant function, algebraic functions, exponential function and differential functions etc.

In this section, we introduce the complex SEE integral transform of some special functions:

Theorem 2.1. Let α > 0, β > 0, Re (α) > 0, Re (β) > 0 and n ϵ ℤ. Then we have the following relation:

$$S^C[E_{\alpha,\beta}(iz)] = \frac{-i}{\alpha.v^{n+1}}\left[E_{\alpha,\beta-1}\left(\frac{1}{v}\right) - (\beta-1)E_{\alpha,\beta}\left(\frac{1}{v}\right)\right]$$

Proof: By using (1.2) and (1.4) and replacing z by *i*z, we have

$$S^C[E_{\alpha,\beta}(iz)] = \frac{1}{v^n}\int_0^\infty E_{\alpha,\beta}(\mathrm{iz})\, e^{-ivz}dz$$

$$= \frac{1}{v^n}\sum_{k=0}^{\infty}\frac{i^k}{\Gamma(\alpha k+\beta)}\int_0^\infty z^k\, e^{-ivz}dz = \frac{1}{v^n}\sum_{k=0}^{\infty}\frac{i^k}{\Gamma(\alpha k+\beta)}\frac{\Gamma(k+1)}{(iv)^{k+1}}$$

$$= \frac{-i}{v^{n+1}}\sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(\alpha k+\beta)}\frac{1}{v^k} = \frac{-i}{v^{n+1}}\sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(\alpha k+\beta+1-1)}\frac{1}{v^k}$$

$$= \frac{-i}{\alpha v^{n+1}}\sum_{k=0}^{\infty}\frac{\alpha\left(k+\frac{\beta-1}{\alpha}-\frac{\beta-1}{\alpha}\right)\Gamma(k)}{(\alpha k+\beta-1)\Gamma(\alpha k+\beta-1)}\frac{1}{v^k}$$

$$= \frac{-i}{\alpha v^{n+1}}\left[\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta-1)}\frac{1}{v^k} - (\beta-1)\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta)}\frac{1}{v^k}\right]$$

$$= \frac{-i}{\alpha.v^{n+1}}\left[E_{\alpha,\beta-1}\left(\frac{1}{v}\right) - (\beta-1)E_{\alpha,\beta}\left(\frac{1}{v}\right)\right]$$

Thus according to the definition (1.2), we show up at the outcome (2.1), which finishes the verification of our hypothesis. Special case: if we put α=1, β =1, we get the result in the form of exponential function, the relation is given as:

$$S^C[E_{1,1}(iz)] = \frac{-i}{v^{n+1}}\sum_{k=0}^{\infty}\frac{1}{(k-1)!}\frac{1}{v^k}$$

$$= \frac{-i}{v^{n+1}}\, e^{\frac{1}{v}}$$

Theorem 2.2. Let **ρ**, **η**, r ϵ ℂ and nϵ ℤ. Then we have the following relation

$$S^C\left[G_{\rho,\eta,r}[a,iz]\right] = \frac{-iv^{\eta-r\rho}}{v^n\Gamma(r)}\sum_{k=0}^{\infty}\frac{\Gamma(r+k)}{k!}\left(\frac{a}{v^\rho}\right)^k$$

Proof: On using equation (1.2) and (1.5) and replacing z by *i*z we have,

$$S^C\left[G_{\rho,\eta,r}[a,iz]\right] = \frac{1}{v^n}\int_0^\infty G_{\rho,\eta,r}[a,iz]\, e^{-ivz}dz$$

$$= \frac{1}{v^n}\sum_{k=0}^{\infty}\frac{(r)_k a^k i^{r\rho+k\rho-\eta-1}}{\Gamma(r\rho+k\rho-\eta)k!}\int_0^\infty z^{r\rho+k\rho-\eta-1}\, e^{-ivz}dz$$

$$= \frac{1}{v^n}\sum_{k=0}^{\infty}\frac{(r)_k a^k i^{r\rho+k\rho-\eta-1}}{\Gamma(r\rho+k\rho-\eta)k!}\,\frac{\Gamma(r\rho+k\rho-\eta)}{(iv)^{r\rho+k\rho-\eta}}$$

$$= \frac{-iv^{\eta-r\rho}}{v^n} \sum_{k=0}^{\infty} \frac{(r)_k}{k!} \left(\frac{a}{v^\rho}\right)^k$$

This directly completes the proof of theorem 2.2

3. APPLICATIONS OF COMPLEX INTEGRAL TRANSFORM ON PIECEWISE CONTINUOUS FUNCTION

This section will show how to apply the complex integral transform to the ODE $L_y = f$ with piecewise continuous function.

Problem 3.1. Consider the function

$$f(t) = \begin{cases} -1, t \le 4, \\ 1, t > 4 \end{cases} \tag{3.1}$$

There is only one point of discontinuity, that is t = 4, at which f (t) has the discontinuity of the first type, or jump discontinuity. We have

$$\lim_{t \to -4} f(t) = -1$$

$$\lim_{t \to 4} f(t) = 1$$

They both exist and are not equivalent. The value of jump is 2. Then we can find the complex SEE integral transform of f (t) by using the definition (1.2)

$$S^c[f(t)] = \frac{1}{v^n} \int_0^\infty f(t) e^{-ivt}\, dt$$

$$= \frac{1}{v^n} \left[\int_o^4 (-1) e^{-ivt}\, dt + \int_4^\infty 1 e^{-ivt}\, dt \right]$$

$$= \frac{1}{v^n(-iv)} \left[-e^{-ivt}\Big|_0^4 + e^{-ivt}\Big|_4^\infty \right]$$

$$= \frac{-i}{v^{n+1}} [2e^{-4iv} - 1]$$

This is the required result of our problem (3.1).

Problem 3.2. Consider the function

$$f(t) = \begin{cases} \sin(t), 0 < t < \pi, \\ 0, t > \pi \end{cases} \tag{3.2}$$

Using equation (1.2)

$$S^c[f(t)] = \frac{1}{v^n} \int_0^\infty f(t) e^{-ivt}\, dt$$

$$= \frac{1}{v^n}\left[\int_0^{\pi} \sin te^{-ivt}dt + \int_{\pi}^{\infty} 0e^{-ivt}dt\right]$$

Solving $\int_0^{\pi} \sin te^{-ivt}dt$ integration by parts and after simple calculations we get

$$\int_0^{\pi} \sin te^{-ivt}dt = \frac{2e^{-ivt}}{(1-iv)^2}$$

Using above equation we reach to the solution of (3.2) given below

$$S^c[f(t)] = \frac{2e^{-ivt}}{v^n(1-iv)^2}$$

Problem 3.3. Consider the piecewise continuous function,

$$f(t) = \begin{cases} 2, 0 < t \leq 5 \\ 0, 5 < t \leq 10 \\ e^{4t}, 10 < t < \infty \end{cases} \tag{3.3}$$

Using equation (1.2), we have

$$S^c[f(t)] = \frac{1}{v^n}\int_0^{\infty} f(t)e^{-ivt}\,dt$$

$$= \frac{1}{v^n}\left[\int_0^{5} 2e^{-ivt}dt + \int_5^{10} 0e^{-ivt}dt + \int_{10}^{\infty} e^{4t}e^{-ivt}dt\right]$$

$$= \frac{1}{v^n}\left[2\left|\frac{e^{-ivt}}{-iv}\right|_0^5 + \left|\frac{e^{-(iv-4)}}{(4-iv)}\right|_{10}^{\infty}\right]$$

Thus, after simple computations we reach at the final solution that is

$$= \frac{1}{v^n}\left[\frac{2-2e^{-5iv}}{iv} + \frac{e^{-10iv}e^{40}}{iv-4}\right]$$

This completes the proof.

4. COMPLEX SEE INTEGRAL TRANSFORM OF INTEGRAL

Proposition 1. Complex SEE integral transform of integral: The complex SEE integral transform of integrals of $f(t)$ is presented in this paragraph

$$S^c\left[\int_0^{t} f(t)dt\right] = \frac{T(v)}{iv} \tag{4.1}$$

Proposition 2. Second shifting theorem (Heaviside shifting theorem): Consider the function g (t) and a be any constant such that

$$g(t) = \begin{cases} f(t-a), t > a \\ 0, t < a \end{cases} \tag{4.2}$$

Then, using (1.2) and (4.2), we have the following relation

$$S^c[g(t)] = \frac{1}{v^n}\left[\int_a^\infty f(t-a)e^{-ivt}\,dt\right]$$

Put t-a =t_1, then

$$S^c[g(t)] = \frac{1}{v^n}\left[\int_0^\infty f(t_1)e^{-iv(a+t_1)}\,dt_1\right]$$
$$= e^{-iva}\,F(v)$$

This completes the proof.

5. APPLICATION OF COMPLEX SEE INTEGRAL TRANSFORM IN MECHANICAL ENGINEERING

The complex SEE integral transform is widely used in the field of mechanical engineering to determine a particular system's transfer function by solving differential equations that arise in mathematical modeling of mechanical systems. The example below explains how to calculate a transfer system by implementing the complex SEE integral transform [10].

Example: The tank in the Fig. 1 [10] is initially empty (t=0). A constant rate of flow is Q_i implemented for t>0. Fluid exits the tank at a rate of $Q_o = CH$. A is the region of the tank's cross-section. For the head H, determine the differential equation. Recognize the system's time constant and also find out the transfer equation of this system.

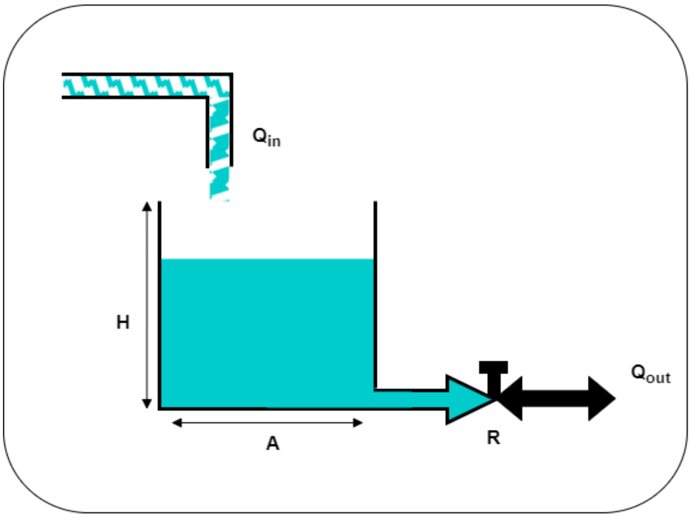

Figure 1. Liquid level Tank with Constant Outflow.

Solution:

Given that $Q_o = CH$

Let, M= Mass of fluid; e = Density of fluid

$$\therefore Mass = M = Volume \times Density$$
$$= AH \times e$$

$$\therefore Mass\ flow\ rate = \dot{M} = \frac{dM}{dt}$$

$$= \frac{d}{dt}(AH \times e) = eA \times \frac{dH}{dt}$$

We know that, Mass flow rate into tank = Mass inflow minus Mass outflow.

$$\therefore eA\frac{dH}{dt} = eQ_i - eQ_o$$

$$\therefore A\frac{dH}{dt} = Q_i - Q_o$$

$$\therefore A\frac{dH}{dt} = Q_i - CH \ \ldots\ldots\ldots \{Q_o = CH\}$$

$$\therefore Q_i = A\frac{dH}{dt} + CH$$

The above equation represents head H's differential equation. Now, applying complex SEE integral transform on both sides,

$$S^C[Q_i] = A_v S^C\left[\frac{dH}{dt}\right] + C_v S^C[H]$$

$$\therefore Q_i(iv) = A_{iv}\left\{\frac{-H(0)}{v^n} + iv\, H(iv)\right\} + C_{iv}\, H(iv)$$

$$\therefore Q_i(iv) = (ivA_v + C_v)H(iv) \ldots\ldots \{H(0) = 0\} \tag{5.1}$$

$$\therefore \frac{H(iv)}{Q_i(iv)} = \frac{1}{ivA_{iv} + C_{iv}}$$

but, $Q_o = CH$.

By applying complex SEE integral transform, we have

$$S^C[Q_o] = C_{iv}S^C[H]$$
$$\therefore H(iv) = \frac{Q_o(v)}{C_{iv}}$$

Using this equation in (5.1), we get

$$\frac{Q_o(v)}{C_v Q_i\,(v)} = \frac{1}{ivA_{iv} + C_{iv}}$$
$$\therefore \frac{Q_o(v)}{Q_i\,(v)} = \frac{1}{iv\left(\frac{A_{iv}}{C_{iv}}\right) + 1} \tag{5.2}$$

Thus, the transfer equation of the system is represented by above (5.2) equation. Hence, the time constant is given by,

$$\therefore time\ constant, \tau = \frac{A_{iv}}{C_{iv}}.$$

6. PROBLEM TO BEAMS

A hinged beam with x = 0 and x = N carries a constant loud w_o per unit length. Any point P can have a deflection; find it.

Solutions: The ordinary differential equation and boundary conditions are:

$$\frac{d^4y}{dx^4} = \frac{w_0}{EI}, 0 < x < N \tag{6.1}$$

$$y(0) = y''(0) = 0, y(N) = y''(0) = 0 \tag{6.2}$$

EI is referred to as the flexural rigidity of the beam, where E represents the young's modulus, cross-section's moment of inertia about an axis perpendicular to the plane of bending is represented by I.

According to the application, certain physical quantities include:

$$y'(x), M\,(x) = EIy''(x) and\ S(x) = M'(x)EIy''(x)$$

which, in turn, stand for the “slope”, “bending moment”, and “shear” at a point P:

Taking complex SEE transform of both ideas of equation (6.1), we get, if

$$S^C\{y(x)\} = T(iv),$$
$$(iv)^4T(iv) - \frac{1}{v^n}[y'''(0) + ivy''(0) + (iv)^2y'(0) + (iv)^3y(0)] = \frac{w_0}{EI}\left(\frac{-i}{v^{n+1}}\right)$$
$$(iv)^4T(iv) - \frac{1}{v^n}[C_2 + (iv)^2C_1] = \frac{w_0}{EI}\left(\frac{-i}{v^{n+1}}\right)$$
$$(iv)^4T(iv) = \frac{-w_0 i}{EIv^{n+1}} + \frac{1}{v^n}[C_2 + (iv)^2C_1]$$
$$S^C\{y(x)\} = T(iv) = \frac{-w_0 i}{EIv^{n+5}} + \frac{C_2}{v^{n+4}} - \frac{C_1}{v^{n+2}}$$

Taking the inverse complex SEE transform of above equation,

$$y(x) = C_1x + C_2\frac{x^3}{3!} + \frac{w_0}{EI}\frac{x^4}{4!}$$

$$y(x) = C_1 x + C_2 \frac{x^3}{6} + \frac{w_0}{EI}\frac{x^4}{24}$$

From the last two conditions in equation (6.2), we find

$$C_1 = \frac{w_0 N^3}{24EI}, C_2 = \frac{w_0 N}{24EI}$$

Thus, the required deflection is:

$$y(x) = \frac{w_0}{24EI} x(N - x)(N^2 - Nx - x^2).$$

Any point P on the beam, but particularly the ends, the bending moment and shear can be calculated.

7. CONCLUSION

The results obtained in this paper are new and can be further modified, which are used in various areas of engineering, higher order differential equations and physics.

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