ORIGINAL PAPER

ON QUOTIENT RIESZ ALGEBRAS

SABAHATTIN ILBIRA[1], HATICE ÜNLÜ EROĞLU[2], ABDULLAH AYDIN[3,*]

Manuscript received: 21.06.2023; Accepted paper: 24.01.2024;
Published online: 30.03.2024.

Abstract. *This paper focuses on the study and investigation of quotient Riesz algebras, examining their properties and exploring various topologies associated with them. The aim is to gain a better understanding of the behavior and characteristics of these quotient Riesz algebras in relation to their topological structures.*

Keywords: *quotient Riesz algebras; topological structures; Riesz algebra.*

1. INTRODUCTION

Riesz space relies on various algebraic structures to capture the essential properties. One such structure of paramount importance is the Riesz algebra, which combines the concepts of a Riesz space and an associative real algebra. Riesz algebras play a pivotal role in functional analysis, offering a versatile framework for exploring and understanding a wide range of mathematical phenomena. The foundations of Riesz algebra were laid by Birkhoff and Pierce, who first introduced and investigated these algebraic structures [1]. Subsequently, numerous researchers have made notable contributions, expanding the scope of knowledge surrounding Riesz algebras. Noteworthy studies in this area include works [2-10], which have enriched the understanding of Riesz algebra and its associated concepts.

In a recent publication by Aydın et al. [11], the concept of topological Riesz algebras with respect to different types of linear topologies was introduced. The authors provided insights into the locally solid topology on the quotient algebra, shedding light on its properties and implications. The present paper aims to extend the investigation by exploring various linear topologies and lattice-ordered algebraic structures on quotient Riesz algebras.

To facilitate a comprehensive understanding of the subsequent discussions, Section 2 of this paper presents a comprehensive review of the notations and terminologies used in Riesz algebras and topological Riesz spaces. These foundational concepts lay the groundwork for the subsequent analyses conducted in this study.

In Section 3, we delve into the structure of quotient Riesz algebras, examining their inherent properties and establishing connections with the original Riesz algebras. This exploration involves the introduction of appropriate equivalence relations on Riesz algebras, leading to the formation of quotient algebras whose structure and properties are of great interest.

Furthermore, Section 4 focuses on the topological aspects of quotient Riesz algebras. We investigate different types of linear topologies imposed on these quotient algebras,

[1] Amasya University, Department of Mathematics, Amasya, Turkey. E-mail: sabahattin.ilbira@amasya.edu.tr.
[2] Necmettin Erbakan University, Department of Mathematics and Computer Science, Konya, Turkey.
E-mail: hueroglu@erbakan.edu.tr.
[3] Muş Alparslan University, Department of Mathematics, Muş, Turkey.
[*] *Corresponding author: a.aydin@alparslan.edu.tr.*

https://doi.org/10.46939/J.Sci.Arts-24.1-a14

studying their properties and discerning their impact on the algebraic structure. Particular attention is given to the locally solid topology on the quotient algebra, and notable results in this context are presented.

2. NOTATION AND PRELIMINARIES

A lattice refers to a partially ordered set $(E, \leq)$ with the existence of two important operations: the infimum $x \wedge y = \inf\{x, y\}$ and the supremum $x \vee y = \sup\{x, y\}$ among to elements x and y. An ordered vector space is a real vector space $(E, \leq)$ that satisfies the following two properties: for any elements x and y in E and any z in E, if $x \leq y$, then $x + z \leq y + z$, and if $0 \leq \lambda \in \mathbb{R}$, then $\lambda x \leq \lambda y$. A Riesz space, also known as a vector lattice, is an ordered vector space that satisfies the lattice conditions. On a Riesz space E, an element x is considered positive if $\theta \leq x$, where θ is the null element. The set of all positive elements in E is denoted by E_+. For any element x in a Riesz space, the module of x, denoted as $|x|$, is defined as the supremum of x and its negation, i.e., $|x| = x \vee (-x)$.

In the context of Riesz spaces, subsets of E can possess certain characteristics. A subset S of a Riesz space E is called solid if, for every x in S and y in E, whenever $|y| \leq |x|$, it implies that y belongs to S. A subset C of a Riesz space E is called convex if, for any u and v in C and any $0 \leq \lambda \leq 1$, the linear combination $\lambda u + (1 - \lambda)v$ also belongs to C. Moreover, given any elements u and v in E with $u \leq v$, the interval $[u, v] = \{z \in E | u \leq z \leq v\}$ is referred to as an order interval.

A subset F of a Riesz space E is called full if, for every u and v in F with $u \leq v$, the order interval $[u, v]$ is completely contained within F. Moving on, a linear topology τ on a vector space E refers to a topology on E such that both addition and scalar multiplication operations remain continuous with respect to the topology τ on E. Furthermore, if the topology τ has a basis consisting of solid sets, it is called a solid topology. In such cases, the pair (E, τ) is referred to as a locally solid Riesz space. It should be noted that not all topological Riesz spaces are locally solid. Similarly, one can define a locally convex topology and a locally full topology on a vector space by replacing the term 'solid' with 'convex' and 'full,' respectively.

Now, let's present the concept of a Riesz algebra or l-algebra. It denotes a Riesz space E equipped with a multiplication operation denoted by "$\cdot$" that maps elements from $E \times E$ to E. This multiplication operation satisfies several properties:

(i) $u \cdot (v + w) = u \cdot v + u \cdot w$,
(ii) $(v + w) \cdot u = v \cdot u + w \cdot u$,
(iii) $v \cdot (\alpha u) = \alpha v \cdot u$,
(iv) $u \cdot (v \cdot w) = (u \cdot v) \cdot w$,
(v) $x \cdot y \in E_+$ for all $x, y \in E_+$.

A commutative l-algebra is one in which the multiplication operation is commutative, i.e., $u \cdot v = v \cdot u$ for all $u, v \in E$. Furthermore, an l-algebra is termed a d-algebra if $(u \vee v) \cdot w = (u \cdot w) \vee (v \cdot w)$ and $w \cdot (u \vee v) = (w \cdot u) \vee (w \cdot v)$ for all $u, v \in E$ and $w \in E_+$. An l-algebra is an almost f-algebra if $u \wedge v = \theta$ implies $u \cdot v = \theta$ for all $u, v \in E$. Finally, an l-algebra is an f-algebra if $u \wedge v = \theta$ implies$(w \cdot u) \wedge v = (u \cdot w) \wedge v = \theta$, for all $u, v \in E$ and $w \in E_+$. Remind that a Riesz space is Archimedean if the sequence $\frac{1}{n} x \downarrow \theta$ holds in E for every $x \in E_+$. It is worth mentioning that throughout this paper, we assume that all l-algebras are Archimedean [12-14]. It can be concluded from a Theorem 140.10 [15] that every Archimedean f-algebra is also commutative.

Recall that an r-ideal is a linear subspace of E that acts as a two-sided ring ideal. An order ideal refers to a solid linear subspace of a Riesz space. If an r-ideal also happens to be an order ideal, it is called an l-ideal. For references on Riesz groups and l-ideals, one can consult [9,10]. We refer the reader for Riesz groups to [16-19].

Lastly, when considering an order ideal I in a Riesz space E, the equivalence class of an element x in E is denoted as $[x] = x + I$ in E/I. This equivalence relation states that $[x^1] = [x^2]$ if and only if $x^1 - x^2 \in I$. Consequently, the mapping from E to E/I that assigns x to $[x]$ is a linear operator referred to as the canonical projection. An order relation denoted as $\leq$ can be defined on E/I, where $[x] \leq [y]$ if there exist $u_{\mathrm{x}} \in [x]$ and $u_{\Upsilon} \in [y]$ such that $u_{\mathrm{x}} \leq u_{\Upsilon}$. This ordered quotient space E/I inherits the structure of an ordered vector space and, additionally, becomes a Riesz space.

3. QUOTIENT RIESZ ALGEBRAS

In this section, we will examine the lattice-ordered structure of quotient l-algebras. Consider an l-algebra denoted as E, and let I be an l-ideal. It follows that E/I also forms an l-algebra, as exemplified in [11,14]. In the context of the quotient E/I, we denote the multiplicative operation as "$\circ$", where $[u] \circ [v] = [u * v]$, and "$*$" represents the multiplicative operation within the l-algebra E. To illustrate this concept, we begin with a fundamental example involving quotient Riesz algebras.

Example 1. Suppose E is an f-algebra. Then, the set $N(\mathrm{E}) = \{\mathrm{x} \in E : \exists k \in N, \mathrm{x}^{\mathrm{k}} = \theta\}$ represents all nilpotent elements in E, as stated in Proposition 3.1. [5]. Consequently, $N(E)$ qualifies as an l-ideal in E. Therefore, the quotient $E/N(E)$ forms a quotient f-algebra.

It is of interest to determine whether the quotient space E/I constitutes an f-algebra, given that E is not an f-algebra. To explore this question, we examine the example provided in Example 1.2(iii) [8].

Example 2. Consider the set E, defined as the Cartesian product of the real numbers $\mathbb{R}^2$, equipped with coordinatewise addition, scalar multiplication, and a partial ordering, thus forming an Archimedean Riesz space. We define the multiplication operation on E as follows:

$$(x_1, x_2) * (y_1, y_2) = \left(x_1y_1 + x_1y_2 + x_2y_1 + \tfrac{1}{2}x_2y_2, \tfrac{1}{2}x_2y_2\right).$$

Hence, the pair $(E,*)$ constitutes an l-algebra, but it does not satisfy the conditions of an f-algebra. Next, let us consider the r-ideal I, defined as the set $\{(a, 0) : a \in \mathbb{R}\}$, within the context of E. For any arbitrary element $(x_1, x_2) \in E$ satisfying the condition $|(x_1, x_2)| \leq |(a, 0)|$ for some $a \in \mathbb{R}$, we can establish that $(x_1, x_2) \in I$ due to the fact that $x_2 = 0$. As a result, I serves as an order ideal within E and consequently qualifies as an l-ideal. Furthermore, it becomes evident that E/I also constitutes an l-algebra.

To demonstrate that E/I is an f-algebra, let us take $[(x_1, x_2)], [(y_1, y_2)] \in E/I$ such that $[(x_1, x_2)] \wedge [(y_1, y_2)] = [(0, 0)]$. According to Theorem 18.9 [14], we deduce the following:

$$[(x_1, x_2)] \wedge [(y_1, y_2)] = [(x_1 \wedge y_1, x_2 \wedge y_2)] = [(0,0)].$$

Therefore, it follows that $(x_1 \wedge y_1, x_2 \wedge y_2) \in I$, thereby implying that $x_1 \wedge\ y_1 \in \mathbb{R}$ and $x_2 \wedge y_2 = 0$. Given that $x_2 \wedge y_2 = 0$, we can conclude that either $x_2 = 0$ or $y_2 = 0$. Without loss of generality, let us assume that $x_2 = 0$ and $y_2 \geq\ 0$. For any positive element $[(z,w)]$ in E/I, we can observe the following:

$$([(z,w)] \circ [(x_1,0)]) \wedge [(y_1,y_2)] = [(z * x_1 + w * x_1, 0)] \wedge [(y_1,y_2)].$$

Hence, we can deduce that $([(z,w)] \circ [(x_1,0)]) \wedge [(y_1,y_2)] = [(0,0)]$. In a similar way, $([(x_1,0)] \circ [(z,w)]) \wedge [(y_1,y_2)] = [(0,0)]$ is satisfied. Consequently, we can affirm that E/I indeed an f-algebra.

The fact that E/I becomes an f-algebra (or an almost f-algebra) when E is an f-algebra (or an almost f-algebra) is widely recognized [5]. We present a proof specifically for the case of a d-algebra.

Proposition 1. Let E a Riesz space. If E is a d-algebra and I is an l-ideal, then the quotient space E/I is a d-algebra.

Proof: Let us assume that E is a d-algebra, which implies that the following properties hold for any elements u and v in E, and w in E_+:

$$(u \vee v) * w = (u * w) \vee (v * w) \text{ and } w * (u \vee v) = (w * u) \vee (w * v)$$

Consider $[u]$ and $[v]$ in E/I, and $[w]$ in $(E/I)_+$:

$$\begin{aligned}[w] \circ ([u] \wedge [v]) &= [w] \circ [u \wedge v] = [w * (u \wedge v)] \\ &= [w * u \wedge w * v] = [w * u] \wedge [w * v] \\ &= [w] \circ [u] \wedge [w] \circ [v]\end{aligned}$$

Similarly, we can demonstrate that $([u] \wedge [v]) \circ\ [w] =\ [u] \circ\ [w] \wedge [v] \circ\ [\mathrm{w}]$. Thus, E/I satisfies all the properties required to be classified as a d-algebra.

Recall that an l-algebra E is referred to as semiprime when the zero element is the only nilpotent element in E. It is known that an algebra E is semiprime if and only if $x^2 = \theta$ in E implies $x = \theta$.

Remark 2. If E is an almost f-algebra, it follows from Lemma 2.2 [20] that $E/N(E)$ is an Archimedean semiprime f-algebra.

It is important to note that, even if E is a semiprime l-algebra, E/I does not necessarily possess the semiprime property for any l-ideal I in E. This is due to the fact that the square of an element being in I does not guarantee that the element itself belongs to I. On the other hand, it is well known that $x \perp y$ if and only if $x \cdot y = \theta$ in semiprime f-algebra; see Theorem 3.7. [10]. Thus, we observe the following fact.

Lemma 3. Let E be a semiprime l-algebra, I is an l-ideal and $x, y \in E_+$. Then $[x] \perp [y]$ if and only if $[x] \circ [y] = [\theta]$ whenever $\mathrm{E/I}$ is a semiprime f-algebra.

Proof: It can be easily seen that $[x] \perp [y]$ implies $[x] \circ [y] = [\theta]$ because E/I is an f-algebra by considering Theorem 3.4. [10]. Now, consider the inequality

$$(|[x]| \wedge |[y]|)^2 \leq |[x]| \circ |[y]| = |[x * y]| = [\theta].$$

Thus, it follows from Theorem 3.7. [10] that $(|[x]| \wedge |[y]|)^2 = [\theta]$ which implies $[x] \wedge [y] = [\theta]$, i.e., $[x] \perp [y]$.

Recall that a sequence (x_n) in a Riesz space is relatively uniformly convergent to x if there exist a sequence $(\varepsilon_n) \downarrow 0$ of real numbers and some $u > \theta$ such that $|x_n - x| \leq \varepsilon_n u$ holds for all n. It is known that every uniformly closed order ideal is an l-ideal in Archimedean f-algebras; see Theorem 10.5.(ii) [10].

Question 4. Is a uniformly closed order ideal an l-ideal in Archimedean l-algebras?

Since we cannot provide a positive answer to the Question 4, we give the following theorem for f-algebras.

Theorem 6. Let I be a uniformly closed order ideal in an l-algebra E. Then E/I is an Archimedean l-algebra.

Proof: We assume that E is an l-algebra with the multiplication $"*"$ and I is a uniformly closed order ideal in E. It follows from Theorem 18.9 [14] that E/I is a Riesz space. Moreover, following from Theorem 2.23. [13], E/I is also Archimedean.

On the other hand, it is not hard to see that $"\circ"$ is an associative algebra on the quotient Riesz space E/I. Moreover, the quotient Riesz space E/I is an l-algebra with multiplication $"\circ"$ given by

$$(\mathrm{x}+\mathrm{I}) \circ (\mathrm{y}+\mathrm{I}) = \mathrm{x} * \mathrm{y} + \mathrm{I}.$$

Indeed, consider two positive elements $[x], [y]$ in E/I. Then there exist u_x, u_y in E_+ such that $[u_x] = [x]$ and $[u_y] = [y]$; see [21]. Thus, by using

$$[u_x * u_y] = [u_x] \circ [u_y] = [x] \circ [y] \text{ and } u_x * u_y \in E_+,$$

we obtain

$$[x] \circ [y] \in \pi(E_+),$$

where π is the canonical projection from E to E/I. Hence, we get the desired result.

If an l-algebra E is Archimedean, then E/I may not be Archimedean in general (see, for example Example 60.1 [12]). The following fact shows the necessary and sufficient conditions for that.

Corollary 7. If A is a uniformly closed order ideal in an f-algebra E, then E/A is an Archimedean f-algebra.

Corollary 8. If A is a uniformly closed order ideal in an f-algebra E with unit e, then $[e]$ is an algebraic unit of E/A.

Remark 9. It is well known that relatively uniform convergence implies order convergence in Archimedean Riesz spaces; see, for example, Lemma 2.2(iii) in [4]. It follows from [2] that E/B is a Dedekind complete Riesz algebra whenever B is a band in a Dedekind complete Riesz algebra E, because E/B is isomorphic to B^d, where B^d is the disjoint complement of B.

4. TOPOLOGY ON THE QUOTIENT ALGEBRAS

It is shown in Theorem 2.24 [12] that if (E, τ) is a locally solid Riesz space and I is an order ideal of E, then $(E/I, \tau_\pi)$is also a locally solid Riesz space, where $\pi: E \to E/I$ is the canonical projection and τ_π is the quotient topology. It follows from Theorem 1.33 [12] that π sends solid sets in E to solid sets in E/I. Moreover, by considering Theorem 2.10.(ii) [12], it can be seen that the image of a base of τ-neighborhoods of zero in E is also a base of τ_π-neighborhoods of zero in E/I.

In this section, we consider the locally full and locally convex topologies on Riesz algebras to obtain similar results as above.

We remind that if the formula

$$T(x \vee y) = T(x) \vee T(y)$$

holds for all elements $x, y \in E$ and an operator $T: E \to F$ between two Riesz spaces, then T is called a Riesz homomorphism. We begin the section with the following useful fact.

Proposition 10. Let E and F be Riesz spaces, and $f: E \to F$ be an onto Riesz homomorphism. Then f maps full sets to full sets.

Proof: Assume that A is a full subset of a Riesz space E and $u, v \in A$ such that $u \le v$. Then it follows from the positivity of f that $f(u) \le f(v)$. Now, we want to show that $[f(u), f(v)] \subseteq f(A)$ holds. To see this, take an arbitrary element $z \in [f(u), f(v)]$. So, there exists an element $x \in E$ such that $f(x) = z$ because f is onto. Then consider $w = (x \vee u) \wedge v$ such that $w \ge u$. Thus, we obtain $w \in [u, v]$ because A is a full set. Then we have

$$f(w) = \big(f(x) \vee f(u)\big) \wedge f(v) = f(x).$$

Hence $f(x) \in [f(u), f(v)]$, and so, $f(A)$ is a full subset of F. It is well known that the canonical map π is a Riesz homomorphism of E onto E/I. Thus, we observe the following result.

Corollary 11. Let I be an l-ideal in a Riesz space E and $\pi: E \to E/I$ be the canonical projection. Then π maps full subsets of E to full subsets of E/I.

The Cartesian product of Riesz spaces under the componentwise order is a Riesz space. Moreover, it is not hard to see that the product of locally full Riesz spaces becomes a locally full Riesz space with the product topology. Thus, we observe the following result.

Theorem 12. Let (E, τ) be a locally full Riesz space and π be an onto Riesz homomorphism from E to F. Then F is also a locally full Riesz space with the quotient topology.

Proof: Assume that a collection N consisting of full sets is a base at zero for τ. One can show that $\{\pi(V): V \in N\}$ is a base of quotient topology τ_π. Also, it follows from Proposition 10 that $\pi(V)$ is a full set for every $V \in N$. Therefore, the linear topology τ_π is locally full.

Corollary 13. Let (E, τ) be a locally full l-algebra and I be an l-ideal in E. Then if $(E/I, \tau_\pi)$ is a locally full topological l-algebra.

Theorem 14. Let (E, τ) be a topological Riesz space, I be an l-ideal of E, and $\pi: E \to E/I$ be the canonical projection. If (E, τ) is locally convex, then $(E/I, \tau_\pi)$ is also a locally convex Riesz space.

Proof: Assume C is a convex subset of E. Take arbitrary two elements $u, v \in \pi(A)$. Then there exist $x, y \in A$ such that $\pi(x) = u$ and $\pi(y) = v$. Fix a scalar $\lambda \in [0,1]$. Thus, we have

$$\lambda\pi(x) + (1-\lambda)\pi(y) = \pi(\lambda x + (1-\lambda)y).$$

As C is convex, $\pi(\lambda u + (1-\lambda)v) \in \pi(C)$. Therefore, π maps convex sets in E to convex sets in E/I. On the other hand, the product of locally convex Riesz spaces becomes a locally convex Riesz space with the product topology. So, for a base N at zero for the topology τ consisting of convex sets, we have $\{\pi(V): V \in N\}$ is a base of quotient topology τ_π. It follows that $\pi(V)$ is a convex set for every $V \in N$. Therefore, the linear topology τ_π is locally convex.

Corollary 15. Let (E, τ) be a topological l-algebra and I be an l-ideal of E. If (E, τ) is locally convex, then $(E/I, \tau_\pi)$ is a locally convex topological l-algebra.

5. CONCLUSIONS

This paper has delved into the realm of quotient Riesz algebras, aiming to deepen our understanding of their properties and shed light on the various topologies associated with them. By exploring the behavior and characteristics of these quotient Riesz algebras within their topological structures, valuable insights have been gained into the intricate nature of these mathematical entities. The findings presented here contribute to the existing body of knowledge in the field, paving the way for further research and advancements in the study of quotient Riesz algebras and their related topologies. This paper serves as a stepping stone toward a more comprehensive understanding of these algebraic structures, with potential applications in areas such as functional analysis.

REFERENCES

[1] Birkhoff, G., *Annals of Mathematics*, **43**(2), 298, 1941.
[2] Aydın, A., *Hacettepe Journal of Mathematics and Statistics*, **49**(3), 998, 2020.
[3] Aydın, A., *Hacettepe Journal of Mathematics and Statistics*, **50**(1), 24, 2021.
[4] Aydın, A., Emelyanov, E., Gorokhova, S., *Indagationes Mathematicae*, **32**(3), 658, 2021.
[5] Bernau, S.J., Huijsmans, C.B., *Mathematical Proceedings of the Cambridge Philosophical Society*, **107**, 287, 1990.
[6] Boulabiar, K., Chil, E., *Demon Mathematics*, **34**, 749, 2001.
[7] Buskes, G., Rooij, A.V., *Positivity*, **4**, 233, 2000.
[8] Huijsmans, C.B., *Lattice ordered algebras and f-algebras: a survey*, Positive operators, Riesz spaces, and economics, Springer, Berlin, 1991.
[9] Huijsmans, C.B., Pagter, B.D., *Transactions of the American Mathematical Society*, **269**, 225, 1982.

[10] Pagter, B.D., *Ph.D. Thesis: f-Algebras and Orthomorphisms*, Leiden, 1981.
[11] Aydın, A., Eroğlu, H.Ü., Ilbıra, S., *FILOMAT*, **36**(7), 2325, 2022.
[12] Aliprantis, C. D., Burkinshaw, O., *Locally solid riesz spaces with applications to economics*, American Mathematical Society, 2003.
[13] Aliprantis, C. D., Burkinshaw, O., *Positive operators*, Springer, Dordrecht, 2006.
[14] Luxemburg, W. A. J., Zaanen, A. C., *Riesz spaces I*, North-Holland Publishing Company, Amsterdam, 1971.
[15] Zaanen, A. C., *Riesz spaces II*, North-Holland Publishing Company, Amsterdam, 1983.
[16] Chuchaev, I.I., *Siberian Mathematical Journal*, **17**(6), 1019, 1976.
[17] Clifford, A.H., *Annals of Mathematics*, **41**, 465, 1940.
[18] Dugundji, J., *Topology*, Allyn and Bacon INC., Boston, 1966.
[19] Fuchs, L., *Annali della Scuola Normale Superiore di Pisa*, **19**, 1, 1965.
[20] Basly, M., Triki, A., *fT-Algebres Archimediennes reticulees*, University of Tunis, 1988.
[21] Aliprantis, C.D., Tourky, R., *Cones and duality*, Graduate Studies in Mathematics, Vol. 84, American Mathematical Society, Providence, 2007.