ORIGINAL PAPER

THE TRANSFORMATION OF THE EVOLUTE CURVES USING LIFTS ON R^3 TO TANGENT SPACE TR^3

HAŞİM ÇAYIR[1], SÜLEYMAN ŞENYURT[2]

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Abstract. *In this paper, firstly, we define the evolute curve of any curve concerning the vertical, complete, and horizontal lifts on space R^3 to its tangent space $TR^3 = R^6$. Secondly, we examine the Frenet-Serret apparatus $\{T^*(s), N^*(s), B^*(s), \kappa^*, \tau^*\}$ and the Darboux vector W^* of the evolute curve α^* according to the vertical, complete and horizontal lifts on TR^3 by depend on the lifting of Frenet-Serret aparatus $\{T(s), N(s), B(s), \kappa, \tau\}$ of the first curve α on space R^3. In addition, we include all special cases the curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ of the Frenet-Serret aparatus $\{T^*(s), N^*(s), B^*(s), \kappa^*, \tau^*\}$ of the evolute curve α^* with respect concerning complete and horizontal lifts on space R^3 to its tangent space TR^3. As a result of this transformation on space R^3 to its tangent space TR^3, we could have some information about the features of the volute curve of any curve on space TR^3 by looking at the characteristics of the first curve α. Moreover, we get the transformation of the evolute curves using bifts on R^3 to tangent space TR^3. Finally, some examples are given for each curve transformation to validate our theoretical claims.*

Keywords: *Vector fields; evolute curve; vertical lift; complete lift; horizontal lift; tangent space.*

1. INTRODUCTION

In differentiable geometry, the lift method has an important role because it is possible to generalize differentiable structures on any space (resp. manifold) to extended spaces (resp. extended manifolds) using the lift function [1-6]. Thus, it may be extended the following theorem given on space R^3 to its tangent space TR^3(see extended frames and curves [7-11]). The evolution of a given curve is a well-known concept in R^3. We can say that evolute and involute are methods of deriving a new curve based on a given curve. Let α and α* be the curves in Euclidean $3-$space. The tangent lines to a curve α generate a surface called the tangent surface of α. If the curve α*, which lies on the tores intersects the tangent lines orthogonally, is called an involute of α. If a curve α* is an involute of α, then by definition α is an evolute of α*. Hence, given α, its evolutes are the curves whose tangent lines intersect α orthogonally. By using the similar method, we produce a new ruled surface based on the other ruled surface. It is well-known that, if a curve is diferentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. These vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define

[1] Giresun University, Faculty of Arts and Sciences, Department of Mathematics, 28100 Giresun, Turkey.
E-mail: hasim.cayir@giresun.edu.tr.
[2] Ordu University, Faculty of Arts and Sciences, Department of Mathematics, Ordu, Turkey.
E-mail: ssenyurt@odu.edu.tr.

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the curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve α, is called the Frenet-Serret apparatus of the curves.

Let Frenet vector fields be $T(s), N(s), B(s)$ of α and let the curvature and torsion of the curve $\alpha(s)$ be κ and τ, respectively. The quantities $\{T(s), N(s), B(s), D, \kappa, \tau\}$ are collectively Frenet-Serret aparatus of the curves. Let a rigid object move along a regular curve described parametrically by α(s). For any unit speed curve α, in terms of the Frenet-Serret aparatus, the Darboux vector can be expressed as [12]

$$\mathrm{D(s)} = \tau(\mathrm{s})\mathrm{T(s)} + \kappa(\mathrm{s})\mathrm{B(s)} \tag{1.1}$$

where curvature and torsion function are defined by $\kappa(s) = \| T^{'}(s) \|$ and $\tau(s) = -\left\langle B^{'}(s), N(s)\right\rangle$. Let a vector field be

$$\widetilde{D}(s) = \frac{\tau}{\kappa}(s)T(s) + B(s) \tag{1.2}$$

along α(s) under the condition that $\kappa(s) \neq 0$ and it is called the modified Darbox vector field of α [13].

Definition 1. Let the unit speed regular curve $\alpha: I \to R^3$ and $\alpha^*: I \to R$ be given and let $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$ be the Frenet frames of the curves α and α^*, respectively. For ∀s ∈ I, if the tangent of curve $\alpha^*(s)$ at the point of $\alpha^*(s)$ pass through point of $\alpha(s)$ and $\langle T^*(s), T(s)\rangle = 0$, curve $\alpha^*(s)$ is called a evolute of curve $\alpha(s)$. The relations between the Frenet frames

$$\{T(s), N(s), B(s)\} \text{and} \{T^*(s), N^*(s), B^*(s)\}$$

are as follows [14, 15]:

$$\begin{aligned} T^*(s) &= \cos(\varphi + c)N - \sin(\varphi + c)B \\ N^*(s) &= -T(s) \\ B^*(s) &= \sin(\varphi + c)N + \cos(\varphi + c)B \end{aligned} \tag{1.3}$$

In addition, the relation between curvatures and the torsions are:

$$\begin{aligned} \kappa^* &= \frac{\kappa^3\cos^3(\varphi + c)}{\kappa\tau\sin(\varphi + c) - \kappa'\cos(\varphi + c)}, \\ \tau^* &= -\frac{\kappa^3\sin(\varphi + c)\cos^2(\varphi + c)}{\kappa\tau\sin(\varphi + c) - \kappa'\cos(\varphi + c)}, \end{aligned} \tag{1.4}$$

where $\measuredangle(\alpha^*(s) - \alpha(s), N(s)) = \varphi(s) + c$ and for $c \in R$

$$\varphi(s) = \int \tau(u)du \tag{1.5}$$

and the evolute of curve $\alpha^*(s)$ is defined by [14, 15]

$$\alpha^*(s) = \alpha(s) + \rho(s)N(s) - \rho(s)\tan[\varphi(s) + c]B(s) \tag{1.6}$$

Theorem 1. [17] Let α^* be evolute of curve α. There are following relation between B binormal vector of curve α at point $\alpha(s)$ and W^* Darboux vector of curve α^* at point $\alpha^*(s)$.

$$W^* = \frac{\kappa^3\cos(\varphi + c)}{\kappa\tau\sin(\varphi + c) - \kappa'\cos(\varphi + c)} B, \tag{1.7}$$

where T, N, B, κ, τ is respectively tangent vector, normal vector, binormal vector, curvature, torsion of the curve $\alpha(s)$.

The paper is structured as follows. Firstly, we define the evolute curve of any curve with respect to the vertical, complete and horizontal lifts on space R^3 to its tangent space $TR^3 = R^6$. Secondly, we examine the Frenet-Serret aparatus $\{T^*(s), N^*(s), B^*(s), \kappa^*, \tau^*\}$ of the evolute curve α^* with respect to the vertical, complete and horizontal lifts on TR^3 by depend on the lifting of Frenet-Serret aparatus $\{T(s), N(s), B(s), \kappa, \tau\}$ of the first curve α on space R^3. Finally, it is revealed the vertical, complete and horizontal lifts of the Darboux vector W^* by depend on the lifting of Frenet-Serret aparatus of the curve α. In this paper, we include all special cases the curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ of the Frenet-Serret aparatus $\{T^*(s), N^*(s), B^*(s), \kappa^*, \tau^*\}$ of the evolute curve α^* with respect to the vertical, complete and horizontal lifts on space R^3 to its tangent space TR^3.

In this study, all geometric objects will be assumed to be of class C^∞ and the sum is taken over repeated indices. Also, v, c and H denote the vertical, complete and horizontal lifts any differentiable geometric structures defined on R^3 to its tangent space TR^3, respectively.

2. LIFT OF VECTOR FIELD

The vertical lift of a vector field X on space R^3 to extended space $TR^3 (= R^6)$ is the vector field $X^v \in \chi(TR^3)$ given as [1,6]:

$$X^v (f^c)=(Xf)^v, \forall f \in F(R^3) \tag{2.1}$$

The vector field $X^c \in \chi(TR^3)$ defined by

$$X^c(f^c) = (Xf)^c, \forall f \in F(R^3) \tag{2.2}$$

is called the complete lift of a vector field X on R^3 to its tangent space TR^3.
The horizontal lift of a vector field X on space R^3 to TR^3 is the vector field $X^H \in \chi(TR^3)$ determined by

$$X^H(f^v) = (Xf)^v, \forall f \in F(R^3) \tag{2.3}$$

the general properties of vertical, complete and horizontal lifts of a vector field on R^3 are as follows:

Proposition 1. Let $f, g \in F(R^3)$ and $X, Y \in \chi(R^3)$. Then the following equalities are satisfied [5, 6, 17].

$$\begin{gathered} (X+Y)^v = X^v + Y^v, (X+Y)^c = X^c + Y^c, (X+Y)^H = X^H + Y^H, \\ (fX)^v = f^v X^v, (fX)^c = f^c X^v + f^v X^c, X^v(f^v) = 0, (fg)^H = 0, \\ X^c(f^v) = X^v(f^c) = (Xf)^v, X^c(f^c) = (Xf)^c, X^H(f^v) = (Xf)^v, \end{gathered} \tag{2.4}$$

$$\chi(\mathrm{U}) = \mathrm{Sp}\left\{\frac{\partial}{\partial x^\alpha}\right\}, \chi(\mathrm{TU}) = \mathrm{Sp}\left\{\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\alpha}\right\},$$
$$\left(\frac{\partial}{\partial x^\alpha}\right)^c = \frac{\partial}{\partial x^\alpha}, \left(\frac{\partial}{\partial x^\alpha}\right)^v = \frac{\partial}{\partial y^\alpha}, \left(\frac{\partial}{\partial x^\alpha}\right)^H = \frac{\partial}{\partial x^\alpha} - \chi\Gamma_\beta^\alpha \frac{\partial}{\partial y^\alpha}.$$

where Γ_β^α are Christoper symbols, U and TU are respectively topolgical opens of R^3 and TR^3, $f^v, f^c \in F(TR^3)$,$X^v, Y^v, X^c, Y^c, X^H, Y^H \in \chi(TR^3)$,$1 \leq \alpha, \beta \leq 3$.

3. LIFTING OF THE EVOLUTE CURVE $\boldsymbol{\alpha}^*(\mathbf{s})$

In this section, we compute the vertical, complete and horizontal lifts of the involute curve $\alpha^*(s)$ with the curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ on space R^3 to TR^3.

3.1. The vertical lifting of the evolute curve $\alpha^*(s)$

Theorem 2. Let $\alpha^*(s)$ be the evolute of the curve α defined by (1.6) on space R^3. Then, we get the following equalities with respect to vertical, complete and horizontal lifts on TR^3.
i)$\alpha^*(s)_1 = (\alpha^*(s))^v = \alpha(s) + \rho(s)N^v(s) - \rho(s)\tan[\varphi(s) + c]B^v(s)$.This is general equation for all evolute curves $\alpha^*(s)$ of curve $\alpha(s)$ on space R^3 to TR^3.
ii)
a)$\alpha^*(s)_2 = (\alpha^*(s))^c = \alpha^c(s) + \rho^c(s)N^c(s) - \rho^c(s)(\tan[\varphi(s) + c])^c B^c(s)$. This is general equation for curves whose curvature κ is not constant on space R^3 to TR^3. For example general helix curve [18], anti Salkowski curve [19].
b)$\alpha^*(s)_2 = (\alpha^*(s))^c = \alpha^c(s)$.This is general equation for curves whose curvature κ is constant on space R^3 to TR^3. For example circular helis curve [20], Salkowski curve [19].
iii)$\alpha^*(s)_3 = (\alpha^*(s))^H = 0$.This is general equation for all evolute curves $\alpha^*(s)$ of curve $\alpha(s)$ on space R^3 to TR^3, where $c = \mathrm{constant}, \forall s \in I$.

Proof.
i) Suppose that $\alpha^*(s)_1 = (\alpha^*(s))^v$ is a unit speed curve $\alpha^*(s)_1 = (\alpha^*(s))^v$ on TR^3 with curvature $(\kappa^*(s))^v$ and torsion $(\tau^*(s))^v$ on TR^3. Also let fbe a function and c be a constant. Then we write $f^v = f$ and $c^v = c$ with respect to vertical lifts on tangent bunde TR^3 [6]. If we apply vertical lift to both sides the equation (1.6), we get
$(\alpha^*(s))^v = \alpha^v(s) + \rho^v(s)N^v(s) - \rho^v(s)(\tan[\varphi(s) + c])^v B^v(s)$
$= \alpha(s) + \rho(s)N^v(s) - \rho(s)\tan[\varphi(s) + c]B^v(s)$.
ii)
a) Let $\alpha^*(s)_2 = (\alpha^*(s))^c$ be a unit speed curve on TR^3 with curvature $(\kappa^*(s))^c$ and torsion $(\tau^*(s))^c$ on TR^3. Let f be a function and c =constant. Then we write $c^c = 0$ with respect to complete lifts on tangent bunde TR^3[6]. If we apply complete lift to both sides the equation (1.6), we get
$(\alpha^*(s))^c = \alpha^c(s) + \rho^c(s)N^c(s) - \rho^c(s)(\tan[\varphi(s) + c])^c B^c(s)$
b) If the curvature κ =constant, we write $\rho^c = (\frac{1}{\kappa})^c = 0$. So, we get $(\alpha^*(s))^c = \alpha^c(s)$.
iii) Suppose that $\alpha^*(s)_3 = (\alpha^*(s))^H$ is a unit speed curve on TR^3 with curvature $(\kappa^*(s))^H$ and torsion $(\tau^*(s))^H$ on TR^3. Let f be a function and c =constant. Then we write $f^H = 0$ and $c^H = 0$ with respect to horizontal lifts on tangent bunde TR^3 [6]. If we take the horizontal lift to both sides the equation (1.6), we get

$$\begin{aligned}(\alpha^*(s))^H &= \alpha^H(s)+\rho^H(s)N^H(s)-\rho^H(s)(\tan[\varphi(s)+c])^H B^H(s)\\ &= \alpha^H(s)+0.N^H(s)-0.0.B^H(s)\\ &= 0.\end{aligned}$$

Theorem 3. Let $\alpha^*(s)$ be the evolute of the curve α on R^3 and a unit speed curve $\alpha^*(s)_1=(\alpha^*(s))^v$ on TR^3 with curvature $(\kappa^*(s))^v$ and torsion $(\tau^*(s))^v$ on TR^3. Then, we get the following equalities with respect to vertical lifts on tangent bunde TR^3.

$$\begin{aligned}(T^*(s))^v &= (\cos(\varphi+c))^v N^v-(\sin(\varphi+c))^v B^v,\\ (N^*(s))^v &= -T^v(s),\\ (B^*(s))^v &= (\sin(\varphi+c))^v N^v+(\cos(\varphi+c))^v B^v,\end{aligned}\tag{3.1}$$

where T, N, B, κ, τ is respectively tangent vector, normal vector, binormal vector, curvature, torsion of the curve $\alpha(s)$.

Proof. If we take the derivative and vertical lift to both sides the equation (1.6), we get

$$\begin{aligned}((\alpha^*(s))')^v &= T^v+(\rho')^v N^v+\rho^v(N')^v-(\rho')^v(\tan(\varphi+c))^v B^v\\ &\quad -\rho^v(\varphi')^v[1+\tan^2(\varphi+c)]^v B^v-\rho^v(\tan(\varphi+c))^v(B')^v\\ &= T^v+(\rho')^v N^v+\rho^v(-\kappa T+\tau B)^v-(\rho')^v(\tan(\varphi+c))^v B^v\\ &\quad -\rho^v(\tau)^v[1+\tan^2(\varphi+c)]^v B^v-\rho^v(\tan(\varphi+c))^v(-\tau N)^v\\ &= [\rho'+\rho\tau\tan(\varphi+c)]^v N^v-(\tan(\varphi+c))^v[\rho'+\rho\tau\tan(\varphi+c)]^v B^v]\\ &= [\rho'+\rho\tau\tan(\varphi+c)]^v[N^v-(\tan(\varphi+c))^v B^v]\\ &= \frac{[\rho'+\rho\tau\tan(\varphi+c)]^v}{(\cos(\varphi+c))^v}[\cos(\varphi+c)N-\sin(\varphi+c)B]^v.\end{aligned}$$

It is easily obtained

$$\|((\alpha^*(s))')^v\| = \frac{[\rho'+\rho\tau\tan(\varphi+c)]^v}{(\cos(\varphi+c))^v}.\tag{3.2}$$

Due to the equality $(T^*(s))^v=\frac{((\alpha^*(s))')^v}{\|((\alpha^*(s))')^v\|}$, we get the following equation

$$(T^*(s))^v = (\cos(\varphi+c))^v N^v-(\sin(\varphi+c))^v B^v.\tag{3.3}$$

If we calculate the $((T^*(s))')^v$ from (3.2), we get

$$\begin{aligned}((T^*(s))')^v &= -((\varphi'))^v(\sin(\varphi+c))^v N^v+(\cos(\varphi+c))^v(-\kappa T+\tau B)^v\\ &\quad -((\varphi'))^v(\cos(\varphi+c))^v+(\sin(\varphi+c))^v\tau^v N^v\\ &= -\kappa^v(\cos(\varphi+c))^v T^v\end{aligned}$$

From the equation $((T^*(s))')^v=\|((\alpha^*(s))')^v\|(\kappa^*)^v(N^*)^v$, we get

‖((α^* (s))^')^v ‖(κ^*)^v (N^*)^v=-κ^v (cos(φ+c))^v T^v (3.4)

Hence, N^* and T vectors are a unit length vector, we have

$$(N^*(s))^v = -T^v\tag{3.5}$$

Furthermore, from $(B^*(s))^v = (T^*(s))^v \times (N^*(s))^v$ we get

$$(B^*(s))^v = (\sin(\varphi + c))^v N^v + (\cos(\varphi + c))^v B^v. \tag{3.6}$$

Corollary 1. Let the curvature κ and torsion τ of the curve $\alpha(s)$ on R^3 be constant or non-constant functions and $\alpha^*(s)$ be the evolute of the curve $\alpha(s)$ on R^3. For a unit speed curve $\alpha^*(s)_1 = (\alpha^*(s))^v$ on with curvature $(\kappa^*(s))^v$ and torsion $(\tau^*(s))^v$ on TR^3, we say the curve $(\alpha^*(s))^v$ on TR^3 is similar structure and apperance to R^3 with respect to vertical lifts.

Example 1. Let $\alpha^*(s)$ be the evolute of the a circular helis curve $\alpha(s)$ on R^3. Then, $\alpha^*(s)$ has similar appearance with the curve $\alpha^*(s)_1 = (\alpha^*(s))^v$ on TR^3. Because of the curvature κ and torsion τ of a circular helis curve is constant, we write $\kappa^v = \kappa$ and $(\tau)^v = \tau$. From (1.4), we get $(\kappa^*(s))^v = \kappa^*(s)$ and $(\tau^*(s))^v = \tau^*(s)$ on TR^3. Then, we can say the curve $\alpha^*(s)_1 = (\alpha^*(s))^v$ has the same $\kappa^*(s)$ and $\tau^*(s)$ on TR^3.

3.2 The complete and horizontal lifting of the evolute curve $\alpha^*(s)$

Theorem 4. Let $\alpha^*(s)$ be the evolute of the curve α on R^3 and a unit speed curve $\alpha^*(s)_2 = (\alpha^*(s))^c$ on TR^3 with curvature $(\kappa^*(s))^c$ and torsion $(\tau^*(s))^c$ on TR^3. Then, we get the following equalities with respect to complete lifts on tangent bunde TR^3.

$$\begin{aligned} &(T^*(s))^c = (\cos(\varphi + c))^c N^c - (\sin(\varphi + c))^c B^c, \\ &(N^*(s))^c = -T^c(s), \\ &(B^*(s))^c = (\sin(\varphi + c))^c N^c + (\cos(\varphi + c))^c B^c, \end{aligned} \tag{3.7}$$

where T, N, B, κ, τ is tangent vector, normal vector, binormal vector, curvature, torsion of the curve $\alpha(s)$, respectively.

Proof. Similarly to vertical lifts, the theorem easily proved with respect to complete lift.

Corollary 2. Let the curvature κ and torsion τ of the curve $\alpha(s)$ on R^3 be non-constant functions (for example general helix curve [18]) and $\alpha^*(s)$ be the evolute of the curve $\alpha(s)$ on R^3. For a unit speed curve $\alpha^*(s)_2 = (\alpha^*(s))^c$ on with curvature $(\kappa^*(s))^c$ and torsion $(\tau^*(s))^c$ on TR^3, we say the curve $(\alpha^*(s))^c$ on TR^3 is similar structure and apperance to R^3 with respect to complete lifts.

Theorem 5. Let the curvature κ and torsion τ of the curve $\alpha(s)$ be constant or non-constant functions on R^3. Everytime, the horizontal lift of basic normal vector of the evolute curve $\alpha^*(s)$ is the same direction with the horizontal lift of the tangent vector $T(s)$ of the principal curve $\alpha(s)$ on TR^3 (the horizontal lift of basic normal vector $(N^*(s))^H$ linear dependency $T^H(s)$ on TR^3), where $\alpha^*(s)$ be the evolute of the curve α on R^3 and a unit speed curve $\alpha^*(s)_3 = (\alpha^*(s))^H$ on TR^3 with curvature $(\kappa^*(s))^H$ and torsion $(\tau^*(s))^H$ on TR^3.

Proof. Let the curvature κ and torsion τ of the curve $\alpha(s)$ be constant or non-constant functions on R^3. For all functions on R^3, we write $f^H = 0$ with respect to horizontal lifts [6]. So, we get $(\cos(\varphi + c))^H = (\sin(\varphi + c))^H = 0$ and $(N^*(s))^H = -T^H(s)$ on TR^3.

Theorem 6. Let $\alpha^*(s)$ be the evolute curve with curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ on R^3. Then, the curvature $(\kappa^*(s))^v$ and torsion $(\tau^*(s))^v$ of a unit speed curve $(\alpha^*(s))^v$ is defined by the following equations with respect to vertical lifts on TR^3.

$$
\begin{aligned}
(\kappa^*)^v &= \frac{(\kappa^3)^v(\cos^3(\varphi+c))^v}{(\kappa)^v(\tau)^v(\sin(\varphi+c))^v-(\kappa')^v(\cos(\varphi+c))^v},\\
(\tau^*)^v &= -\frac{(\kappa^3)^v(\sin(\varphi+c))^v(\cos^2(\varphi+c))^v}{(\kappa)^v(\tau)^v(\sin(\varphi+c))^v-(\kappa')^v(\cos(\varphi+c))^v},
\end{aligned}
\tag{3.8}
$$

where κ,τ is curvature and torsion of the curve $\alpha(s)$, respectively.

Proof. N^* and T vectors are a unit length vector. From the equation (3.4), we have

$$\|((\alpha^*(s))')^v\|(\kappa^*)^v = -\kappa^v(\cos(\varphi+c))^v. \tag{3.9}$$

If we put the equation (3.2) in the equation (3.9), we get

$$
\begin{aligned}
(\kappa^*)^v &= \frac{\kappa^v(\cos(\varphi+c))^v}{\|((\alpha^*(s))')^v\|}\\
&= \frac{\kappa^v(\cos(\varphi+c))^v}{\frac{[\rho'+\rho\tau\tan(\varphi+c)]^v}{(\cos(\varphi+c))^v}}\\
&= \frac{(\kappa^3)^v(\cos^3(\varphi+c))^v}{(\kappa)^v(\tau)^v(\sin(\varphi+c))^v-(\kappa')^v(\cos(\varphi+c))^v}
\end{aligned}
$$

Similarly, from the equation (3.6), we have

$$((B^*(s))')^v = -\kappa^v(\sin(\varphi+c))^v T^v \tag{3.10}$$

If we put the equation (3.10) in the equation

$$((B^*(s))')^v = -\|((\alpha^*(s))')^v\|(\tau^*)^v(N^*)^v,$$

we get

$$
\begin{aligned}
(\tau^*)^v &= \frac{-\kappa^v(\sin(\varphi+c))^v}{\|((\alpha^*(s))')^v\|}\\
&= \frac{-\kappa^v(\sin(\varphi+c))^v}{\frac{[\rho'+\rho\tau\tan(\varphi+c)]^v}{(\cos(\varphi+c))^v}}\\
&= -\frac{(\kappa^3)^v(\sin(\varphi+c))^v(\cos^2(\varphi+c))^v}{(\kappa)^v(\tau)^v(\sin(\varphi+c))^v-(\kappa')^v(\cos(\varphi+c))^v}
\end{aligned}
$$

Theorem 7. Let $\alpha^*(s)$ be the evolute curve with curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ on R^3. Then, the curvature $(\kappa^*(s))^c$ and torsion $(\tau^*(s))^c$ of a unit speed curve $(\alpha^*(s))^c$ is defined by the following equations with respect to complete lifts on TR^3.

$$
\begin{aligned}
(\kappa^*)^c &= \frac{(\kappa^3)^c(\cos^3(\varphi+c))^c}{(\kappa)^c(\tau)^c(\sin(\varphi+c))^c-(\kappa')^c(\cos(\varphi+c))^c},\\
(\tau^*)^c &= -\frac{(\kappa^3)^c(\sin(\varphi+c))^c(\cos^2(\varphi+c))^c}{(\kappa)^c(\tau)^c(\sin(\varphi+c))^c-(\kappa')^c(\cos(\varphi+c))^c},
\end{aligned}
\tag{3.11}
$$

where κ, τ is curvature and torsion of the curve $\alpha(s)$, respectively.

Proof. Similarly to vertical lifts, the theorem easily proved with respect to complete lift.

Corollary 3. If a curve α has a constant curvature κ (for example Salkowski curve [19] and circular helix curve [20]), then, complete lifts of the curvature (κ^*) and torsion (τ^*) of the evolute curve α^* is already zero on TR^3.

Corollary 4. If a curve α has a constant torsion τ (for example anti Salkowski curve [19] and circular helix curve [20]), then, complete lifts of the curvature (κ^*) and torsion (τ^*) of the evolute curve α^* given by the following equalites on TR^3

$$\begin{aligned}(\kappa^*)^c &= (\frac{\kappa^3}{\kappa'})^c(\cos^2(\varphi+c))^c,\\ (\tau^*)^c &= -(\frac{\kappa^3}{\kappa'})^c(\sin 2(\varphi+c))^c\end{aligned} \tag{3.12}$$

Corollary 5. Let $\alpha^*(s)$ be the evolute of the curve α on R^3. Then, the horizontal lifts of the curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ of the evolute curve are already zero on TR^3.

Proof. For all functions on R^3, we write $f^H = 0$ with respect to proporties of the horizontal lifts on TR^3 [6]. So, we can write $(\kappa^*(s))^H = (\tau^*(s))^H = 0$.

3. 3. The Darboux vector of the evolute curve with recpect to vertical, complete and horizontal lifts on TR^3

Theorem 8. Let W^* be the Darboux vector of the evolute curve $\alpha^*(s)$ with curvature $\kappa^*(s)$ and torsion $\tau^*(s)$. Then, we get the Darboux vector $(W^*)^V$ the following equation with respect to vertical lifts on TR^3.

$$(W^*)^v = \frac{(\kappa^3)^v(cos(\varphi+c))^v}{(\kappa)^v(\tau)^v(sin(\varphi+c))^v - (\kappa')^v(cos(\varphi+c))^v}B^v, \tag{3.13}$$

where B, κ, τ is respectively, binormal vector, curvature, torsion of the curve $\alpha(s)$.

Proof. Substituting (3.1) and (3.8) into $(W^*)^v = (\tau^*)^v(T^*)^v + (\kappa^*)^v(B^*)^v$ and using the lift properties, we get the following equation with respect to vertical lifts on TR^3

$$(W^*)^v = \frac{(\kappa^3)^v(cos(\varphi+c))^v}{(\kappa)^v(\tau)^v(sin(\varphi+c))^v - (\kappa')^v(cos(\varphi+c))^v}B^v. \tag{3.14}$$

Corollary 6. Let the curvature κ and torsion τ of the curve $\alpha(s)$ on R^3 be constant or non-constant functions and $\alpha^*(s)$ be the evolute of the curve $\alpha(s)$ on R^3. For a unit speed curve $\alpha^*(s)_1 = (\alpha^*(s))^v$ on with curvature $(\kappa^*(s))^v$ and torsion $(\tau^*(s))^v$ on TR^3, we say the Darboux vector $(W^*)^v$ on TR^3 is similar structure and apperance W^* with respect to vertical lifts.

Example 2. Let $\alpha^*(s)$ be the evolute of the a circular helis curve $\alpha(s)$ on R^3. we say the Darboux vector $(W^*)^v = W^*$ with respect to vertical lifts on TR^3. Because of the curvature κ

and torsion τ of a circular helis curve is constant, we write $\kappa^v = \kappa$ and $(\tau)^v = \tau$. So, we can say the curve $\alpha^*(s)_1 = (\alpha^*(s))^v$ has the same apperance W^* on TR^3.

Theorem 9. Let W^* be the Darboux vector of the evolute curve $\alpha^*(s)$ with curvature $\kappa^*(s)$ and torsion $\tau^*(s)$. Then, we get the Darboux vector $(W^*)^c$ the following equation with respect to complete lifts on TR^3

$$(W^*)^c = \frac{(\kappa^3)^c (cos(\varphi + c))^c}{(\kappa)^c (\tau)^c (sin(\varphi + c))^c - (\kappa')^c (cos(\varphi + c))^c} B^v, \tag{3.15}$$

where κ, τ is curvature and torsion of the curve $\alpha(s)$, respectively.

Proof. Similarly to vertical lifts, the theorem easily proved with respect to complete lift.

Corollary 7. Let the curvature κ and torsion τ of the curve $\alpha(s)$ on R^3 be non-constant functions (for example general helix curve [18]) and $\alpha^*(s)$ be the evolute of the curve $\alpha(s)$ on R^3. For a unit speed curve $\alpha^*(s)_2 = (\alpha^*(s))^c$ with curvature $(\kappa^*(s))^c$ and torsion $(\tau^*(s))^c$ on TR^3, we say the Darboux vector $(W^*)^c$ on TR^3 is similar structure and apperance (W^*) on R^3 with respect to complete lifts.

Corollary 8. Let the curvature κ and torsion τ be constant (for example circular helix curve [20]), we get $\kappa^c = 0$ and $(\tau)^c = 0$. So, $(W^*)^c = 0$. Then, the Darboux vector $(W^*)^c$ is point with respect to complete lifts on TR^3.

Corollary 9. Let the curvature κ and torsion τ of the curve $\alpha(s)$ on R^3 be constant and non-constant functions, respectively (for example Salkowski curve [19]) and $\alpha^*(s)$ be the evolute of the curve $\alpha(s)$ on R^3. Then, the Darboux vector $(W^*)^c$ is a point with respect to complete lifts on TR^3.

Proof. Because of the curvature $\kappa =$ constant of the curve $\alpha(s)$ on R^3, we can write $\kappa^c = 0$. If we put on the equation (3.15), we get $(W^*)^c = 0$.

Corollary 10. Let the curvature κ and torsion τ of the curve $\alpha(s)$ on R^3 be non-constant and constant functions, respectively (for example anti Salkowski curve [19]) and $\alpha^*(s)$ be the evolute of the curve $\alpha(s)$ on R^3. Then, the Darboux vector $(W^*)^c$ on TR^3 is the same direction with the complete lifts of binormal vector of the principal curve $\alpha(s)$ (the Darboux vector $(W^*)^c$ linear dependency B^c on TR^3).

Proof. Because of the torsion $\tau =$ constant of the curve $\alpha(s)$ on R^3, we can write $\tau^c = 0$. If we put on the equation (3.15), we get

$$(W^*)^c = \frac{(\kappa^3)^c}{(\kappa')^c} B^c, \tag{3.16}$$

Theorem 10. Darboux vector $(W^*)^H$ with respect to horizontal lifts on TR^3 is a point everytime .

Proof. From Theorem 5, we get $(\kappa)^H = (\tau)^H = 0$. So, $(W^*)^H = 0$ with respect to horizontal lifts on TR^3. The theorem is proved.

4. CONCLUSION

In this study, using lifting methods, we see that it may be generalized the evolute curve $\alpha^*(s)$ and Frenet-Serret aparatus of the evolute curve $\alpha^*(s)$ given by (1.3),(1.4),(1.7) with recpect to vertical, complete and horizontal lifts on TR^3. As a result of this transformation on space R^3 to its tangent space TR^3, we could have some information about the features of evolute curve of any curve on space TR^3 by looking at the characteristics of the first curve α. In addition, it can be created various corresponding examples from space R^3 to its tangent space $TR^3 = R^6$.

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