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THEORETICAL ANALYSIS AND NUMERICAL SOLUTION OF LINEAR AND NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. *In this paper, a collocation method based on the Haar wavelet is presented for the solution of both linear and nonlinear first-order neutral delay differential equations. The Haar functions are used to approximate the first-order derivative, and the approximate solution is obtained by using initial condition and integration. Some examples from the literature are used to test the suggested method efficiency and applicability. A comparison of exact and approximate solutions is given in figures for different numbers of collocation points. The root mean square and maximum absolute errors are calculated for different numbers of collocation points. The rate of convergence is calculated which is approximately equal to 2. The comparison of the present method with the other numerical methods is also given. The results demonstrate that the Haar wavelet collocation method is simple and effective for solving first-order linear and nonlinear neutral delay differential equations.*

Keywords: *Delay differential equations; neutral delay differential equations; Haar wavelet; numerical approximation.*

1. INTRODUCTION

One of the most important types of delay differential equations is neutral type differential equations with proportional delays. To approximate the solution of neutral delay differential equations (NDDEs), several numerical algorithms have been developed in the literature. Chen and Wang [1] used an accurate numerical method termed the variational iteration method, {for the numerical} solution of the neutral functional delay differential equation (NFDDE). They were able to obtain an approximate analytical solution by only following a few steps in the procedure. They compared the efficiency of the method to that of specific Runge-Kutta and one-leg methods. They put this strategy to the test in a variety of scenarios to see how well it worked. Based on the Bezier surface form, Ghumanjani and Farahi [2] developed an effective approach for computing the approximate solution of DDEs. They determined the optimal residual function control points that resolve the approximate DDE solution. They present numerical examples to investigate the correctness and effectiveness of the developed method. Ibis and Baryam [3] proposed a collocation approach {in terms of} Hermite polynomials to {approximate solution} of NFDDEs. They used collocation points and said polynomials to convert these equations and the provided condition into a linear system with unknown coefficients. They solved this system of linear equations to find these unknown coefficients. They checked various examples and compared the method to existing ways to demonstrate its effectiveness.

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For the numerical solution of functional integro DDEs, Gumgum et al. [4] presented the numerical matrix-collocation approach. They used residual error analysis to promote the achieved answers and evaluated certain cases to ensure the validity of the method. Giyas Sakar [5] utilized the residual error function to enhance the homotopy analysis approach with the optimal determination of auxiliary parameters for the numerical solution of NFDEs. He discovered that this method is so simple and useful after performing error estimation and convergence analysis on it, as well as comparing it to other ways and testing it on several samples. Barton et al. [6] proposed two collocation approaches, direct discretization of NDDE and related DDE with a difference equation, for the periodic solutions of NDDEs. They applied several examples to check the order of convergence of these methods. Hussien [7] suggested a numerical scheme based on the Chebyshev series to approximate the numerical solution of NDDEs, which is applicable to neutral and advanced DDEs with single or multiple delay terms. Li and Zhang [8] reviewed discontinuous Galerkin methods L_{∞} error estimates, for the analytical solution of DDEs. They justified the theoretical settlements with numerical problems. Maghami and Ulsoy [9] presented a new analytical approach for obtaining the entire solution for DDE systems based on the concept of Lambert functions.

Aziz and Amin [10] developed a numerical scheme in terms of Haar wavelet for the approximate solution of a specific family of DDEs. This method may be used to approximate linear and nonlinear DDEs, and systems comprising these DDEs. They also improved the method for solving delayed PDEs numerically. Lee and Kung [11] used the finite-dimensional shifted Legendre polynomial expansion to find the solution to time-invariant linear systems with time delay. They obtained an integration matrix and a delay matrix for the shifted Legendre vector, to minimize the linear time delay state equation's solution to the linear algebraic equation's solution. For approximating delay systems with inverse time, Tao Wang [12] proposed hybrid functions using Legendre polynomials and universal block pulse functions. He extended the delay and inverse time functions using hybrid functions and therefore computed the numerical solution. Staelen et al. [13 developed a finite difference approach, to approximate the solution of partial DDE. They checked the convergence, stability, and order of approximation of the developed method. Zhong and Zhang [14] used the linearized compact difference technique to approximate the numerical solution of nonlinear delay PDE. They calculated the approach stability and unconditional convergence. They also used numerical examples to defend the theoretical settlements.

Yan and Zhang [15] used the block boundary value method to solve nonlinear functional differential and functional equations. Both authors [16] also developed a technique for solving a nonlinear hybrid system with distributed delay using a combination of block boundary value approaches and reducible quadrature rules. In [17] Zhang and Yan expand the class of block boundary value methods to approximate the solutions of nonlinear DDEs with algebraic constraints and piecewise continuous arguments. Yan et al. [18] found nonlinear DDEs with proportional delay that are exponentially transformed into nonlinear DDEs with constant delay.

Xu and Lin [19] proposed a simpler reproducing kernel approach for numerically solving fractional DDEs with starting value problems. The proposed method was significant since it created a new reproducing kernel space that confirms the original circumstances. They employed the simple replicating kernel method (SRKM) to arrive at a valid estimate. Liu et al. [20] proposed an approach for the stability of analysis of the system of NDDEs, which includes a descriptor model transformation. They began by proving that the original and converted sets of NDDEs and DDAEs respectively are equivalent. Then the influence on stability analysis is numerically evaluated using the Chebyshev discretization of the characteristic equations and a delay-independent stability criterion. The discontinuous solution of NDDEs was discussed by Baker and Paul [21]. Since discontinuous derivatives may arise in the solutions of implicit and explicit NDDEs, however, it has not been recognized enough that the solutions of NDDEs and as a result solutions of DDAEs do not have to be continuous. They demonstrated and discussed how discontinuities emerge, as well as gave out some computational strategies for dealing with these issues. For implicit nonlinear NDDEs, a stable numerical scheme was proposed by Vermiglio and Torelli [22]. To develop efficient numerical systems with stability qualities, they use implicit nonlinear NDDEs. Their hypothesis on the actual problem allows us to use the theory of stability concerning the forcing term to investigate boundedness and asymptotics of true numerical solutions.

In this paper, we will discuss the Haar wavelet collocation method (HWCM) for the numerical solution of first-order linear and nonlinear NDDE. Consider the following linear NDDE

$$Q^{/}(t)a_1(t) = Q(t)a_2(t) + Q^{/}(t-\xi)a_3(t) + Q(t-\xi)a_4(t) + f(t),\ 0 \le t \le 1, \tag{1}$$

and nonlinear NDDE

$$Q^{/}(t) - Q(t-\xi)Q^{/}(t-\xi)c(t) = -Q(t)a(t) + g(Q(t-\xi))b(t),\ 0 \le t \le 1, \tag{2}$$

with initial condition $Q(0) = Q_0$, and delay condition is

$$Q(t) = \phi(t), \quad \text{for} \quad t \in (-1, 0), \tag{3}$$

where a_λ, for $\lambda = 1, 2, 3, 4$, $a(t)$, $b(t)$, $c(t)$, and $g(t)$ are given functions.

2. THEORETICAL ANALYSIS

We derive some results using a fixed-point technique for the problem (2), which addresses the existence and uniqueness of the solution. Problem (2) is now written as

$$Q^{/}(t) - Q(t-\xi(t))Q^{/}(t-\xi)c(t) = -Q(t)a(t) + g(Q(t-\xi(t)))b(t),\ 0 \le t \le 1, \tag{4}$$

where a, b, c and $f : [0, \infty] \to R$ are continuous functions, the function g satisfies a Lipschitz condition; i.e. $\exists\ L > 0$ and $l > 0$ such that g satisfies

$$|g(x) - g(y)| \le L|x - y|, \quad \text{for} \quad x, y \in [-l, l], \tag{5}$$

An initial condition for the NDDE (3) is defined as $Q(0) = Q_0$, and the delay condition as $Q(t) = \varphi(t)$ for $t \in (r_0, 0)$, where $\varphi \in C([r_0, 0], R)$. Here $C([r_0, 0], R)$ denotes the set of all continuous functions $\varphi : [r_0, 0] \to R$ with the supremum norm $\|.\|$. For $\varphi \in C([r_0, 0], R)$, we call a continuous function $Q(t, \varphi)$ to be a solution of Eq. (2) with the initial condition if $Q : [r_0, a) \to R$ for some positive constant $a > 0$ satisfies

$$\frac{d}{dt}\left(Q(t) - \frac{2\left(1-\xi^{/}(t)\right)}{Q(t-\xi)^2 c(t)}\right) = -Q(t)a(t) + g\left(Q\left(t-\xi(t)\right)\right)b(t) - \frac{d}{dt}\left(\frac{c(t)}{2\left(1-\xi^{/}(t)\right)}\right)Q^2\left(t-\xi(t)\right), \tag{6}$$

on $[0, a)$ and $Q = \varphi$ on $[\xi_0, 0]$. Now we define

$$S_{\varphi}^{l}=\left\{\varphi\in C\left(\left[\xi_0,\infty\right]\right):\|\varphi\|=\sup_{t\geq\xi_0}|\varphi(t)|\leq l,\ \varphi(t)=\phi(t)\right\}\text{for t}\in\left[\xi_0,0\right],\varphi(t)\to 0\ \text{ as } t\to\infty$$

Then S_{φ}^{l} is a complete metric space with metric $\rho(x,y)=\sup_{t\geq\xi_0}\{|Q(t)-y(t)|\}$. Let z(t) = ϕ (t) on [ξ_0 , 0], and let Q(t) = p(t)z(t) for t $_0$. If z satisfies

$$z'(t)=-\left(a(t)-\frac{p'(t)}{p(t)}\right)z(t)+\frac{c(t)p(t-\xi(t))}{p(t)}z^2(t-\xi(t))+\frac{c(t)p^2(t-\xi(t))}{p(t)}z(t-\xi(t))z'(t)$$
$$+\frac{b(t)p(t-\xi(t))}{p(t)}g\left(z\left(t-\xi(t)\right)\right), \tag{7}$$

then it can be verified that Q satisfies Eq. (6).

Definition 2.1. [23] The zero solution of Eq. (2) is said to be stable if, for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\varphi : [\xi_0, 0] \to (-\delta, \delta)$ implies that $Q(t) < \epsilon$ for $t \geq 0$.

Definition 2.2. *[*23] The zero solution of Eq. (2) is said to be asymptotically stable if it is stable and there exists a $\delta > 0$ such that for any initial function $\varphi : [\xi_0, 0] \to (-\delta, \delta)$, the solution Q(t) with Q(t) = φ(t) on [ξ_0, 0] tends to zero as $t \to \infty$.

Theorem 2.3. [23] Consider the NDDE (2) and the following conditions are satisfied:
(i) $\xi(t)$ is twice differentiable with $\xi'(t)\neq 1$and $t-\xi(t)\to\infty\ as\ t\to\infty$,
(ii) $\exists$ a bounded function $p:[\xi_0,\infty)\to(0,\infty)$ with p(t) = 1 for $t\in[\xi_0,0)$ such that p$'$(t) exists on $[\xi_0,0)$ and there exists a constant $\alpha\in(0,1)$ and an arbitrary continuous function $v:[\xi_0,1)\to R$ such that

$$l\left\{|p^2(t\xi(t))c(t)p(t)(1-\xi(t))|+\int_0^t|\bar{k}(s)-2b_1(s)|e^{-\int_0^t v(u)du}ds\right\}+L\int_0^t e^{-\int_x^t v(u)du}\frac{|b(s)|p(s-\xi(s))^{\gamma}}{p(s)}ds$$
$$+\int_{t-\xi(t)}^t|v(s)-a(s)\frac{p'(s)}{p(s)}|ds+\int_0^t e^{-\int_x^t v(u)du}|v(s)|\int_{s-\xi(s)}^s|v(u)-a(u)-\frac{p'(u)}{p(u)}|duds$$
$$+\int_0^t e^{-\int_s^t v(u)du}|v(s(\xi(s))-a(s-\xi(s))-\frac{p'(s-\xi(s))}{p(s-\xi(s))}|1-\xi'(s)|ds$$
$$\leq\alpha,$$

where

$$\bar{k}(s)=\frac{\left[\bar{c}(s)v(s)+\bar{c}'(s)\right](1-\xi'(s))+\bar{c}''(s)}{\xi}(1-\xi'(s))^2, \tag{8}$$

$$\bar{c}(s)=\frac{c(s)p^2(s-\xi(s))}{p(s)},\qquad b_1(s)=\frac{c(s)p(s-\xi(s))p'(s-\xi(s))}{p(s)}, \tag{9}$$

and the constants l, L are defined as in Eq. (5),

(iii) and such that and such that $\lim_{t\to\infty}\int_0^t v(s)ds > -\infty$, then the zero solution of Eq. (2) isasymptotically stable if and only if $\int_0^t v(s)ds \to \infty,\ as\ t \to \infty$.

Proof: We just need to show that the zero solution of Eq. (7) is asymptotically stable because $p(t)$ is a positive bounded function. If $e^{\int_0^t v(s)ds,}$ is multiplied on both and then integrate from 0 to t, we get

$$\begin{aligned} z(t) &= \phi(0)e^{-\int_0^t v(s)ds} + \int_0^t \left(v(s) - a(s) - \frac{p'(s)}{p(s)} \right) e^{-\int_s^t v(u)du} z(s)ds \\ &+ \int_0^t e^{-\int_s^t v(u)du} \frac{c(s)\,p(s-\xi(s))\,p'(s-\xi(s))}{p(s)} z^2(s-\xi(s))ds \\ &+ \int_0^t e^{-\int_s^t v(u)du} \frac{c(s)\,p^2(s-\xi(s))}{p(s)} z(s-\xi(s))z'(s-\xi(s))ds \\ &+ \int_0^t e^{-\int_s^t v(u)du} \frac{b(s)\,p(s-\xi(s))\gamma}{p(s)} g(z(s-\xi(s)))ds, \end{aligned} \tag{10}$$

Let $\int_0^t e^{-\int_s^t v(u)du} \frac{c(s)\,p^2(s-\xi(s))}{p(s)} z(s-\xi(s))z'(s-\xi(s))ds$

$$= \int_0^t e^{-\int_s^t v(u)du} \frac{c(s)\,p^2(s-\xi(s))}{p(s)} z(s-\xi(s))z'(s-\xi(s))ds \times (1-\xi'(s)) \frac{1}{1-\xi'(s)} \tag{11}$$

using integration by parts the right-hand side of (10), we have

$$\int_0^t e^{-\int_s^t v(u)du} \frac{c(s)\,p^2(s-\xi(s))}{p(s)} z(s-\xi(s))z'(s-\xi(s))ds$$

$$\begin{aligned} &= \frac{p^2(t-\xi(s))}{2p(t)} \frac{c(t)}{1-\xi'(t)} z^2(t-\xi(t)) - \frac{p^2(-\xi(0))}{2p(0)} \frac{c(0)}{1-\xi'(0)} \phi^2(--\xi(0)) \times e^{-\int_0^t v(s)ds} \\ &- \frac{1}{2}\int_0^t e^{-\int_{ss}^t v(u)du} \bar{k}(s) z^2(s-\xi(s))ds, \end{aligned} \tag{12}$$

where $\bar{k}$ (s) is given in Eq. (7). Using integration by parts, we have

$$= \int_0^t e^{-\int_s^t v(u)du} \frac{c(s)\,p^2(s-\xi(s))}{p(s)} z(s-\xi(s))z'(s-\xi(s))ds \times (1-\xi'(s)) \frac{1}{1-\xi'(s)} \tag{13}$$

$$\int_0^t \left(v(s) - a(s) - \frac{p^{/}(s)}{p(s)} \right) e^{-\int_s^t v(u)du} z(s)ds = \int_0^t e^{-\int_s^t v(u)du} d\left(\int_{s-\xi(s)}^{s} v(u) - a(u) - \frac{p^{/}(u)}{p(u)} z(u)du \right)$$

$$= \int_0^t e^{-\int_s^t v(u)du} d\left(\int_{s-\xi(s)}^{s} v(u) - a(u) - \frac{p^{/}(u)}{p(u)} z(u)du \right)$$
$$+ \int_0^t e^{-\int_s^t v(u)du} \left(v(s) - \xi(s) - a(s-\xi(s)) - \frac{p^{/}(s-\xi(s))}{p(s-\xi(s))} \right) \left(1 - \xi^{/}(s)\right) z(s-\xi(s))ds,$$
$$= \int_{t-\xi(t)}^{t} \left(v(s) - a(s) - \frac{p^{/}(s)}{p(s)} \right) z(s)ds \tag{14}$$
$$= -\int_0^t e^{-\int_s^t v(u)du} v(s) \int_{s-\xi(s)}^{s} \left(v(u) - a(u) - \frac{p^{/}(u)}{p(u)} \right) z(u)duds$$
$$+ \int_0^t e^{-\int_s^t v(u)du} \left(v(s-\xi(s)) - a(s-\xi(s)) - \frac{p^{/}(s-\xi(s))}{p(s-\xi(s))} \right) \left(1 - \xi^{/}(s)\right) z(s-\xi(s))ds,$$

Combining Eqs. (9), (11), and (13), we obtain the solution of Eq. (6).

3. NUMERICAL METHOD

In this section, we consider a linear NDDE defined in Eq. (1) and we will develop HWCM for its numerical solution. Initially, suppose that $Q^{/}(t)$ is a square integrable function and we can write it as a sum of following Haar functions

$$Q^{/}(t) = \sum_{\lambda=1}^{N} a_\lambda h_\lambda(t), \tag{15}$$

by process of integration, we get

$$Q(t) = Q(0) + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t), \tag{16}$$

using initial condition, we have

$$Q(t) = Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t). \tag{17}$$

Case 1. Applying the Haar approximation, the Eq. (1) become

$$\sum_{\lambda=1}^{N} a_\lambda h_\lambda(t) a_1(t) - \left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t)\right) a_2(t)$$
$$= \begin{cases} \phi^{/}(t-\xi) a_3(t) + \phi(t-\xi) a_4(t) + f(t), & \text{for } t<0, \\ \sum_{\lambda=1}^{N} a_\lambda h_\lambda(t-\xi) a_3(t) + \left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t-\xi)\right) a_4(t) + f(t), & \text{for } t>0, \end{cases} \quad (18)$$

After simplification, we have

$$\sum_{\lambda=1}^{N} a_\lambda h_\lambda(t) a_1(t) - \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t) a_2(t)$$
$$= \begin{cases} Q_0 a_2(t) + \phi^{/}(t-\xi) a_3(t) + \phi(t-\xi) a_4(t) + f(t), & \text{for } t<0, \\ Q_0 a_2(t) + \sum_{\lambda=1}^{N} a_\lambda h_\lambda(t-\xi) a_3(t) + \left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t-\xi)\right) a_4(t) + f(t), & \text{for } t>0, \end{cases} \quad (19)$$

Now putting collocation points (CPs) t_j, j = 1, 2, 3, . . . , N , the above equation becomes

$$\sum_{\lambda=1}^{N} a_\lambda h_\lambda(t_j) a_1(t_j) - \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t_j) a_2(t_j)$$
$$= \begin{cases} Q_0 a_2(t_j) + \phi^{/}(t_j-\xi) a_3(t_j) + \phi(t_j-\xi) a_4(t_j) + f(t_j), & \text{for } t_j<0, \\ Q_0 a_2(t_j) + \sum_{\lambda=1}^{N} a_\lambda h_\lambda(t_j-\xi) a_3(t_j) + \left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t_j-\xi)\right) a_4(t_j) + f(t_j), & \text{for } t_j>0, \end{cases} \quad (20)$$

In matrix notation this system can be written as,

$$KA = B, \quad (21)$$

where

$$B=[b\lambda]_{N\times1}, \quad A=[a\lambda]_{N\times1}, \quad K=[k\lambda j]_{N\times N},$$

and

$$k_{\lambda j} = \begin{cases} h_\lambda(t_j) a_1(t_j) - p_{\lambda,1}(t_j) a_2(t_j), & \text{for } t_j<0 \\ h_\lambda(t_j) a_1(t_j) - p_{\lambda,1}(t_j) a_2(t_j) - h_\lambda(t_j-\xi) a_3(t_j) - p_{\lambda,1}(t_j-\xi) a_4(t_j), & \text{for } t_j>0. \end{cases}$$

$$b_j = \begin{cases} Q_0 a_2(t_j) + \phi^{/}(t_j-\xi) a_3(t_j) + \phi(t_j-\xi) a_4(t_j) + f(t_j), & \text{for } t_j<0, \\ Q_0 a_2(t_j) + Q_0 a_4(t_j) + f(t_j), & \text{for } t_j>0. \end{cases}$$

Hence a_λ, λ = 1, 2, 3, . . . , N can be calculated as $A = K^{-1}B$. Finally, by substituting a_λ, λ = 1, 2, 3, . . . N in Eq.(17) we obtained approximate solution at the CPs.

Case 2. In this section, we consider a nonlinear NDDE defined in Eq. (2) and develop HWCM for its solution. Suppose that $Q^{/}(t)$ is a square integrable function and can write as a sum of following Haar functions

$$Q^{/}(t) = \sum_{\lambda=1}^{N} a_\lambda h_\lambda(t), \quad (24)$$

by process of integration, we get

$$Q(t) = Q(0) + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t), \tag{25}$$

using initial condition, we have

$$Q(t) = Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t). \tag{26}$$

Applying the Haar approximation, the Eq. (3) become

$$\sum_{\lambda=1}^{N} a_\lambda h_\lambda(t) - \phi(t-\xi)\phi'(t-\xi)c(t) = \left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t)\right)a(t) + g\left(\phi'(t-\xi)\right)b(t), \quad \text{for } t<0, \tag{27}$$

$$\sum_{\lambda=1}^{N} a_\lambda h_\lambda(t) - \left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t-\xi)\right)\sum_{\lambda=1}^{N} a_\lambda h_\lambda(t-\xi)c(t) = -\left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t)\right)a(t) + g\left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t-\xi)\right)b(t), \quad \text{for } t>0, \tag{28}$$

Let

$$F = \begin{cases} \sum_{\lambda=1}^{N} a_\lambda h_\lambda(t) - \phi(t-\xi)\phi'(t-\xi)c(t) + \left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t)\right)a(t) - g\left(\phi'(t-\xi)\right)b(t) = 0, & \text{for } t<0, \\ \sum_{\lambda=1}^{N} a_\lambda h_\lambda(t) - \left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t-\xi)\right)\left(\sum_{\lambda=1}^{N} a_\lambda h_\lambda(t-\xi)\right)c(t) + \left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t)\right)a(t) - g\left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t-\xi)\right)b(t) = 0, & \text{for } t<0, \end{cases} \tag{29}$$

Discretizing this equation at CPs t_j , j = 1, 2, . . , N, we have

$$F = \begin{cases} \sum_{\lambda=1}^{N} a_\lambda h_\lambda(t_j) - \phi(t_j-\xi)\phi'(t-\xi)c(t_j) + \left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t_j)\right)a(t_j) - g\left(\phi'(t_j-\xi)\right)b(t_j) = 0, & \text{for } t_j<0, \\ \sum_{\lambda=1}^{N} a_\lambda h_\lambda(t_j) - \left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t_j-\xi)\right)\left(\sum_{\lambda=1}^{N} a_\lambda h_\lambda(t_j-\xi)\right)c(t_j) + \left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t_j)\right)a(t_j) - g\left(Q_0 + \sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t_j-\xi)\right)b(t_j) = 0, & \text{for } t_j<0, \end{cases} \tag{30}$$

To solve this system by Broyden's method we obtain the Jacobian matrix J which is the partial differentiation of the above equation with respect to unknown a_λ, $\lambda = 1, 2, 3, \ldots, N$ as follows

$$J = \begin{cases} h_\lambda(t_j) - p_{\lambda,1}(t_j)a(t_j) = 0, & \text{for } t_j<0, \\ h_\lambda(t_j) - \left(p_{\lambda,1}(t_j-\xi)\sum_{\lambda=1}^{N} a_\lambda h_\lambda(t_j-\xi)\right)c(t_j) - \left(\sum_{\lambda=1}^{N} a_\lambda p_{\lambda,1}(t_j-\xi)h_\lambda(t_j-\xi)\right)c(t_j) + p_{\lambda,1}(t_j)a(t_j) - g\left(p_{\lambda,1}(t_j-\xi)\right)b(t_j) = 0, & \text{for } t_j<0, \end{cases} \tag{31}$$

Finally, by substituting a_λ, $\lambda = 1, 2, 3, \ldots, N$ in Eq. (26) we obtained approximate solutionat the CPs.

4. NUMERICAL EXPERIMENTS

In this section, we use HWCM to solve some problems. To demonstrate the method efficiency, we compare approximate solutions with exact solutions and the results are shownin tables and figures for each example. If Q_{ex} denote the exact solution and Q_{ap} denote the approximate solution at N CPs. Then the maximum absolute error (MAE) is defined as

$$MAE = \max |Q_{ex} - Q_{ap}|,$$

and root mean square error (RMSE) is defined by

$$RMSE = \sqrt{\frac{\sum_{\lambda=1}^{N}\left(Q_{ex}(t_j) - Q_{ap}(t_j)\right)^2}{N}}.$$

Also we calculate experimental rate of convergence $R_c(N)$ which is given as

$$R_c(N) = \frac{\log\left[\frac{MAE\left(\frac{N}{2}\right)}{MAE(N)}\right]}{\log(2)}.$$

Problem 1. Consider the following first-order NDDE with proportional delay

$$Q'(t) = -Q(t) + 0.1Q(0.8t) + 0.5Q'(0.8t) + (0.32t - 0.5)e^{-0.8t} + e^{-t}, 0 \le t \le 1,$$

with initial condition Q(0) = 0 and which has the exact solution $Q(t) = e^{-t}$ [24-26].

Table 1. MAE, RMSE, R_c(N) and CPU time for Problem 1

J	$N=2^{J+1}$	MAE	R_c(N)	RMSE	R_c(N)
1	4	1.225690×10^{-02}	---	7.357118×10^{-03}	---
2	8	3.453724×10^{-03}	1.8274	1.853621×10^{-03}	1.9888
3	16	9.178345×10^{-04}	1.9118	4.643077×10^{-04}	1.9972
4	32	2.366578×10^{-04}	1.9554	1.161334×10^{-04}	1.9993
5	64	6.009072×10^{-05}	1.9776	2.903688×10^{-05}	1.9999
6	128	1.514016×10^{-05}	1.9888	7.259441×10^{-06}	1.9999
7	256	3.799832×10^{-06}	1.9944	1.814874×10^{-06}	1.9999
8	1012	9.518139×10^{-07}	1.9972	4.537194×10^{-07}	1.9999
9	2024	2.381859×10^{-07}	1.9986	1.134299×10^{-07}	1.9999

Table 2. Comparisons of MAE errors of present method with other methods for Problem 1

Present method	Runge Kutta method [24]	One-leg θ method [25]	variational iteration method [26]
1.13×10^{-07}	8.68×10^{-04}	4.65×10^{-03}	1.30×10^{-03}

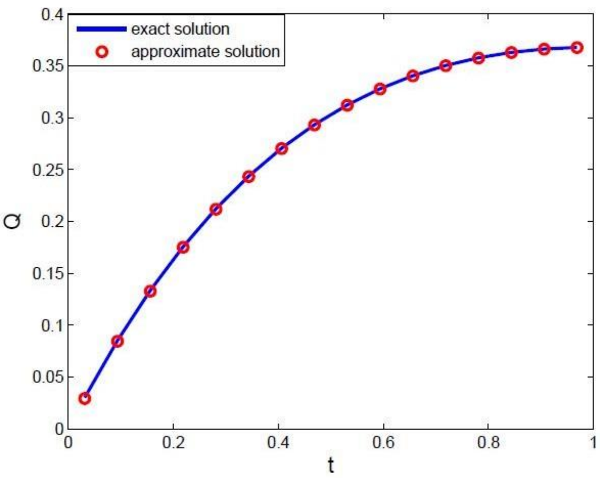

Figure 1. Comparison of exact and approximate solution for 16 CPs of Problem 1

Problem 2. Consider the following first order nonlinear NDDE

$$Q'(t) = Q^2(t) + \cos(0.25t)Q(0.2t) - \sin(0.2t)Q'(0.25t) + \cos(t) - \sin^2(t), \qquad 0 \le t \le 1,$$

with initial condition $Q(0) = 0$ and which has the exact solution $Q(t) = \sin(t)$.

Table 3. MAE, RMSE, $R_c(N)$ and CPU time for Problem 2

J	$N=2^{J+1}$	MAE	$R_c(N)$	RMSE	$R_c(N)$	CPU time(seconds)
1	4	8.499963×10^{-03}	---	4.187245×10^{-03}	---	0.013119
2	8	1.934085×10^{-03}	1.8471	1.043434×10^{-03}	2.0047	0.081571
3	16	5.139050×10^{-04}	1.9121	2.617939×10^{-04}	1.9948	0.574093
4	32	1.311445×10^{-04}	1.9703	6.560273×10^{-05}	1.9966	3.402875
5	64	3.036939×10^{-05}	2.1105	1.605393×10^{-05}	2.0308	9.930585
6	128	5.063225×10^{-06}	2.5845	3.609328×10^{-06}	2.1531	13.326847

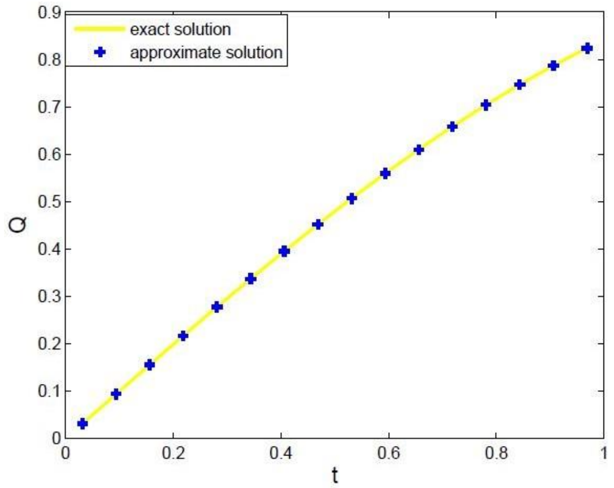

Figure 2. Comparison of exact and approximate solution for 16 CPs of Problem 2

Problem 3. Consider the following first order nonlinear NDDE

$$Q'(t)Q(t) + \sqrt{\cos(t)}Q'(t) + \left(\sin(t) + e^t\right)Q\left(\sin(t)\right) = 2e^t + \sqrt{\cos(t)}e^{\sqrt{t}} + \left(\sin(t) + e^t\right)e^{\sin(t)}, \qquad 0 \le t \le 1,$$

with initial condition $Q(0) = 0$ and which has the exact solution $Q(t) = e^t$.

Table 4. MAE, RMSE, $R_c(N)$ and CPU time for Problem 3

J	$N=2^{J+1}$	MAE	$R_c(N)$	RMSE	$R_c(N)$	CPU time(seconds)
1	4	8.499963×10^{-03}	---	7.818018×10^{-03}	---	0.015475
2	8	2.159772×10^{-03}	1.9766	1.935762×10^{-03}	2.0139	0.127520
3	16	5.461964×10^{-04}	1.9834	4.809678×10^{-04}	2.0088	1.484438
4	32	1.305248×10^{-04}	2.0651	1.141202×10^{-04}	2.0754	10.481233
5	64	3.825854×10^{-05}	1.7705	3.226524×10^{-05}	1.8225	15.12185
6	128	1.640562×10^{-05}	1.2216	9.838679×10^{-06}	1.7134	20.08573

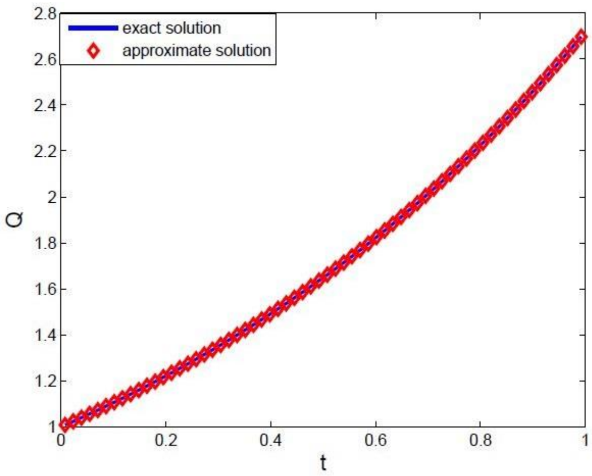

Figure 3. Comparison of exact and approximate solution for 64 CPs of Problem 3

5. RESULTS AND DISCUSSION

The HWCM is applied to some examples available in the literature. The first order derivative is approximated by the Haar function and the process of integration is used for expression of the approximate solution. By putting the CPs in given NDDEs, we obtain a system of algebraic equations. The Gauss elimination method is used for the solution of this linear system. By solving this system we obtain the unknown Haar coefficients $a_i, i = 1, 2, \ldots N$.

Two errors MAE and RMSE are calculated for different numbers of CPs. Tables show that by increasing the number of CPs both the errors are decreased. The rate of convergence is also calculated which is approximately equal to 2 which confirms the theoretical results proved by Majak et al [27]. Table 2 shows the comparison of the present method with the Runge-Kutta method [24], one-leg θ-method [25], and variational iteration method [26] of absolute errors. The comparison of exact and approximate solution for different number of CPs is also shown in figure. The figures show that the approximate solution is close to the exact solution.

6. CONCLUSIONS

In this work, we consider first order linear and nonlinear NDDEs studied HWCM for solution of these equations. The NDDEs are converted to the system of algebraic equations. This system is solved by Gauss elimination method. The MAE and RMSE are calculated for each example. A comparison of our technique with Runge-Kutta method [24], one-leg θ-method [25], and variational iteration method [26] is also given. The comparison of exact and approximate solutions is given in figure. All computational calculations are done in Matlab software.

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