ORIGINAL PAPER

OPTICAL ART OF A PLANAR IDEMPOTENT DIVISOR GRAPH OF COMMUTATIVE RING

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Abstract. *The idempotent divisor graph of a commutative ring R is a graph with a vertex set in R* = R-{0}, where any distinct vertices x and y are adjacent if and only if x.y = e, for some non-unit idempotent element e^2 = e ∈ R, and is denoted by Л(R). Our goal in this work is to transform the planar idempotent divisor graph after coloring its regions into optical art by depending on the reflection of vertices, edges, and planes on the x or y-axes. That is, we achieve Op art solely through pure mathematics in this paper.*
Keywords: *Zero divisor graph; idempotent elements; planar graph; idempotent divisor graph and Op Art.*

1. INTRODUCTION

Let R be a finite commutative ring with identity 1≠ 0. In 1988 Beck [1], introduced the relationship between ring and graph theories when he studied the coloring of commutative ring. Later in 1999, Anderson and Livingston [2], modified this model by studying the zero divisor graph Γ(R) which has vertices Z(R)* = Z(R)-{0} and for x_1, x_2 ∈ Z(R)* with $x_1 \neq x_2$, {x_1, x_2} is an edge in Γ(R) if and only if $x_1 \cdot x_2 = 0$ where Z(R) be the set of all zero divisor elements of R. Many authors studied this notion [3-4]. Also, there are many other definitions that connect these two theories of graph and ring, for example see [5-7]. In 2021, Mohammad and Shuker [8], gave the definition of the idempotent divisor graph of commutative ring R with identity 1≠ 0, it is a graph denoted by Л(R) which has vertices set in R* = R-{0}, and for any distinct vertices x and y, x adjacent with y if and only if x.y = e for some non-unit idempotent element e^2 = e in R. If R is a local ring, then the only non-unit idempotent element in R is zero, they assumed that V(Л(R))=$Z(R)^*$, this means that Л(R)= Γ(R). Furthermore they presented some fundamental properties of this graph, and they studied planarity of this graph. In 2022 [9], we find the clique number, the chromatic number and the region chromatic number for every planar idempotent divisor graphs of commutative rings. We obtained to beautiful graphs after coloring the regions of these graphs; this led to do this paper. Mathematics and art painting are two worlds that may seem completely separate, the first one involving logic, accuracy and truth, and the other involving sensual expression, emotion, and aesthetics. But what unites in these two worlds is more interconnected than some think. Art is closely related to mathematics through the reliance of most painters on purely mathematical concepts to draw their paintings, such as their use of the golden ratio, the Pythagorean Theorem, the concepts of reflection and other mathematical concepts. On the other hand,

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there have recently appeared mathematical studies aimed at analyzing the quality of artistic paintings or obtaining artistic paintings by mathematical methods [10-11].

In a graph theory, for the connected simple graph G. Let H be a graph, G∪H is a graph with V(G∪H)=V(G)∪V(H) and E(G∪H)=E(G)∪E(H). G+H is a graph with V(G∪H)=V(G)∪V(H) and E(G+H)= E(G)∪E(H) ∪{{u,v}: u∈V(G),v∈V(H)}. Let u and v are non-adjacent vertices in the graph G, then G+{{u,v}} is a graph with V(G+{{u,v}})=V(G) and E(G+{{u,v}})=E(G)∪{u,v}. A graph is called complete if every two of its vertices are adjacent, and the complete graph of order n is denoted by K_n, and K_1 is a graph which has only one vertex. A cycle graph of order n is denoted by C_n is a graph with $V(C_n)=\{v_i$: i=1,2,…,n} and $E(C_n)=\{\{v_j,v_{j+1}\}$: j=1,2,…, n-1}∪{(v_1,v_n)}. A planar graph G is said to be n-region colorable if the regions of G can be colored with n or fewer colors so that adjacent regions are colored differently. The region chromatic number $\chi^*(G)$ of a planar graph G is the minimum n for which G is n-region colorable.” For more details see [12].

In a ring theory, let R be a finite commutative ring with identity. R is called to be local if it contains a single maximal ideal. The ring R is a direct product of the rings R_i for i=1,2,…,n, if $R\cong R_1\times R_2\times \ldots \times R_n=\{(r_1,r_2,\ldots,r_n) : r_i \in R_i\}$. If R is a non-local then R is direct product of local rings. Z_n, F_q are the ring of integer modulo n and a field of order q respectively, where n is a positive integer and q is a power of prime number [13].

In abstract art, optical art is a current in the plastic arts in modern art that has been called *Op Art* (short for optical art) and it appears in what has been called “abstract visual agitations” i.e. works that provoke in the viewer a psycho-physiological reaction by using the artist’s drawings with wavy effects that excite the eye It confuses it to suggest a movement based on the principle of deceiving the eye based on the overlapping of geometric shapes, which are usually black and white. Beginning in 1965 Bridget Riley began to produce color-based op art; however, other artists, such as Julian Stanczak and Richard Anuszkiewicz, were always interested in making color the primary focus of their work. Josef Albers taught these two primary practitioners of the “Color Function” school at Yale in the 1950s. Often, colorist work is dominated by the same concerns of figure-ground movement, but they have the added element of contrasting colors that produce different effects on the eye. For instance, in Anuszkiewicz’s “temple” paintings (Fig. 1), the juxtaposition of two highly contrasting colors provokes a sense of depth in illusionistic three-dimensional space so that it appears as if the architectural shape is invading the viewer’s space.

a) **b)**

Figure 1. Intrinsic Harmony, by Richard Anuszkiewicz (1965). For more details see [14-15].

In section two, we presented the conditions we adopted for drawing the graph and coloring its regions depending on the algebraic properties of the vertices of this graph, for any planar idempotent divisor graph. Finally, in the third section we drew all planar idempotent divisor graphs of order n (for 4≤n≤ 26) depending on the conditions of the second section.

2. PLANAR IDEMPOTENT DIVISOR GRAPH OF COMMUTATIVE RING

We begin by the following interesting lemmas which have been obtained in [8].

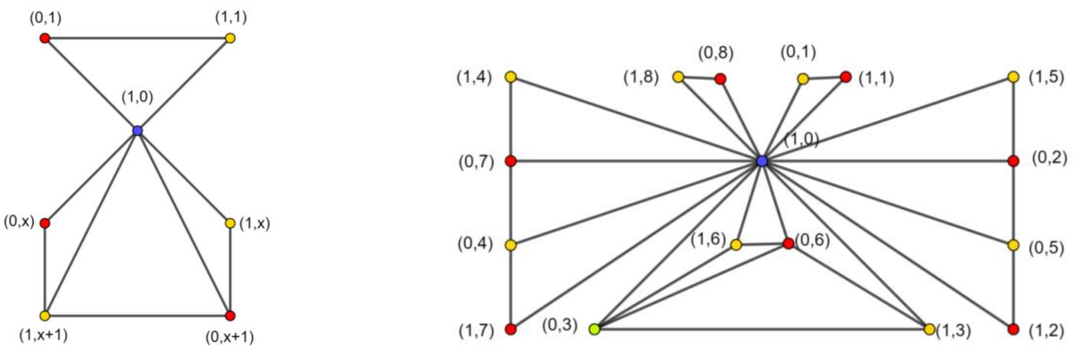
Lemma 2.1. For any local ring R, a graph Л(R) is planar if and only if R is isomorphic to one of the following rings of the Table 2.1:

Table 2.1. The planar idempotent divisor graphs of local rings.

Ring(s) type	Graph
Z_4 or $F_2[Y]/(Y^2)$	K_1
Z_9 or $F_3[Y]/(Y^2)$	K_2
$F_2[Y_1, Y_2]/(Y_1, Y_2)^2$, $Z_4[Y] / (2, Y)^2$, $F_4[Y]/(Y^2)$ or $Z_4[Y]/(Y^2 + Y + 1)$	K_3
Z_8, $Z_4[Y]/(2Y, Y^2 - 2)$ or $F_2[Y]/(Y^3)$	$K_{1,2}$
Z_{25} or $F_5[Y]/(Y^2)$	K_4
Z_{27}, $F_3[Y]/(Y^3)$ or $Z_9[Y]/(3Y,\ Y^2 \pm 3)$	$K_2 + 6K_1$
Z_{16}, $F_2[Y]/(Y^4)$, $Z_4[Y]/(2Y, 2Y^2, Y^3 - 2)$, $Z_4[Y]/(\ Y^2 - 2Y - 2)$, or $Z_4[Y]/(Y^2 - 2)$	$K_1 + (4K_1 \cup K_2)$
$Z_8[Y]/(2Y, Y^2 - 4)$, $Z_4[Y]/(Y^2 - 2Y)$, $F_2[Y_1, Y_2]/(Y_1^2, Y_2^2 - Y_1Y_2)$ or $Z_4[Y_1, Y_2]/(Y_1^2, Y_2^2 - Y_1Y_2, Y_1Y_2 - 2, 2Y_1, 2Y_2)$	$K_1 + (K_2 \cup C_4)$
$Z_4[Y_1, Y_2]/(Y_1^2, Y_2^2, Y_1Y_2 - 2, 2Y_1, 2Y_2)$, $Z_4[Y]/(Y^2)$ or $F_2[Y_1, Y_2]/({Y_1}^2, {Y_2}^2)$.	$K_1 + 3K_2$"

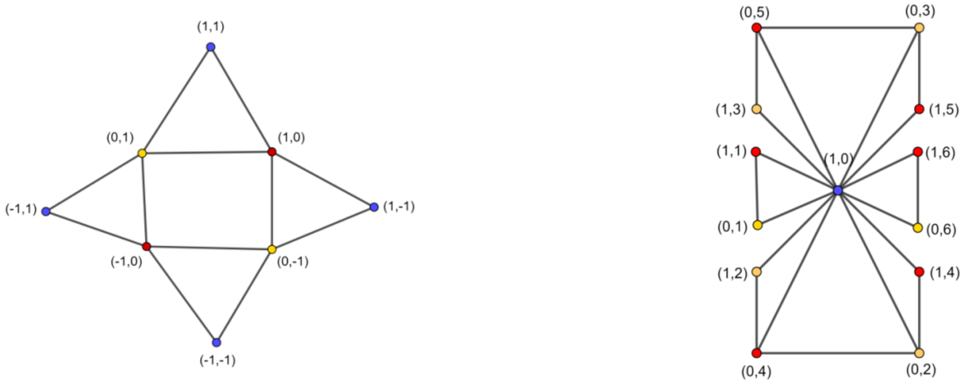
Lemma 2.2. [8] Let R is a non-local ring, then the graph Л(R) is planar graph if and only if R isomorphic one of the rings: $F_2 \times F_q, F_3 \times F_q, F_2 \times Z_4, F_2 \times F_2[Y]/(Y^2)\ , F_2 \times Z_9$ or $F_2 \times F_3[Y]/(Y^2)$, where q is a power of prime.

Example 2.3. Let R≅ $F_2 \times F_4$, $F_2 \times Z_9$, $F_3 \times F_3$, $F_2 \times F_7$, $F_3 \times F_4$, $F_3 \times F_5$ or $F_3 \times F_8$, then the idempotent divisor graph of R as follows in Figs. 2-5:



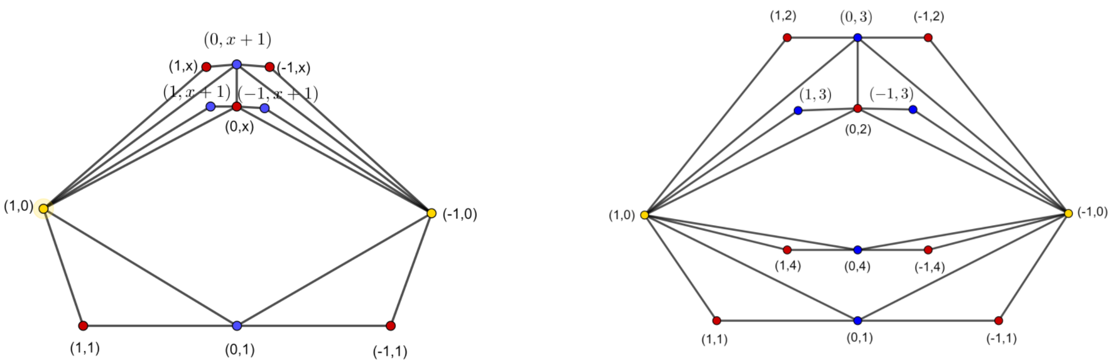
(a) (b)

Figure 2. (a) Л($F_2 \times F_4$) (b) Л($\mathbf{F_2}$ **x** $\mathbf{Z_9}$) ≅ Л($\mathbf{F_2}$ **x** $\mathbf{F_3[Y]}$ / $\mathbf{(Y^2)}$)



(a) (b)

Figure 3. (a) Л$(\mathbf{F_3} \times \mathbf{F_3})$ **(b) Л$(\mathbf{F_2} \times \mathbf{F_7})$**



(a) (b)

Figure 4. (a) Л$(\mathbf{F_3} \times \mathbf{F_4})$ **(b) Л$(\mathbf{F_3} \times \mathbf{F_5})$**

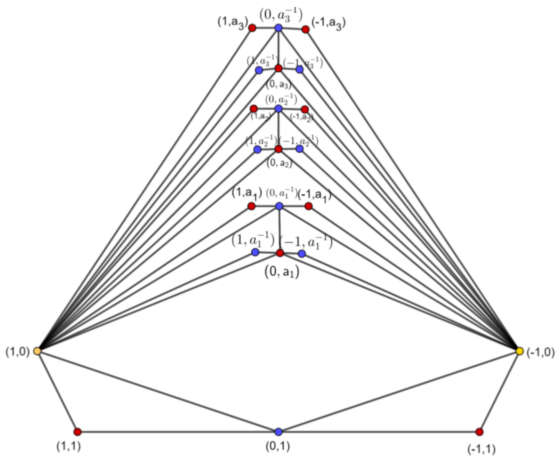


Figure 5. Л$(\mathbf{F_3} \times \mathbf{F_8})$

The following table we get it in [9]:

Table 2.4. The region coloring for the planar idempotent divisor graphs of commutative rings.

The ring R	Type of R	The order of Л(R)	χ^*(Л(R))
Z_4 or $F_2[Y]/(Y^2)$	Local	1	1
Z_9 or $F_3[Y]/(Y^2)$	Local	2	1
$F_2[Y_1, Y_2]/(Y_1, Y_2)^2$, $Z_4[Y]\ /\ (2, Y)^2$, $F_4[Y]/(Y^2)$ or $Z_4[Y]/(Y^2 + Y + 1)$	Local	3	2

The ring R	**Type of R**	**The order of Л(R)**	$\chi^*(Л(\mathbf{R}))$
Z_8, $Z_4[Y]/(2Y, Y^2-2)$ or $F_2[Y]/(Y^3)$	Local	3	1
Z_{25} or $F_5[Y]/(Y^2)$	Local	4	4
Z_{16}, $F_2[Y]/(Y^4)$, $Z_4[Y]/(2Y, 2Y^2, Y^3-2)$, $Z_4[Y]/(Y^2-2Y-2)$, or $Z_4[Y]/(Y^2-2)$.	Local	7	2
$Z_8[Y]/(2Y, Y^2-4)$, $Z_4[Y]/(Y^2-2Y)$, $F_2[Y_1, Y_2]/(Y_1^2, Y_2^2-Y_1Y_2)$ or $Z_4[Y_1, Y_2]/(Y_1^2, Y_2^2-Y_1Y_2, Y_1Y_2-2, 2Y_1, 2Y_2)$.	Local	7	3
Z_{27}, $F_3[Y]/(Y^3)$ or $Z_9[Y]/(3Y, Y^2\pm 3)$.	Local	8	3
$Z_4[Y_1, Y_2]/(Y_1^2, Y_2^2, Y_1Y_2-2, 2Y_1, 2Y_2)$, $Z_4[Y]/(Y^2)$ or $F_2[Y_1, Y_2]/(Y_1{}^2, Y_2{}^2)$.	Local	7	2
$F_2 \times Z_4$ or $F_2 \times F_2[Y]\ /\ (Y^2)$	Non-local	7	2
$F_2 \times Z_9$ or $F_2 \times F_3[Y]\ /\ (Y^2)$	Non-local	17	3
$F_2 \times F_2$	Non-local	3	2
$F_2 \times F_3$	Non-local	5	2
$F_3 \times F_3$	Non-local	8	2
$F_2\times F_{p^\alpha}$, for any prime number p and any positive integer α with $p^\alpha \geq 4$.	Non-local	$2p^\alpha - 1$	3
$F_3\times F_{p^\alpha}$, for any prime number p and any positive integer α with $p^\alpha \geq 4$.	Non-local	$3p^\alpha - 1$	3”

Now, from the properties of graph and ring theories, we give the following remarks.

Proposition 2.5. Let R be any ring. Then for any distinct vertices a,b∊V(Л(R)), the vertex a is adjacent with the vertex b if and only if –a adjacent with –b.

Proof: Let a — b, then $\exists 1 \neq e \in I(R)$ such that ab=e, then we have $(-a)(-b) = -(-(ab))$ =ab= e. The converse is clear.

Remark 2.6. Let R be any ring and a,b∈V(Л(R)), If 0≠a≠ −a and 0≠b≠ −b then we can draw the graph Л(R) such that the vertices a and b are images of the reflection of the vertices –a and –b respectively on the Y-axis in the graph Л(R), also the edge {a,b} is an image of the reflection of the edge{–a,–b} on the Y-axis (or X-axis) in the graph Л(R). Also, any region in Л(R) is an image of the reflection on the Y-axis for deferent region in this graph. For example, if R=Z_{27}, then the vertices 3,6 and 12 are images of the reflection of the 24, 21 and 15 respectively on the Y-axis in the graph Л(Z_{27}), Fig. 6**.**

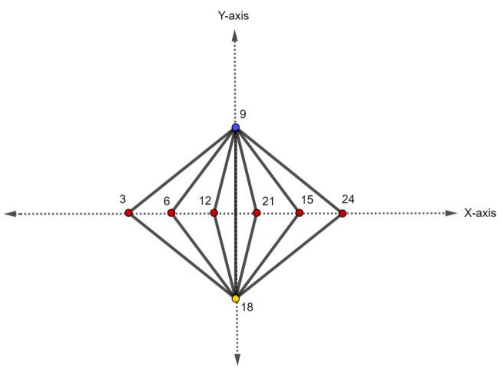
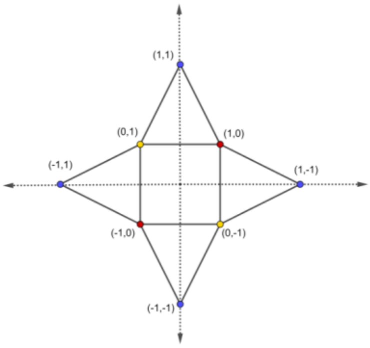


Figure 6. Л(Z_{27})

Remark 2.7. If F_p be a field of order non-even prime number p, then a≠-a for any non-zero a∊F_p. If R≅F_pxF_q where p and q are non-even primes, then by Remark 2.6 we can draw the graph Л(R) such that any vertex or edge of Л(R) is an image of the reflection of such two deferent vertices or edges respectively on the x-axis and y-axis in the graph Л(R). For example, see the following graphs, $Л(F_3 \times F_3)$ and $Л(F_3 \times Z_5)$, Fig. 7.



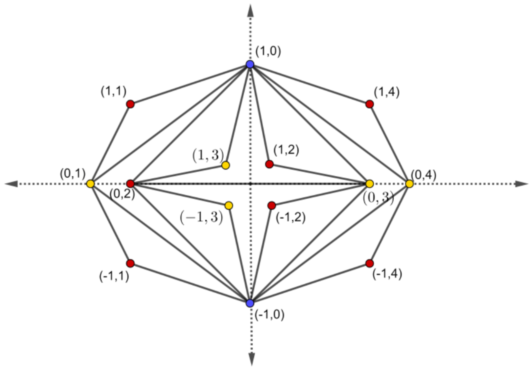


Figure 7. (a) $Л(\mathbf{F_3} \times \mathbf{F_3})$ **(b)** $Л(\mathbf{F_3} \times \mathbf{Z_5})$

Also, any region in $Л(F_p \times F_q)$ is an image of the reflection on the x-axis or y-axis for deferent region in this graph.

Proposition 2.8. Let $R \cong F_2 \times F_q$ where q be a prime number, then:
(1) The vertex (1,0) adjacent with every other vertices of R^* in Л(R).
(2) The vertex (1,y) adjacent with the vertex $(0,y^{-1})$ if and only if the vertex (0,y) adjacent with $(1,y^{-1})$ for any $y \neq 0$.

Proof:
(1) Let $(a,b) \in R^*-\{(1,0)\}$, then (1,0)(a,b)=(a,0), since $a \in F_2$ then $(a,0) \in I(R)-\{1\}$, therefore (1,0) adjacent with every other vertices of R^*.
(2) The proof is clear.

Remark 2.9. Let $R \cong F_2 \times F_q$ where q be a prime number, then for any $-1,0 \neq y \in F_q$ we can draw the graph Л(R) such that the vertices (1,y) and $(0,y^{-1})$ are images of the reflection of the vertices (0,y) and $(1,y^{-1})$ respectively on the Y-axis in the graph Л(R), also the edge $\{(1,y), (0,y^{-1})\}$ is an image of the reflection of the edge$\{(0,y), (1,y^{-1})\}$ on the Y-axis (or X-axis) in the graph Л(R), so that we can put the vertex (1,0) in the origin. Also, any region in Л(R) is an image of the reflection on the Y-axis for deferent region in this graph. For example, if $R=F_2 \times Z_7$ then we can draw the graph Л(R) as following, in Fig. 8:

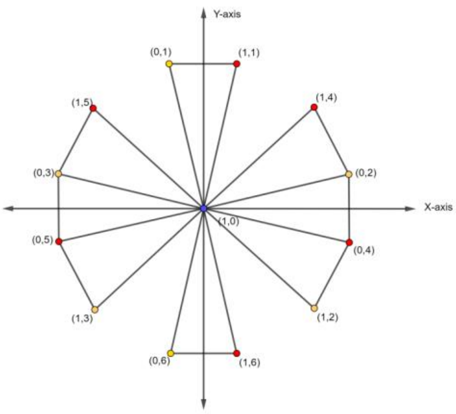


Figure 8. $Л(\mathbf{F_2} \times \mathbf{Z_7})$

3. OPTICAL ART OF A PLANAR IDEMPOTENT DIVISOR GRAPH OF COMMUTATIVE RING

The purpose of this section is to transform the planar idempotent divisor graph after color its region into optical art by depending on Remarks 2.6, 2.7, and 2.8, in the last section. We shall draw and color the regions of all planar idempotent divisor graph of order n, where 4≤n≤ 26. We use the colors red, yellow, blue and green to color the regions of these graphs.

3.1. LOCAL CASE

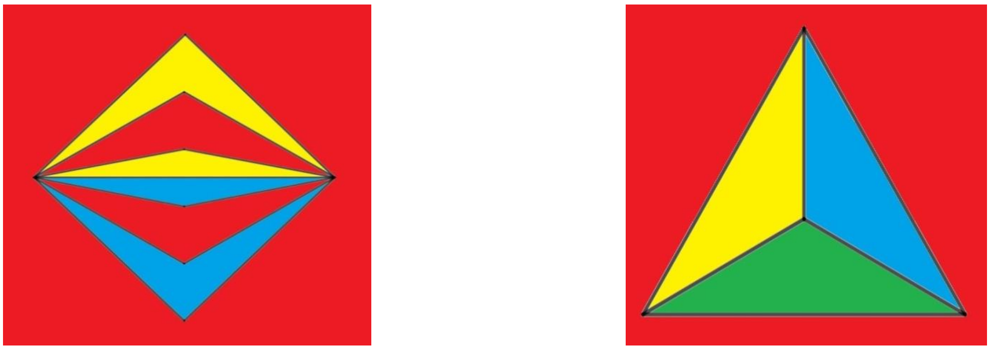


Figure 9. (a) $\mathbf{K_2 + 6K_1}$**; (b)** $\mathbf{K_4}$

We see that the image of the color of regions of $K_2 + 6K_1$ in Fig. 9(a) is a similar as the Image b in Fig. 1.

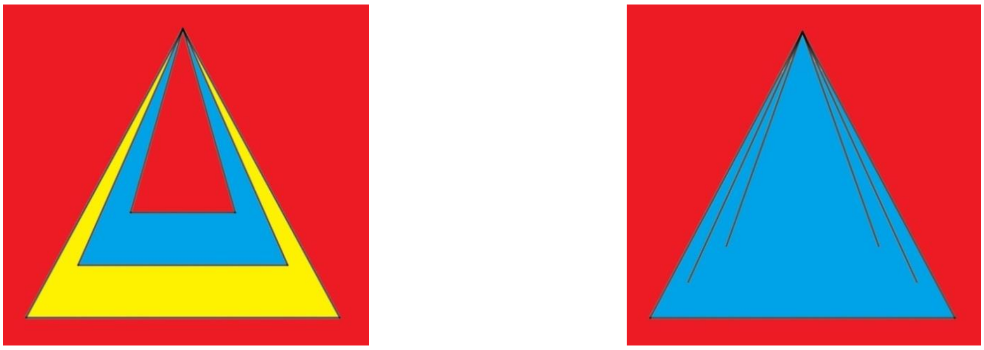


Figure 10. (a) $\mathbf{K_1 + 3K_2}$**; (b)** $\mathbf{K_1 + (4K_1 \cup K_2)}$

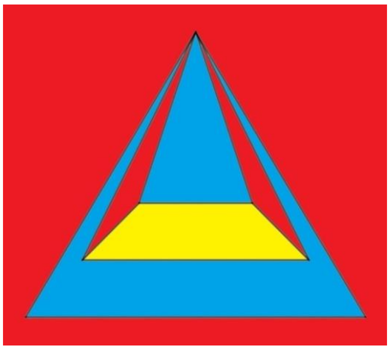


Figure 11. $\mathbf{K_1 + (K_2 \cup C_4)}$

3.2. NON-LOCAL CASE

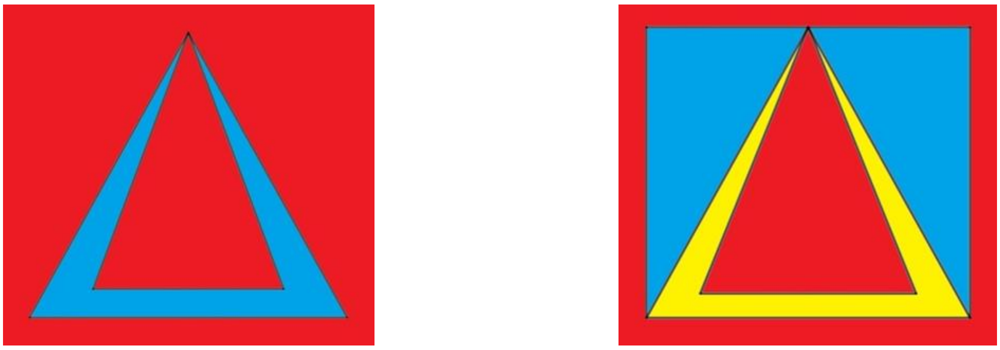


Figure 12. (a) Л$(F_2 \times F_3)$; (b) Л$(F_2 \times F_4)$

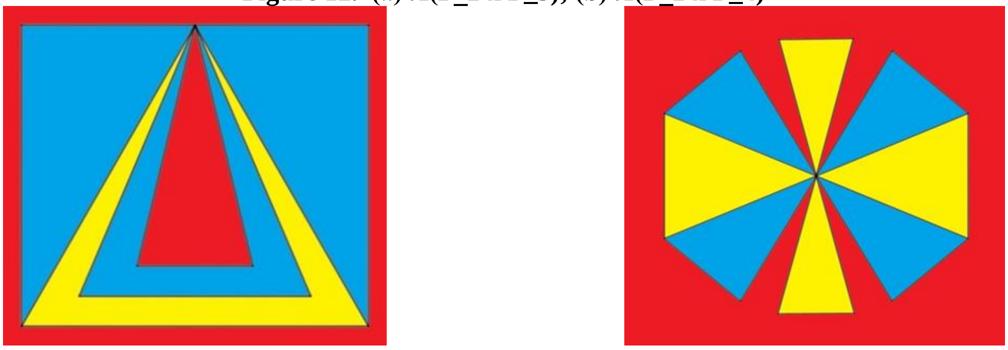


Figure 13. – (a) Л$(F_2 \times F_5)$; (b) Л$(F_2 \times F_7)$

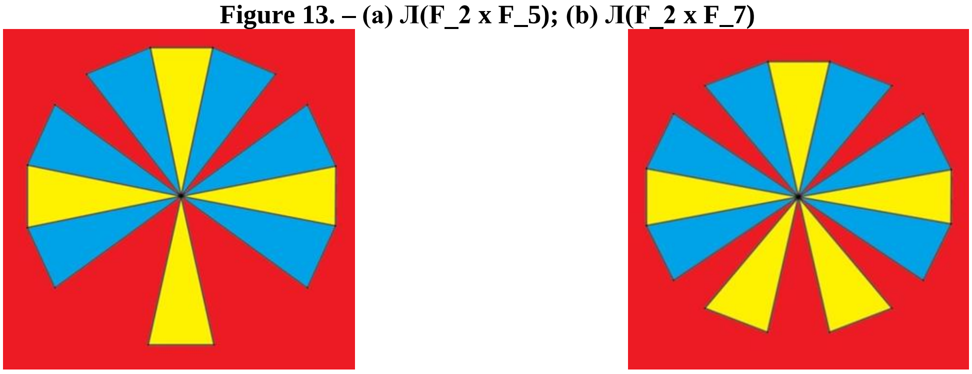


Figure 14. – (a) Л$(F_2 \times F_8)$; (b) Л$(F_2 \times F_9)$

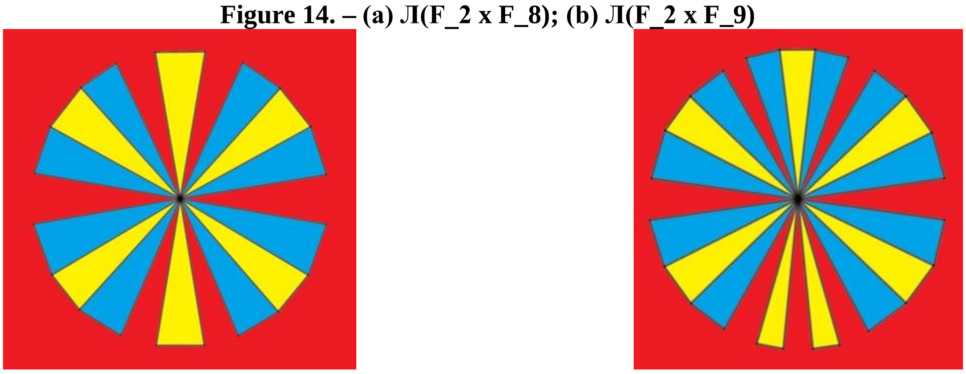


Figure 15. – (a) Л$(F_2 \times F_{11})$; (b) Л$(F_2 \times F_{13})$

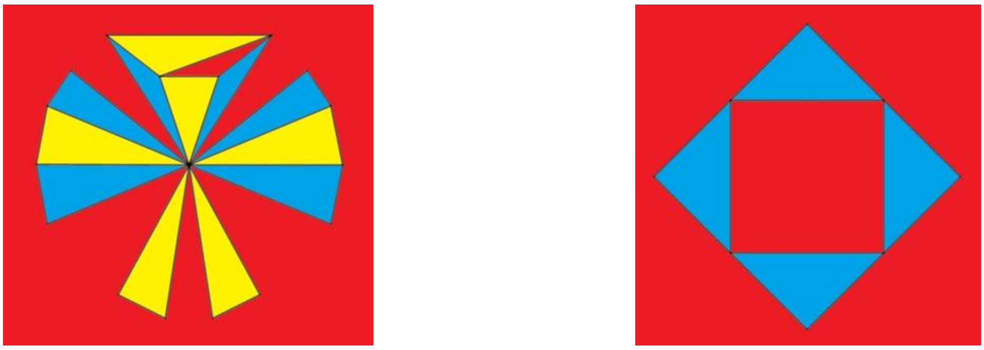


Figure 16. (a) Л(F_2 x Z_9); (b) Л(F_3 x F_3)

We see that the image of the color of regions of Л(F_3 x F_3) in Fig. 15 is similar as the image (a) in Fig. 1.

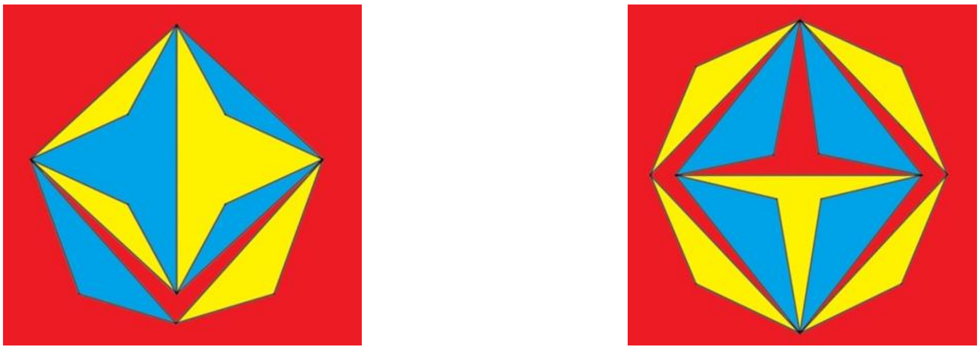


Figure 17. (a) Л(F_3 x F_4); (b) Л(F_3 x F_5)

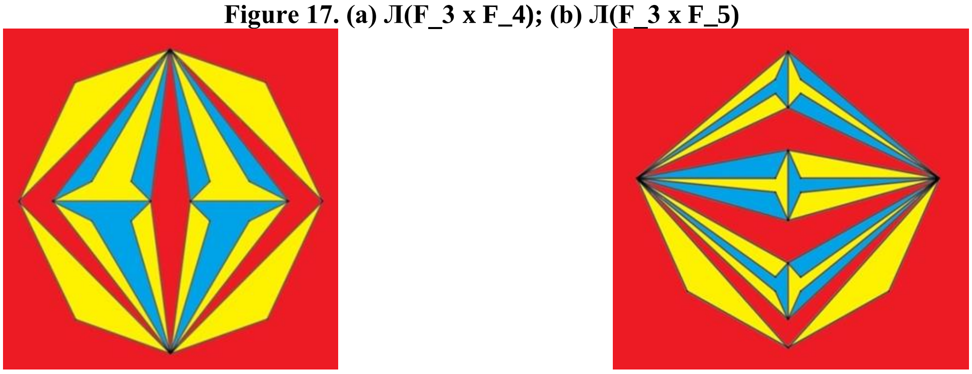


Figure 18. – (a) Л(F_3 x F_7); (b) Л(F_3 x F_8)

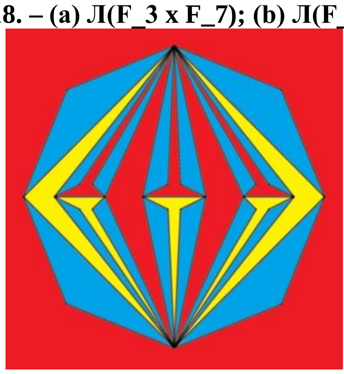


Figure 19. Л(F_3 x F_9)

4. CONCLUSION

In this paper we transform the planar idempotent divisor graph after color its region into optical art.

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