ORIGINAL PAPER

THE PAUL-PAINLEVÉ APPROACH OF THE BLACK SCHOLES MODEL AND ITS EXACT AND NUMERICAL SOLUTIONS

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Abstract. *In this article, we employ the Black Scholes model which plays a vital role in economic operation and financial market management. The Paul-Painlevé approach is used for the first time to achieve the exact wave solution to this equation. Furthermore, the numerical solution to this equation has been constructed by using the variational iteration method.*

Keywords: *The Black Scholes equation; the Paul-Painlevé approach; the variational iteration method; the traveling wave solution; the numerical solution.*

1. INTRODUCTION

The main idea of this paper concentrates on using the Paul-Painlevé approach [1, 2] for the first time to achieve the exact wave solution to the Black-Scholes equation which has not been achieved before. It also aims to achieve the corresponding numerical solutions according to the variational iteration method (VIM).

The Black Scholes model requires five input variables: the strike price of an option, the time to expiration, the current stock price, the risk-free rate, and the volatility. The Black-Scholes model is a pricing model that is used to determine the fair price or theoretical value for a call or a put option based on six variables such as volatility, type of option, underlying stock price, time, strike price, and risk-free rate. The quantum of speculation increases in the case of stock market derivatives, and hence proper pricing of options eliminates the opportunity of any arbitrage. There are two important models for option pricing – the binomial model and the Black-Scholes model. The model is used to determine the price of a European call option, which simply means that the option can only be exercised on the expiration date.

The Black-Scholes equation is a nonlinear partial differential equation that governs the price evolution of a European call or European put. Based on works developed previously by market researchers and practitioners, such as Louis Bachelier, Sheen Kassouf, and Ed Thorp among others, Fischer Black and Myron Scholes [3, 4] demonstrated in 1968 that the dynamic revision of a portfolio removes the expected return of the security, thus inventing the risk-neutral argument. This mathematical model for the dynamics of a financial market that contains derivative investment instruments gives a theoretical estimation of the price of European-style options and shows that the option has a unique price regardless of the risk of the security and its expected return (instead of replacing the expected security's return with the risk-neutral rate). The formula leads to a boom in options trading and provides

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mathematical legitimacy to the activities of the Chicago board options exchange and other options markets around the world [5]. It is widely used, although it contains some adjustments, by options market participants [6]. For the European call or put on an underlying stock paying no dividends, the equation is

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS} = -rSu_S + ru \quad , \tag{1}$$

where S(t) is the price of the underlying asset at time t, u(S, t) is the price of the option as a function of the underlying asset S, at time t; C(S, t) is the price of a European call option and P(S, t) is the price of a European put option, k is the strike price of the option, also it is known as the exercise price, r is the annualized risk-free interest rate. Large r implies a big market-impact of hedging, if r→0 or no hedging demand continuously compounded and σ is the standard deviation of the stock's returns; this is the square root of the quadratic variation of the stock's log price process. Furthermore, the left-hand side consists of a "time decay" term, the change in derivative value with respect to time is called theta; a term that involves the second spatial derivative gamma, and the convexity of the derivative value concerning the underlying value. The right-hand side is the riskless return from a long position in the derivative and a short position which consists of u_S shares of the underlying , u the price of the option as a function of stock price S also t and r the time and risk-free interest rate respectively and σ the volatility of the stock. In general, this inequality does not have a closed-form solution, although an American call with no dividends is equal to a European call, and the Roll–Geske–Whaley method provides a solution to an American call with one dividend [7, 8]. Bjerksund and Stensland [9] provide an approximation based on an exercise strategy corresponding to a trigger price with evidence that indicates that the approximation may be more accurate in pricing long-dated options than Barone-Adesi and Whaley [10]. The formula is readily modified for the valuation of a put option by using put–call parity. By solving the Black–Scholes differential equation via using the boundary condition and the Heaviside function, we finished pricing of options that pay one unit above some predefined strike price and nothing below [11]. Some manners have been demonstrated to solve this model numerically through different authors [12-14]. The key financial insight behind the equation is that, under the model assumption of a frictionless market, one can perfectly hedge the option by buying and selling the underlying asset in just the right way and consequently eliminates the risk. Paul Wilmott [14] illustrated this hedge, in turn, that there is only one right price for the option, as returned by the Black–Scholes formula. Pooe, et-al [15] transformed the Black-Scholes equation to one-dimensional linear heat equation via two sets of transformation, an optimal system of one-dimensional sub algebras for the one-dimensional heat equation which is exploited to obtain two classes of optimal systems of one-dimensional sub algebras for the well-known Black-Scholes equation of the mathematics of finance invariant solutions and conservation laws of the Black-Scholes equation. Emery, et-al. [16] investigated Black–Scholes call and put option thetas and derived upper and lower bounds for thetas as a function of underlying asset value. He showed that the maximum option theta does not occur at that point. Instead, it occurs when the asset value is somewhat above the exercise price. He also showed that option theta is not monotonic in any of the parameters in the Black–Scholes option-pricing model. Gulen et al. [17] captured the discrete behavior of linear and nonlinear Black–Scholes European option pricing models by using a sixth-order finite difference (FD6) scheme in space and a third–order strong stability preserving Runge–Kutta (SSPRK3) over time. Jørgen Veisdal [18] used the Black-Scholes formula and explained introduction to the most famous equation in finance. The key idea behind the model is to hedge the options in an investment portfolio by buying and selling the underlying asset (such as a stock) in just the right way and as a consequence, the risk will be eliminated. Also, Paliathanasis et al. [19] performed a classification of the Lie point symmetries for the Black-Scholes-Merton model for European options with stochastic volatility, σ, in which the last is defined by a stochastic differential equation with an Orstein-Uhlenbeck term. Martin Haugh [20] studied the Black-Scholes Model through notes and used its lemma and a replicating argument to derive the famous Black-Scholes formula for European options. Moreover, he also discussed the weaknesses of the Black-Scholes model and geometric Brownian motion, derived, studied the Black-Scholes Greeks and discussed how they are used in practice to hedge option portfolios. Jayaraman, et-al. [21] who transformed this equation into a diffusion equation and solved it by using mean and covariance propagation techniques which were developed previously in the context of solving Fokker–Planck equations on Lie groups.

2. TECHNIQUE DESCRIPTION OF THE PAUL-PAINLEVÉ APPROACH

To propose the general forlasim of the nonlinear evolution equation, let us introduce R as a function of $\varphi(x,t)$ and its partial derivatives as,

$$R(\varphi,\varphi_x,\varphi_t,\varphi_{xx},\varphi_{tt},......)=0\,, \tag{1}$$

that involves the highest order derivatives and nonlinear terms. With the aid of the transformation $\varphi(x,t)=\varphi(\zeta),\zeta=x-C_0t$ equation (2) can be reduced to the following ODE:

$$S(\varphi',\varphi'',\varphi''',......)=0\,, \tag{3}$$

where, S is a function in $\varphi(\zeta)$ and its total derivatives, while $'=\dfrac{d}{d\xi}$. According to Paul-Painlevé [1-2] the exact solution to the nonlinear ordinary differential equation can be written in the following form,

$$\varphi(\zeta)=A_0+W(X)\,e^{-N\zeta},\,X=R(\zeta)\,, \tag{4}$$

or

$$\varphi(\zeta)=A_0+A_1W(X)e^{-N\zeta}+A_2W^2(X)e^{-2N\zeta},X=R(\zeta)\,, \tag{4}$$

where $X=R(\zeta)=C_1-\dfrac{e^{-N\zeta}}{N}$, and $W(X)$ *in E*q. (4) satisfies Riccati-equation in the form $W_X-AW^2=0$, one can solve this equation to get,

$$W(X)=\frac{1}{AX+X_0}. \tag{6}$$

Consequently,

$$\varphi_\zeta = -N e^{-N\zeta} W(X) + R_\zeta e^{-N\zeta} W_X , \tag{7}$$

$$\varphi_{\zeta\zeta} = N^2 e^{-N\zeta} W(X) - 2NR_\zeta e^{-N\zeta} W_X + R_{\zeta\zeta} e^{-N\zeta} W_X + R^2{}_\zeta e^{-N\zeta} W_{XX} , \tag{8}$$

$$\begin{gathered}\varphi_{\zeta\zeta\zeta} = -N^3 W(X) e^{-N\zeta} + 3N^2 R_\zeta W_X e^{-N\zeta} - 3NR_{\zeta\zeta} W_X e^{-N\zeta} - \\ 3NR_\zeta^2 W_{XX} e^{-N\zeta} + 3R_\zeta R_{\zeta\zeta} W_{XX} + R_{\zeta\zeta\zeta} W_X e^{-N\zeta} + R_\zeta^3 W_{XXX} e^{-N\zeta}\end{gathered} \tag{9}$$

3. APPLICATION

In this section, we will apply the Paul-Painlevé as a new technique to achieve the exact solution for the Black Scholes equation "in terms of some variables", when these variables take specific values the traveling wave solutions can easily be obtained. Now, we will apply the constructed approach to the equation (1) mentioned above,

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS} + rSu_S - ru = 0 \ . \tag{10}$$

by using the transformation $u(S,t) = \varphi(\zeta), \zeta = kS - Ct$ we get,

$$\sigma^2(\zeta + Ct)\varphi_{\zeta\zeta} + [2r(\zeta + Ct) - C)]\varphi_\zeta - r\varphi = 0. \tag{11}$$

The notation that is used throughout this equation will be defined as follows:
According to the proposed method, the suggested solution is;

$$\begin{gathered}\varphi(\zeta) = A_0 + A_1 e^{-N\zeta} W(X) + A_2 e^{-2N\zeta} W(X), \\ X = R(\zeta) = C_1 - \frac{e^{-N\zeta}}{N}\end{gathered} \tag{12}$$

Thus, we can easily obtain,

$$\varphi_\zeta = -Ne^{-N\zeta} W - (A + 2N) e^{-2N\zeta} W^2 - 2Ae^{-3N\zeta} W^3, \tag{13}$$

$$\begin{gathered}\varphi_{\zeta\zeta} = N^2 e^{-N\zeta} W + (3AN + 4N^2) e^{-2N\zeta} W^2 + \\ (2A^2 + 10NA) e^{-3N\zeta} W^3 + 6A^2 \ e^{-4N\zeta} W^4\end{gathered} \tag{14}$$

Substituting for $\varphi, \varphi_\zeta, \varphi_{\zeta\zeta}$ at Eq. (11) and equating the coefficients of different powers of $W(\zeta) e^{-N\zeta}$ to zero, we system of equations which solved analytically to extract this result,

$$N = \frac{\frac{9A}{10} \pm \sqrt{\left(\frac{9A}{10}\right)^2 - \frac{4A^2}{10}}}{2} \ , \tag{15}$$

The obtained result could have been reduced to be:

$$N = \frac{A}{20}\left(9 \pm \sqrt{41}\right). \tag{16}$$

This split into these two results,

$$\begin{aligned} N &= \frac{A}{20}\left(9 + \sqrt{41}\right), \\ N &= \frac{A}{20}\left(9 - \sqrt{41}\right). \end{aligned} \tag{17}$$

Each one of these two results will generate other two sub-results according to the value of A which is either positive or negative.

Case 1. When A is positive say A=20, we have two solutions

$$N = 9 + \sqrt{41} = 15.4\,, \tag{18}$$

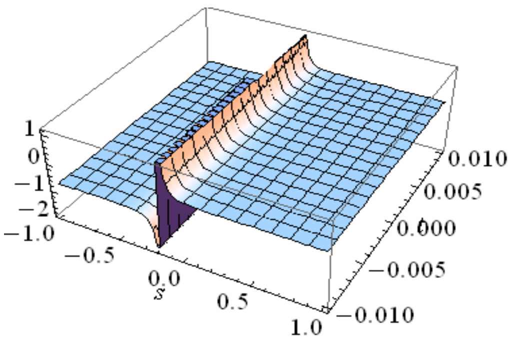
$$N = 9 - \sqrt{41} = 2.6\,. \tag{19}$$

The solutions according to the suggested method is,

$$\varphi(\zeta) = \frac{e^{-N\zeta}}{A\left(1 - \frac{e^{-N\zeta}}{N}\right) + X_0}\,, \tag{20}$$

$$\varphi(\zeta) = \frac{15.4 e^{-15.4\zeta}}{323.4 - 20 e^{-15.4\zeta}}\,, \tag{21}$$

$$\varphi(\zeta) = \frac{2.6 e^{-2.6\zeta}}{52.6 - 20 e^{-2.6\zeta}}\,. \tag{22}$$



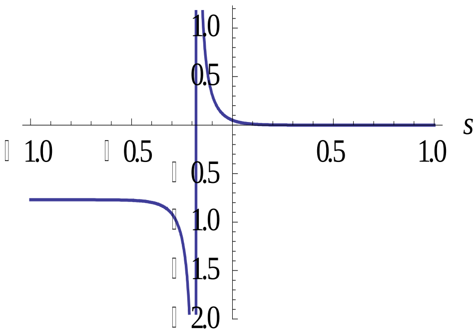


Figure 1. The plot of Eq.(21) in 2D and 3D with values: $N = 15.4, A = 20, C = C_1 = r = \sigma = k = X_0 = 1.$

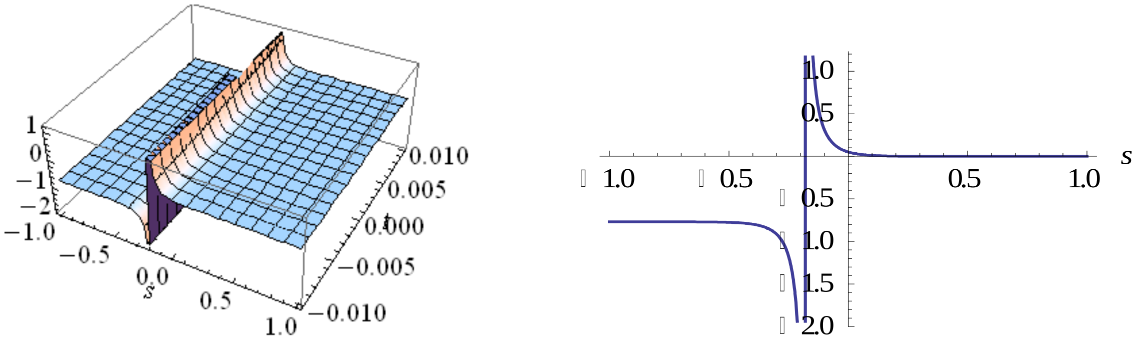


Figure 2. The plot of Eq.(22) in 2D and 3D with values: $N = 2.6, A = 20, C = C_1 = r = \sigma = k = X_0 = 1.$

Case 2. When A is negative say A=-20, we have other two solutions which are,

$$N = -(9+\sqrt{41}) = -15.4\,, \tag{23}$$

$$N = -(9-\sqrt{41}) = -2.6\,. \tag{24}$$

The solutions according to the suggested method are:

$$\varphi(\zeta) = \frac{15.4e^{15.4\zeta}}{-292.6 - 20e^{15.4\zeta}}, \tag{25}$$

$$\varphi(\zeta) = \frac{2.6e^{2.6\zeta}}{-49.4 - 20e^{2.6\zeta}}, \tag{26}$$

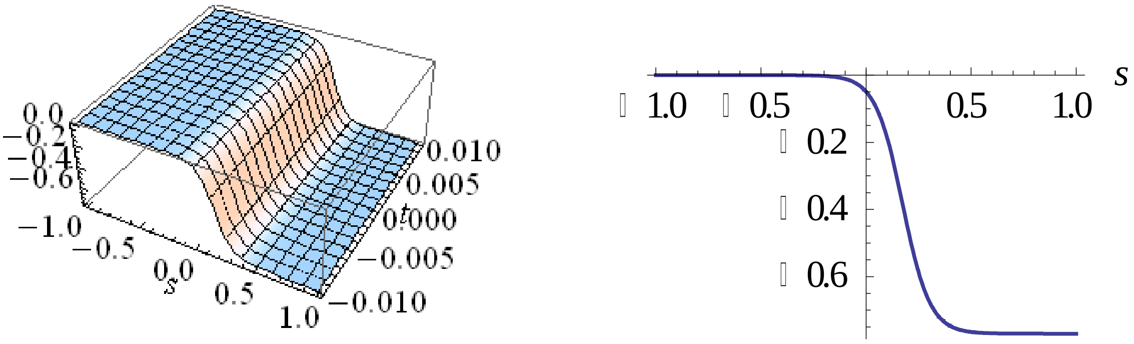


Figure 3. The plot of Eq.(25) in 2D and 3D with values: $N = -15.4, A = -20, C = C_1 = r = \sigma = k = X_0 = 1.$

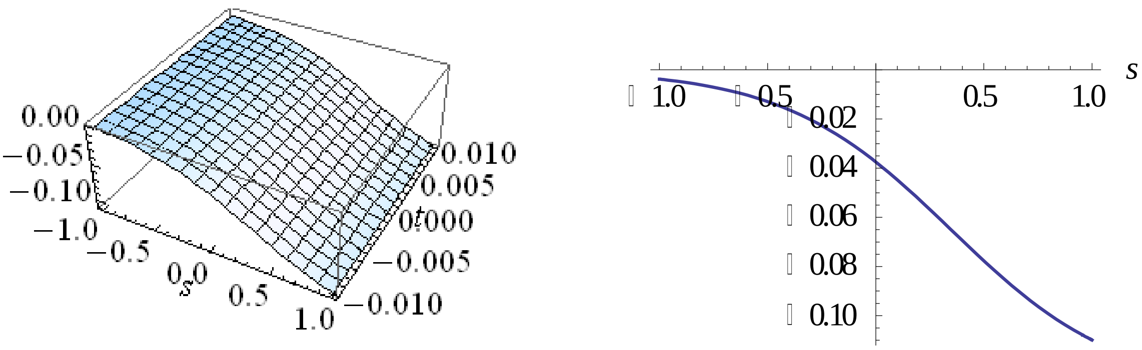


Figure 4. The plot of Eq.(26) in 2D and 3D with values: $N = -2.6, A = -20, C = C_1 = r = \sigma = k = X_0 = 1.$

Also, more significant results for these two cases could be achieved when we choose other values of N, A.

4. THE VARIATIONAL ITERATION METHOD

Considering the differential equation with inhomogeneous term $f(\zeta)$ and R, S denotes the linear and the nonlinear operators respectively become as:

$$LH + NH = f(\zeta), \tag{27}$$

the VIM proposes a functional correction for equation (27) which is;

$$H_{m+1}(\zeta) = H_m(\zeta) + \int_0^{\zeta} \lambda(t)(LH_m(t) + N\,\tilde{H}_m(t) - g(t))dt \cdot \tag{28}$$

where λ is a general Lagrange's multiplier which can be optimally identified via the variational theory, and $\tilde{H}_m$ as a restricted variation which means $\delta\tilde{H}_m$. The Lagrange multiplier λ is crucial and critical in the method, and it can be a constant or function [15]. Having λ determined, an iteration formula should be used for the determination of the successive approximations $H_{m+1}(\zeta)\,;\, n \geq 0$ of the solution $H(\zeta)$. The zeros approximation H_0 can be any selective function. However, using the initial values $H(0)\,;\,H'(0)$ are preferably used for the selective zeros approximation u_0 as will be seen later. Consequently, the solution is given by $H(\zeta) = \lim\limits_{\zeta\to\infty} H_m(\zeta)$. It is interesting to point out that we formally derived the distinct forms of the Lagrange multipliers λ in (27), hence we skip details. We only set a summary of the obtained results. It is important to give a brief of the significant forms of equation (27) according to the Lagrange multipliers in these results, for the 1-st order ODE in the form,

$$H' + q(\zeta)H = p(\zeta),\; H(0) = \rho. \tag{29}$$

We find that $\lambda = -1$, and the correction function gives the iteration formula;

$$H_{m+1}(\zeta) = H_m(\zeta) - \int_0^{\zeta} (H'_m(t) + q(t)H_m(t) - p(t))dt. \tag{30}$$

The 2-nd order ODE in the form is,

$$H''(\zeta) + cH'(\zeta) + dh(\zeta) = g(\zeta), \quad H(0) = \rho, H'(0) = \eta. \tag{31}$$

We find that $\lambda = t - x$, and the correction function gives the iteration formula;

$$H_{m+1}(\zeta) = H_m(\zeta) + \int_0^{\zeta} (t-x)(H''_m(t) + cH'_m(t) + dH_m - g(t))dt. \tag{32}$$

The 3-th order ODE in the form is,

$$H'''(\zeta)+cH''(\zeta)+dH'(\zeta)+eH(\zeta)=g(\zeta),\ H(0)=\rho, H'(0)=\eta, H''(0)=\sigma. \tag{33}$$

We find that $\lambda=-\frac{1}{2!}(t-x)^2$, and the correction function gives the iteration formula

$$H_{m+1}(\zeta)=H_m(\zeta)-\tfrac{1}{2!}\int_0^{\zeta}(t-x)^2(H_m'''(t)+cH_m''(t)+dH_m'(t)+eH_m-g(t))dt. \tag{34}$$

Consecountly, the general form of ODE is:

$$H^{(m)}+f(H',H'',H'''...,H^{(m-1)})=g(\zeta), H(0)=\rho_0, H'(0)=\rho_1, H''(0)=\rho_2..., H^{m-1}(0)=\rho_{m-1}. \tag{35}$$

The lagrange multiplier λ takes the general form as follows $\lambda=\frac{(-1)^m}{(m-1)!}(t-x)^{m-1}$, while the general form of iteration rule becomes,

$$H_{m+1}(\zeta)=H_m(\zeta)+\tfrac{(-1)^m}{(m-1)!}\int_0^{\zeta}(t-x)^{m-1}\left(H^{(m)}+f(H',H'',H''',...,H^{(m-1)})-g(t)\right)dt. \tag{36}$$

Moreover, the zeros approximation $H_0(\zeta)$ can be perfectly selected to be,

$$H_0(\zeta)=H_0(0)+H'(0)\zeta+\tfrac{1}{2!}H''(0)\zeta^2+\tfrac{1}{3!}H'''(0)\zeta^3.......+\tfrac{1}{(m-1)!}H^{m-1}(0)\zeta^{m-1}, \tag{37}$$

where m is the order of the ODE.

For simplicity and similarity, we will apply the VIM to get the numerical solutions corresponding only for the first and the third exact solutions achieved above.

Case 1. The numerical solution corresponding to the first exact solution
According to VIM the initial condition:

$$\varphi(0)=0.05, \varphi'(0)=-0.8. \tag{38}$$

According to VIM the first and the second iterations are: $\varphi_0(\zeta)=\varphi(0)+\zeta\varphi'(0)$, $\varphi_0(\zeta)=0.05-0.8\zeta$

$$\begin{aligned}&\varphi_1(\zeta)=\varphi_0(\zeta)-\int_0^{\zeta}\left((\zeta+t)\varphi_0''+(\zeta+t-1)\varphi_0'-\varphi_0\right)dt\\&\varphi_1=0.05-0.8\zeta-\int_0^{\zeta}[-0.8\zeta-0.8t+0.8-0.05+0.8t]dt=0.05-1.55\zeta+0.8\zeta^2\end{aligned} \tag{39}$$

$$\begin{aligned}
&\varphi_2(\zeta)=\varphi_1(\zeta)-\int_0^{\zeta}\Big(\,(\zeta+t)\varphi_1''+(\zeta+t-1)\varphi_1'-\varphi_1\,\Big)dt\\
&\varphi_2(\zeta)=0.05-1.55\zeta+0.8\zeta^2\\
&\qquad-\int_0^{\zeta}\Big(1.6(\zeta+t)+(\zeta+t-1)(-1.55+1.6t)-0.05+1.55t-0.8t^2\Big)dt\\
&\varphi_2(\zeta)=0.05-3.05\zeta+0.75\zeta^2-1.6\zeta^3
\end{aligned} \tag{40}$$

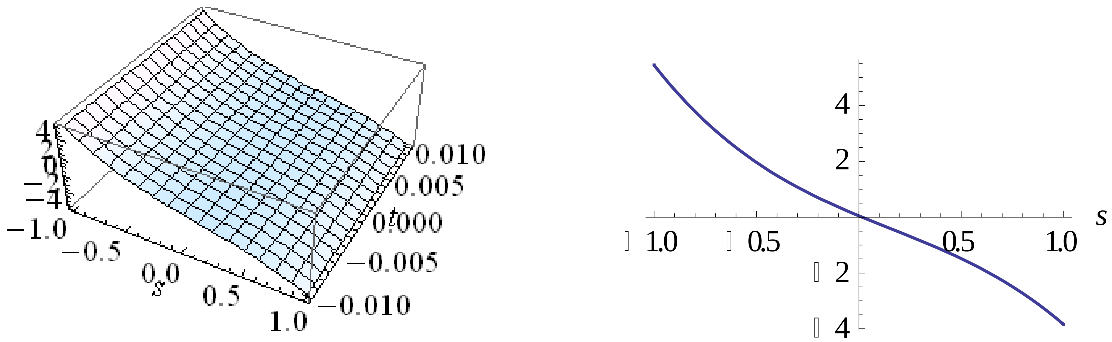


Figure 5. The plot of numerical solution Eq.(40) in 2D and 3D with values: $N=15.4, A=20, C=C_1=r=\sigma=k=X_0=1.$

Case 2. The numerical solution corresponding to the third exact solution
According to VIM the initial condition:

$$\varphi(0)=-0.05, \varphi'(0)=-0.7\,. \tag{41}$$

According to VIM the first and the second iterations are, $\varphi_0(\zeta)=\varphi(0)+\zeta\varphi'(0),$
$\varphi_0(\zeta)=-0.05-0.7\zeta,$

$$\begin{aligned}
&\varphi_1(\zeta)=\varphi_0(\zeta)-\int_0^{\zeta}\Big(\,(\zeta+t)\varphi_0''+(\zeta+t-1)\varphi_0'-\varphi_0\,\Big)dt,\\
&\varphi_1=-0.05-0.7\zeta-\int_0^{\zeta}[-0.7\zeta-0.7t+0.7+0.05+0.7t]dt=-0.05-1.45\zeta+0.7\zeta^2,\\
&\varphi_2(\zeta)=\varphi_1(\zeta)-\int_0^{\zeta}\Big(\,(\zeta+t)\varphi_1''+(\zeta+t-1)\varphi_1'-\varphi_1\,\Big)dt,\\
&\varphi_2(\zeta)=-0.05-1.45\zeta+0.7\zeta^2\\
&\qquad-\int_0^{\zeta}\Big(1.4(\zeta+t)+(\zeta+t-1)(-1.45+1.4t)+0.05+1.45t-0.7t^2\Big)dt,\\
&\varphi_2(\zeta)=-0.05-1.5\zeta+0.75\zeta^2-0.7\zeta^3
\end{aligned} \tag{42}$$

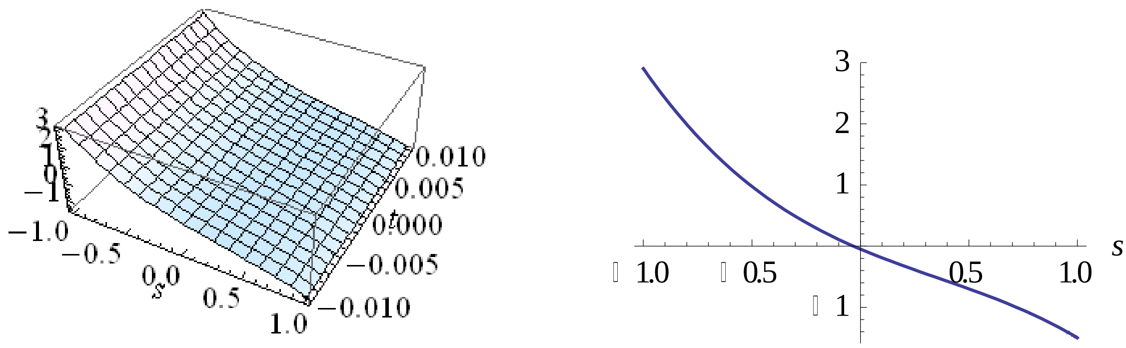


Figure 6. The plot of numerical solution Eq.(42) in 2D and 3D with values: $N=-15.4, A=-20, C=C_1=r=\sigma=k=X_0=1.$

For the two cases of the VIM, the cascading iteration can be obtained as follow,

$$\varphi_3(\zeta)=\varphi_2(\zeta)-\int_0^{\zeta}\Big((\zeta+t)\varphi_2''+[(\zeta+t)-1)]\varphi_2'-\varphi_2\Big)dt,$$

....

$$\varphi_{N+1}(\zeta)=\varphi_N(\zeta)-\int_0^{\xi}\Big((\zeta+t)\varphi_N''+[(\zeta+t)-1)]\varphi_N'-\varphi_N\Big)dt,$$

Using the fact that the exact solution is obtained by using $\varphi(\zeta)=\lim\limits_{\zeta\to\infty}\varphi_N(\zeta)$.

5. RESULTS AND DISCUSSION

According to the obtained results and the corresponding figures, it is clear that there are agreements with the normal form of the curve which represents the value of option. It is also clear that the validity of underlying first assets for the second year approximately doubles the first year “according to the slope of the obtaining curves” although the parameters are not changed. Also, the security risk “risk-free interest rate” which depends on the hazard of the option price lies between %2 and %3 which has never achieved by any methods previously and gives significant accurate value to expected security. It is more accurate than the results obtained by using analytical and numerical methods for pricing financial derivatives that obtained by other authors [12-14, 17, 23]. In addition, the solutions are isomorphic to the European call value by using the Black–Scholes pricing equation for varying asset price SS and time-to-expiry TT in which the strike price is set to one. Furthermore, our approach proposes new exact solutions than those obtained in [15- 19] who used different methods and can be considered a benchmark against any obtained numerical solutions [24-26].

6. CONCLUSIONS

In this work, the exact and hence the solitary wave solutions to the Black Scholes equation “which weren’t previously achieved” have been successfully established. The obtained solutions have been demonstrated for the first time in the framework of the Paul-Painlevé approach “Fig. 1-4”. Moreover, we can establish other new exact solutions for various values of N and A. The 3D-graph explains the relationship among stock price, strike price and time to maturity well. It is clear that the obtained solutions agree with the fact that **t**he Black–Scholes model which assumes that the market consists of at least one risky asset, usually called the stock, and one riskless asset, usually called the money market cash, or bond (riskless rate). The rate of returning riskless asset is constant and thus it is called the risk-free interest rate. Moreover, the benefit for this exact solution “which weren’t realized before” of the nonlinear Black-School model is important to represent the accurate for each one of the European call option and the American call option which depends on two stock price. Furthermore, it is clear that there are agreements between the two cases of the obtained exact solutions figures (3,4) and the corresponding numerical cases figures (5,6) respectively. In addition, our exact solutions are new compared with that obtained by [14-19]. In related subject, the achieved numerical solutions by using VIM is more stable compared with the numerical solutions achieved by using the standard methods of numerical analysis [12-14, 17, 23]. The power of the obtained numerical solutions refers to that its initial conditions are derived from the achieved exact solutions. The stability and symmetry of the obtained curves “which are corresponding to the achieved solutions” with respect to the axes in consecutive intervals proved the powerful and accuracy of the obtained solutions as well as the effectiveness of the suggested methods. Finally, the achieved closed form solution “which were not achieved before” will establish the fact that the American call with no dividends is equal to a European call price and the risk-free interest rate lies between %2 and %3 which has never previously realized by any methods.

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