ORIGINAL PAPER

ASSOCIATED CURVES ACCORDING TO BISHOP FRAME IN 4-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. *Many studies have been doneaccording to different frames of the theory of curves in Euclidean Space. Many scientists have studied frames such as the Frenet frame, Bishop frame, and Adapted frame in this theory. These frames help us in the characterization of curves.In this study, associated curves with the Frenet curve according to the Bishop frame in 4-dimensional Euclidean space are investigated. Direction and rectifying curves of the Frenet curve according to this frame are given.*

Keywords: *Bishop frame; Frenet curves; Frenet frame; Integral curve.*

1. INTRODUCTION

The differential geometry of curves is one of the most important areasthat studied different frames in different spaces. A curve s Frenet frame of 3-dimensional space of Euclid which formed tangent, substantive normal, binormal vector is one of the best known orthonormal framesto work differential geometry. Also, the shape of the space curve well local behavior of this curve is completely specific to its curvature and torsion. So a curve must be at least 3rd order continuously differentiable to be completely examined.

Many researchers use the Frenet frame to characterize of properties of a curve and calculate. But the Frenet frame is defined for only differentiable curves and at some points the 2ndderivative of a curve can be zero. In this situation, we need analternative frame. Therefore L.R. Bishop defined an alternative or parallel translation frame for curves in 3-dimensional Euclid with the help of parallel vector fields in 1975. This parallel translation frame is a well-defined active frame even where the 2nd derivative is completely zero.

Bishop thanks to this frame found an alternative way to define an active frame [1]. This frame has been obtained by keeping the tangent vector constant and rotated at a certain angle [1]. With the discovery of this alternative frame, some researchershave aimed to examine some basic topics according to this new frame and have done a lot of work on the Bishop frame [2-4]

Bukcu and Karacan examine some special curves according to Bishop's frame [5]. Yilmaz, Özyilmaz, and Turgut studied spherical images of the components of this frame [6]. If Babadağ characterized some curves according to Bishop's frame [7]. Macit and Duldul have defined some new associated curves in 4-dimensional Euclid space andcharacterized every associated curve curvature [8]. Buyukkutuk and Öztürk achieved some results on 4-dimensional Euclid space in 2015 [9]. Elzawy studied the Bishop frame and Frenet frame on 4-dimensional Euclid space in his study in 2017 [10]. J.H.Choi and Y.H.Kim defined some associated curves with the help of integral curves in 2012 [11]. We can sort these associated curves as principal directive curve, binormal directive curve, principal donor curve, and

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binormal donor curve [11]. Körpinar has examined some associated curves on 3-dimensional Euclid space [4, 12]. Körpinar has given M_1 - direction curve and M_2-donor curve depending on the Bishop frame on 3-dimensional Euclid space.

In this paper, we will try to get direction curves principal normal vector field and binormal vector field with 1st, 2nd, and 3rd curvature on 4-dimensional Euclid space.

2. MATERIALS AND METHODS

At this stage, some basic concepts about curves in space are given.The Euclidean 4-space is supplied with the standard straight metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^{\,2}.$$

Here (x_1, x_2, x_3, x_4) is a coordinate system of the Euclidean 4-space.The cross product in 4-dimensional Euclidean space is defined as follows:

$$\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} & \boldsymbol{e_4} \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

where $\boldsymbol{a}$ $=(a_1, a_2, a_3, a_4)$,$\boldsymbol{b}$ $=(b_1, b_2, b_3, b_4)$, $\boldsymbol{c}$ $=(c_1, c_2, c_3, c_4) \in E^4$.Considering that $\{\boldsymbol{T(s)}, \boldsymbol{N(s)}, \boldsymbol{B_1(s)}, \boldsymbol{B_2(s)}$ is the Serret-Frenet frame ofα that the following Frenet-Serret equatios can be given.

$$\begin{bmatrix} \boldsymbol{T'} \\ \boldsymbol{N'} \\ \boldsymbol{B_1'} \\ \boldsymbol{B_2'} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B_1} \\ \boldsymbol{B_2} \end{bmatrix}$$

where k_1, k_2, k_3 are the curvature of α .

The Bishop frame, which is referred to as the alternative or parallel frame of the curves depending to parallel vector fields, was introduced by L.R. Bishop in 1975. It is a well-defined alternative approach even in the absence of the second derivative of the curve [22]. The Bishop frame is explained as

$$\begin{bmatrix} \boldsymbol{T'} \\ \boldsymbol{M_1'} \\ \boldsymbol{M_2'} \\ \boldsymbol{M_3'} \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & 0 & 0 \\ -K_2 & 0 & 0 & 0 \\ -K_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{M_1} \\ \boldsymbol{M_2} \\ \boldsymbol{M_3} \end{bmatrix} \tag{2.1}$$

Here, $\{\boldsymbol{T}, \boldsymbol{M_1}, \boldsymbol{M_2}, \boldsymbol{M_3}\}$ is called Bishop Frame of α and K_1, K_2 and K_3 are called Bishop curvatures of$\boldsymbol{\alpha}$

3. ASSOCIATED CURVES ACCORDING TO BISHOP FRAME IN 4-DIMENSIONAL EUCLIDEAN SPACE

3.1 M_1– DIRECTION CURVES

Theorem 3.1.1. *Let β a Frenet curve which curved k_1, k_2, k_3 on E^4 and $\bar{\beta}, M_1$- direction curve ofβ which curved$\bar{k}_1, \bar{k}_2, \bar{k}_3$. The Frenet frame of $\bar{\beta}$ are as follows:*

$$\bar{T} = M_1, \tag{3.1}$$

$$\bar{N} = -T, \tag{3.2}$$

$$\bar{B}_1 = \frac{1}{\sqrt{K_3^2 + K_2^2}}(K_2 M_2 - K_3 M_3), \tag{3.3}$$

$$\bar{B}_2 = \frac{1}{\sqrt{K_3^2 + K_2^2}}(K_3 M_2 - K_2 M_3), \tag{3.4}$$

$$\bar{k}_1 = K_1, \tag{3.5}$$

$$\bar{k}_2 = -\frac{\sqrt{K_3^2 + K_2^2}}{K_1}, \tag{3.6}$$

$$\bar{k}_3 = \frac{1}{K_3^2 + K_2^2}(K_3 K_2' - K_2 K_3'). \tag{3.7}$$

Proof: Let $\{\bar{T}, \bar{N}, \bar{B}_1, \bar{B}_2, \bar{k}_1, \bar{k}_2, \bar{k}_3\}$Frenetaparatus of $\bar{\beta}$. If the M_1- direction curve of $\beta$is $\bar{\beta}$ then the following expressions are found:

$$\bar{\beta}' = \bar{T} = M_1, \bar{T}' = \bar{M}_1' = -K_1 T$$

So we can write M_1- direction curve s 1st curvature,

$$\|\bar{T}'\| = \bar{k}_1 = \sqrt{\langle -K_1 T, -K_1 T \rangle}$$
$$\bar{k}_1 = K_1$$

and principal normal vector field of M_1- direction curve is

$$\bar{N} = \frac{\bar{\beta}''}{\|\bar{\beta}''\|} = \frac{\bar{T}'}{\bar{k}_1} = \frac{\bar{M}_1'}{K_1} = \frac{-K_1 T}{K_1} = -T.$$

Also from

$$\bar{\beta}' \times \bar{\beta}'' \times \bar{\beta}''' = \begin{vmatrix} T & M_1 & M_2 M_3 \\ 0 & 1 & 0 \;\; 0 \\ -1 & 0 & 0 \;\; 0 \\ 0 & K_1 & K_2 \, K_3 \end{vmatrix}$$

$$= K_3M_2 - K_2M_3$$
$$\|\bar{\beta}' \times \bar{\beta}'' \times \bar{\beta}'''\| = \sqrt{\langle K_3M_2 - K_2M_3, K_3M_2 - K_2M_3\rangle}$$
$$= \sqrt{K_3^2 + K_2^2}.$$

Equalities 2nd binorrmal vector fields of M_1- direction curve as found as

$$\bar{B}_2 = \frac{\bar{\beta}' \times \bar{\beta}'' \times \bar{\beta}'''}{\|\bar{\beta}' \times \bar{\beta}'' \times \bar{\beta}'''\|} = \frac{1}{\sqrt{K_3^2 + K_2^2}}(K_3M_2 - K_2M_3)$$

If $\bar{B}_1(s) = \bar{B}_2(s) \times \bar{T}(s) \times \bar{N}(s)$equality is used, then 1st binorrmal vector fields of M_1- direction curve is in the form

$$\bar{B}_1 = \frac{1}{\sqrt{K_3^2 + K_2^2}}\begin{vmatrix} T & M_1 & M_2 & M_3 \\ 0 & 0 & K_3 & -K_2 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}$$
$$= \frac{1}{\sqrt{K_3^2+K_2^2}}(-K_2M_2 - K_3M_3).$$

So 2nd curvature of M_1- direction curve is given by

$$\bar{k}_2 = \frac{\langle \bar{B}_1, \bar{\beta}'''\rangle}{\bar{k}_1} = \frac{1}{K_1}\frac{1}{\sqrt{K_3^2 + K_2^2}}(\langle -K_2M_2 - K_3M_3, K_1M_1 + K_2M_2 + K_3M_3\rangle$$
$$= \frac{1}{K_1\sqrt{K_3^2 + K_2^2}}[-K_3^2 - K_2^2]$$
$$= -\frac{K_3^2 + K_2^2}{K_1\sqrt{K_3^2 + K_2^2}} = -\frac{\sqrt{K_3^2 + K_2^2}}{K_1}$$

With the inclusion of

$$\bar{\beta}^4 = K_1'M_1 + K_2'M_2 + K_3'M_3 - (K_1^2 + K_2^2 + K_3^2)T$$

and

$$\bar{k}_3 = \frac{\langle \bar{B}_2, \bar{\beta}^4\rangle}{\bar{k}_1, \bar{k}_2},$$

The 3rd curvature of M_1- direction curve is found as follows:

$$\bar{k}_3 = \frac{1}{K_3^2+K_2^2}(K_3K_2' - K_2K_3')$$

3.2. M_2– DIRECTION CURVES

Theorem 3.2.1. *Let β a Frenet curve which curved k_1, k_2, k_3 on E^4 and $\bar{\beta}, M_2$- directioncurve of β which curved $\bar{k}_1, \bar{k}_2, \bar{k}_3$ .Then the Frenet frame of $\bar{\beta}$ are as follows:*

$$\bar{T} = M_2, \tag{3.8}$$

$$\bar{N} = -T, \tag{3.9}$$

$$\bar{B}_1 = \frac{1}{\sqrt{K_3^2 + K_1^2}}(K_1M_1 + K_3M_3). \tag{3.10}$$

$$\bar{B}_2 = \frac{1}{\sqrt{K_3^2 + K_1^2}}(-K_3M_1 + K_1M_3) \tag{3.11}$$

$$\bar{k}_1 = K_2 \tag{3.12}$$

$$\bar{k}_2 = -\frac{\sqrt{K_3^2 + K_1^2}}{K_2}, \tag{3.13}$$

$$\bar{k}_3 = -\frac{1}{K_3^2+K_1^2}(K_3K_1' - K_1K_3') \tag{3.14}$$

3.3. M_3– DIRECTION CURVES

Theorem 3.3.1. *Let β a Frenet curve which curved k_1, k_2, k_3 on E^4 and $\bar{\beta}, M_3$- direction curve of β which curved $\bar{k}_1, \bar{k}_2, \bar{k}_3$. Then the Frenet frame of $\bar{\beta}$ are as follows:*

$$\bar{T} = M_3 \tag{3.15}$$

$$\bar{N} = -T, \tag{3.16}$$

$$\bar{B}_1 = \frac{1}{\sqrt{K_1^2 + K_2^2}}(K_1M_1 - K_2M_2), \tag{3.17}$$

$$\bar{B}_2 = \frac{1}{\sqrt{K_1^2 + K_2^2}}(K_1M_2 - K_2M_1), \tag{3.18}$$

$$\bar{k}_1 = K_3 \tag{3.19}$$

$$\bar{k}_2 = -\frac{\sqrt{K_1^2 + K_2^2}}{K_3}, \tag{3.20}$$

$$\bar{k}_3 = -\frac{1}{K_1^2+K_2^2}(K_2K_1' - K_1K_2') \tag{3.21}$$

4. CONCLUSION

In this study, some new associated curves in 4-dimensional Euclidean space are defined according to the Bishop frame. By defining the donor curve of a Frenet curve with the help of the Bishop frame, the relation between the curvatures of these curves is obtained. Furthermore, the principal direction and donor curves in 4-dimensional Euclidean space are obtained and their directional curves are given. Finally, M_1-direction curve, M_2-direction curve, M_3-direction curve for a Frenet curve in 4-dimensional Euclidean space are defined and some characterizations of these curves are obtained.

REFERENCES

[1] Bishop, L.R., *Mathematical Montly*, **82**(3), 246, 1975.
[2] Hanson, A.J., Ma, H., *Techreports* TR425, 1995.
[3] Guven, I.A, Nurkan, S.K., *International Mathematical Forum*, **6**(25), 1209, 2011.
[4] Korpinar, T., Asil, V., Baş, S., *RevistaNotas Matematica*, **7**(1), 37, 2011.
[5] Bukcu, B., Karacan, M.K., *International Scholarly Scientific Research Innovation*, **3**(11), 1010, 2009.
[6] Yilmaz, S., Ozyilmaz, E., Turgut, M., *Analele stiintifice ale Universitatii Ovidius Constanta*, **18**(2), 337, 2010.
[7] Babadag, F., *International Journal New Technology Research*, **2**(4), 59, 2016.
[8] Macit, N., Duldul, M., *Turkish Journal of Mathematics*, **38**, 1023, 2014.
[9] Buyukkutuk, S., Ozturk, G., *New Trends Mathematical Sciences*, **3**(4), 171, 2015.
[10] Elzawy M., *Journal Egptian Mathematical Society*, **25**(3), 268, 2017.
[11] Choi, J.H. ve Kim, Y.H., *Applied Mathematics Computation*, **218**(18), 9116, 2012.
[12] Korpinar, T., Asil, V., *RevistaNotas Matematica*, **7**(1), 37, 2011.
[13] Korpinar, T., Turhan, E., *Boletim Sociedade Paranaense Matematica*, **30**(2), 91, 2012.