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AN APPROACH FOR δ_{ss} −SUPPLEMENTED MODULES WITH IDEALS

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Abstract. *The aim of this paper is to present $J_{\delta_{ss}}$–supplemented modules and investigate their main algebraic properties. Let J be an ideal of a ring S and A be an S −module. We call a module A is $J_{\delta_{ss}}$ −supplemented, provided for each submodule B of A, there exists a direct summand C of A such that $A = B + C$, $B \cap C \leq JC$ and $B \cap C \leq Soc_{\delta}(C)$. We prove that the factor module by any fully invariant submodule remains so, when the module is $J_{\delta_{ss}}$ −supplemented. We show that any direct sum of $J_{\delta_{ss}}$ −supplemented modules preserves its $J_{\delta_{ss}}$ −supplemented property when this direct sum is a duo module. Additionally, we make comparisons of $J_{\delta_{ss}}$ −supplemented modules with other module types.*

Keywords: *semisimple modules; δ_{ss} −supplement submodules; δ_{ss} −supplemented modules; $\bigoplus_{ss}$−supplemented modules; $J_{\delta_{ss}}$ −supplemented modules.*

1. INTRODUCTION

Throughout the study, we denote by S an associative ring with unit, and whole modules under consideration are assumed to be unitary left S −modules. When we use the notation ${}_SS$ (S_S), we refer to the left S −module (the right S −module) over the ring S. Let A be such a module. By the implications $B \leq A$ and $B \leq^{\oplus} A$, we mean that B is a submodule of A and B is a direct summand of A, respectively. $B \leq A$ is said to be *small* in A, denoted as $B \ll A$, if $A \neq B + X$ for each proper $X \leq A$ (see [1]). Dually, $B \leq A$ is said to be *essential* in A, notated as $B \trianglelefteq A$, if $B \cap X \neq 0$ for each nonzero $X \leq A$. A module A is said to be *singular* provided $A \cong A'/B$ for some module A' and $B \trianglelefteq A'$ (see [2-3]). A nonzero module A is said to be *hollow* in case each proper submodule of A is small in A and it is said to be *local* in case the sum of whole proper submodules of A is also a proper submodule of A. A ring S is said to be *local* if ${}_SS$ is a local module (see [4]). For a module A, $Soc(A)$ and $Rad(A)$ indicate the socle and the radical of A, respectively. It can be clearly observed that A is a local module if and only if $Rad(A) \leq A$ is a maximal submodule and $Rad(A) \ll A$ (see [4, 41.4]). A submodule B of a module A is called *d-closed* provided the factor module A/B has a zero socle (see [5]). In [5], a module A is called *D-extending* provided each d-closed submodule of A is a direct summand. A module A is called *semiartinian* provided each nonzero homomorphic image of A includes a simple submodule, that is, $Soc(A/B) \neq 0$ for each proper submodule B of A.

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Let A be a module and $B \leq A$. A submodule C is termed a *supplement* of B in A, if it is a minimal element within the collection of submodules K of A where $A = B + K$. C is a supplement of B in A if and only if $A = B + C$ and $B \cap C \ll C$. A module A is said to be *supplemented* if each submodule of A has a supplement in A. Semisimple, artinian and local modules are supplemented. A module A is said to be *amply supplemented* in case for any $B, C \leq A$ with $A = B + C$, there exists a supplement of B in A that is included in C (for the details, see [4, Section 41]). In recent years, several authors have studied structures similar to supplement submodules through the use of (pre)radicals for the category of left S −modules. Let $B \leq A$ be modules. In [6], a submodule B of A is called to be an sa −*supplement* in A provided there exists a $C \leq A$ such that $A = B + C$ and $B \cap C$ is a semiartinian module. In [7], a submodule B of A is said to have a Z^* −*supplement* C in A provided $A = B + C$ and $B \cap C \leq Z^*(C)$. Here $Z^*(C)$ is the set of elements $c \in C$ for which the cyclic submodule Rc is small in its injective hull.

In [8], Zhou has defined δ −small submodules as a more comprehensive class that includes small submodules, and has emphasized their crucial role within the context of supplements. Let A be a module. The author defines $B \leq A$ as δ −*small* in A if, in cases where $A = B + B'$ and A/B' is singular, so $B' = A$. We signify this description with the notation $B \ll_\delta A$. Each projective semisimple submodule or small submodule of a module A is δ −small in A. Following a similar approach to [8, Lemma 1.5], we will employ the notation $\delta(A)$ to represent the sum of whole δ −small submodules of A. Given that $Rad(A)$ denotes the sum of whole small submodules of A, therefore $Rad(A) \leq \delta(A)$. So for any ring S, $\delta(S) = \delta({}_SS)$.

In [9], a module A is said to be δ −*supplemented* in case each $B \leq A$ has a δ −supplement C in A, that is, $A = B + C$ and $B \cap C \ll_\delta C$. Also in the same paper, a module A is said to be *amply* δ −*supplemented* provided, for any $B, C \leq A$ with $A = B + C$, B has a δ −supplement X in A such that $X \leq C$.

In [10], a submodule C is said to be an *ss-supplement* of B in a module A if $A = B + C$ and $B \cap C \leq Soc_S(C)$. Here $Soc_S(C)$ is the sum of whole small submodules that are simple as defined in [11]. It is proved in [10, Lemma 3] that C is an ss-supplement of B in A if and only if $A = B + C$, $B \cap C$ is semisimple and $B \cap C \ll C$ if and only if $A = B + C$, $B \cap C$ is semisimple and $B \cap C \leq Rad(C)$. Moreover, the authors termed a module A as *ss-supplemented* if each submodule of A has an ss-supplement in A. In the same paper, a module A is defined as *strongly local* if the module A meets two criteria; firstly, A must be local and secondly, $Rad(A)$ must be semisimple. A ring S is said to be *left strongly local* if ${}_SS$ is a strongly local module.

In [12], the concept of $\bigoplus_{ss}$−supplemented module is introduced. A module A is termed as $\bigoplus_{ss}$−*supplemented* if each submodule of A has an ss-supplement which is also a direct summand of A. The author has explored various properties of these modules in the paper.

In the study [13], a module A is said to be δ_{ss} −*supplemented* provided each submodule B of A has a δ_{ss} −supplement C in A, i.e, $A = B + C$, $B \cap C$ is semisimple and $B \cap C \ll_\delta C$. The authors have proved in [13, Lemma 3.3] that C serves as a δ_{ss} −supplement of B in A if and only if the conditions $A = B + C$, $B \cap C$ being semisimple and $B \cap C \leq \delta(C)$ holds. In the same study, for a module A, the submodule $Soc_\delta(A)$ is defined as the sum of whole simple submodules that are δ −small within the module A. [13, Proposition 3.1] presents the fact that $Soc_\delta(A) = Soc(A) \cap \delta(A)$. Hence $C \leq A$ is said to be δ_{ss} −*supplement* of B in A, if $A = B + C$, $B \cap C \leq Soc_\delta(C)$. It can be observed that $Soc_S(A) \leq Soc_\delta(A)$, for any module A.

In [14], a module A is said to be $\oplus - \delta_{ss}$ *−supplemented* provided each submodule of A has a δ_{ss} −supplement that is a direct summand of A. In this paper, the author also determined the rings whose modules are $\oplus - \delta_{ss}$ −supplemented.

Based on the results mentioned above, we can define the concept of $J_{\delta_{ss}}$ −supplemented S −modules, where J is any ideal of the ring S. It is demonstrated that the class of $J_{\delta_{ss}}$ −supplemented modules remains unchanged when considering factor modules by fully invariant submodules. Exactly, it has been established that any direct sum of $J_{\delta_{ss}}$ −supplemented modules is a $J_{\delta_{ss}}$ −supplemented module provided that it is a duo module. Indeed, we draw comparisons between this concept and $\oplus_{ss}$−supplemented modules as well as δ_{ss} −supplemented modules. It is established that for a projective S −module A with the condition $Soc_{\delta}(A) \leq JA$, where J is an ideal of S, the module A is a $J_{\delta_{ss}}$ −supplemented module if and only if it is a δ_{ss} −supplemented module. It is proved that a projective S −module A is a $J_{\delta_{ss}}$ −supplemented module where J is an ideal of S containing $Soc({}_S S)$ if and only if A is a δ_{ss} −supplemented module. It is verified that a $J_{\delta_{ss}}$ −supplemented S −module A, where J is an ideal of S is coatomic if and only if JA is $\delta -$ small in A. It is showed that each fully invariant direct summand B and the factor module A/B of a $J_{\delta_{ss}}$ −supplemented module A are $J_{\delta_{ss}}$ −supplemented.

2. MAIN RESULTS

Proposition 2.1. Let A be a projective S −module. Then A is a δ_{ss} −supplemented module if and only if for each $B \leq A$, there exists a $C \leq^{\oplus} A$, where $A = B + C$, $B \cap C$ is semisimple and $B \cap C \leq \delta(S)C$.

Proof: The proof is derived from [13, Lemma 2.2], [13, Theorem 5.6] and [8, Lemma 1.9].

Considering the fact above, we realize that a new concept has emerged.

Definition 2.2. Let A be an S −module and J be an ideal of S. We call the module A as being $J_{\delta_{ss}}$ *−supplemented*, in case for each submodule B of A, there exists a $C \leq^{\oplus} A$, where $A = B + C$, $B \cap C \leq JC$ and $B \cap C \leq Soc_{\delta}(C)$.

It is explicit that for each ideal J of S, each $J_{\delta_{ss}}$ −supplemented S −module is δ_{ss} −supplemented. In the sequel, we will provide an example of a module which is δ_{ss} −supplemented but not $J_{\delta_{ss}}$ −supplemented for any ideal J of S. Also semisimple S −modules are $J_{\delta_{ss}}$ −supplemented for each ideal J of S.

Lemma 2.3. Let A be an S −module and J be an ideal of S having the condition $JA = 0$. Then A is a $J_{\delta_{ss}}$ −supplemented module if and only if A is a semisimple module.

Proof: Suppose that $B \leq A$. Then by the hypothesis, there is a $C \leq^{\oplus} A$, where $A = B + C$, $B \cap C \leq JC$ and $B \cap C \leq Soc_{\delta}(C)$. Since $JC \leq JA = 0$, then we obtain that $A = B \oplus C$. Thus A is a semisimple module. The other part of the proof is explicit.

Assume that A represents a module over the commutative domain S. Consider the set of whole $x \in A$ for which a nonzero element s of S exists and has the property $sx = 0$, say

$T(A)$. The fact that $T(A)$ is a submodule of A is widely acknowledged. This submodule $T(A)$ of A is named as *torsion submodule* of A. When $T(A) = A$, the module A is said to be *torsion module*. The module A is said to be *torsion-free* in case $T(A) = 0$ (see [3, Chapter 4.8, Exercise 11]).

Analysis to the explanations in [13] yields the following result.

Proposition 2.4. Let A be a torsion-free S −module, where S is a Dedekind domain that is not field. Then A is a semisimple module if and only if A is a $J_{\delta_{ss}}$ −supplemented module.

Proof: To prove the sufficiency, let $B \leq A$. Then by the hypothesis, there exists a $C \leq^{\oplus} A$, where $A = B + C$, $B \cap C \leq JC$ and $B \cap C \leq Soc_{\delta}(C)$. Note that $B \cap C \leq Soc_{\delta}(A)$. Since A is a torsion-free module, then $Soc_{\delta}(A) = 0$. This implies that $A = B \oplus C$. Hence A is semisimple. The rest of the proof is explicit.

Following [15], a module A is said to be δ *−local* if $\delta(A) \ll_{\delta} A$ and $\delta(A) \leq A$ is a maximal submodule. By strengthening the notion of δ *−local* modules, in [13], a module A is said to be *strongly* δ *−local* in cases $\delta(A) \leq A$ is a maximal submodule, $\delta(A) \ll_{\delta} A$ and $\delta(A)$ is semisimple.

Proposition 2.5. Let A be a finitely generated S −module and J be an ideal of S. If A is a $J_{\delta_{ss}}$ −supplemented module, then $A = \sum_{\lambda=1}^{n} A_{\lambda}$, where each A_{λ} is a strongly δ −local module or a projective semisimple module.

Proof: It follows from [13, Corollary 4.11], since $J_{\delta_{ss}}$ −supplemented modules are δ_{ss} −supplemented.

Proposition 2.6. Let A be a $J_{\delta_{ss}}$ −supplemented module with $\delta(A) = A$. Then A is a projective semisimple module.

Proof: By assumption, A is a δ_{ss} −supplemented module with $\delta(A) = A$. Therefore, the result is derived from [13, Proposition 4.17].

Based on the proposition mentioned above, it is evident that a hollow radical S −module cannot be considered as a $J_{\delta_{ss}}$ −supplemented for any ideal J of S. When we combine this observation with [13, Proposition 4.18], we reach the conclusion below:

Corollary 2.7. Let A be a hollow module. If A is a $J_{\delta_{ss}}$ −supplemented module, then it is a strongly local module.

As a reminder from [16] that a module A is said to be *semilocal* if the factor module $A/Rad(A)$ is a semisimple module and a ring S is said to be *semilocal* if ${}_{S}S$ (or S_S) is a semilocal module.

Proposition 2.8. Let A be a projective module. If A is a semilocal and $J_{\delta_{ss}}$ −supplemented module, then A is an ss-supplemented module.

Proof: It follows from [13, Proposition 5.9].

Proposition 2.9. Let A be an S −module and J be an ideal of S with the property for any ideal J of $S, \delta(A) \leq JA$. If A is a strongly δ −local module, then A is a $J_{\delta_{ss}}$ −supplemented module.

Proof: Suppose that $B \leq A$. If $B \leq \delta(A)$, then B is semisimple as $\delta(A) \leq Soc(A)$. Thus $B \ll_{\delta} A$ by [13, Lemma 2.2]. Therefore we have $A = B + A$, $B = B \cap A \leq Soc_{\delta}(A)$. Since $B = B \cap A \leq \delta(A)$, then $B \cap A \leq JA$. Hence A is a $J_{\delta_{ss}}$ −supplemented module. Assume that $B \nleq \delta(A)$. Since $\delta(A) \leq A$ is a maximal submodule, then $A = B + \delta(A)$. This leads to the conclusion that $A = B \oplus P$, where P is a projective semisimple submodule of $\delta(A)$ as $\delta(A) \ll_{\delta} A$ by [8, Lemma 1.2]. Hence A is a $J_{\delta_{ss}}$ −supplemented module.

Proposition 2.10. Let A be a $J_{\delta_{ss}}$ −supplemented S −module. If A is a δ −local module, then A is a strongly δ −local module.

Proof: By assumption, A is a δ_{ss} −supplemented module. Then the result is derived from [13, Proposition 4.5].

Proposition 2.11. Let A be a projective S −module and J be an ideal of S with $Soc_{\delta}(A) \leq JA$. Then A is a $J_{\delta_{ss}}$ −supplemented module if and only if A is a δ_{ss} −supplemented module.

Proof: The first direction of the proof is explicit. Conversely, let A be δ_{ss} −supplemented module and $B \leq A$. Then B has a δ_{ss} −supplement C with $C \leq^{\oplus} A$ as A is projective by [13, Theorem 5.6]. Thus we conclude that $A = B + C$, $B \cap C \leq Soc_{\delta}(C)$. Note that $B \cap C \leq Soc_{\delta}(A)$. Then by assumption, $B \cap C \leq JA$. Therefore, we conclude that $B \cap C \leq C \cap JA = JC$ by [17, Lemma 3.4]. Hence A is a $J_{\delta_{ss}}$ −supplemented module.

Corollary 2.12. Let S be a ring and A be an S −module. If either A is a projective module, or $S/\delta(S)$ is a semisimple ring, then A is a $J_{\delta_{ss}}$ −supplemented module for an ideal J of S containing $\delta(S)$ if and only if A is a δ_{ss} −supplemented module.

Proof: If a module A is projective, then $\delta(A) = \delta(S)A$ by [8, Lemma 1.9]. Also note that if $S/\delta(S)$ is a semisimple ring, then $\delta(A) = \delta(S)A$ for each S −module A by [8, Theorem 1.8]. It follows from $Soc_{\delta}(A) \leq \delta(A)$ that we can deduce the result from Proposition 2.11.

Corollary 2.13. Let A be a projective S −module and J be an ideal of S which includes $Soc({}_SS)$. Then A is a $J_{\delta_{ss}}$ −supplemented module if and only if A is a δ_{ss} −supplemented module.

The next step will show an example of a module that is δ_{ss} −supplemented although it is not $J_{\delta_{ss}}$ −supplemented with respect to any ideal J of S.

Example 2.14. Let $Q = \prod_{\lambda=1}^{\infty} Q_{\lambda}$, where $Q_{\lambda} = \mathbb{Z}_2$. Suppose that S is the subring of Q generated by $\bigoplus_{\lambda=1}^{\infty} Q_{\lambda}$ and 1_Q. Say $A = {}_SS$. Then A is a regular module that is not semisimple. Thus $Soc(A)$ is a maximal submodule. By [8, Example 4.1], $Soc(A) = \delta(A) \ll_{\delta} A$. A is a δ_{ss} −supplemented module by [13, Lemma 4.1]. On the other hand, since $Rad(S)A = 0$, A is not a $Rad(S)_{\delta_{ss}}$ −supplemented module by Proposition 2.11.

Example 2.15. Assume that t is a prime integer, $i \geq 3$ and A denotes the local $\mathbb{Z}$ −module $\mathbb{Z}_{t^i}$. By [13, Example 4.4(2)], A is not a δ_{ss} −supplemented module. Hence for at least one ideal J of $\mathbb{Z}$, A can not be a $J_{\delta_{ss}}$ −supplemented module.

Proposition 2.16. Let A be an S −module and J be an ideal of S. If A is a $J_{\delta_{ss}}$ −supplemented module, then A/JA is a semisimple module.

Proof: Let $JA \leq B \leq A$. Then by the hypothesis, there is a $C \leq^{\oplus} A$ provided that $A = B + C$, $B \cap C \leq JC$ and $B \cap C \leq Soc_{\delta}(C)$. Thus $A/JA = B/JA + (C + JA)/JA$. Therefore we obtain that $B \cap (C + JA) = JA + (B \cap C) = JA$. This implies that $B/JA \leq^{\oplus} A/JA$. Hence A/JA is a semisimple module.

A module A is called *coatomic* if, each proper submodule is included in a maximal submodule of A (see [18]). Semisimple modules and finitely generated modules can be given as examples of coatomic modules. It is a commonly established fact that coatomic modules have small radical.

Corollary 2.17. Let A be a $J_{\delta_{ss}}$ −supplemented S −module, where J is an ideal of S. Then JA is δ −small in A if and only if A is a coatomic module.

Proof: ($\Longrightarrow$) If $JA = A$, then $A \ll_{\delta} A$, and hence A is a projective semisimple module. Suppose that $JA \neq A$ and $B \leq A$. If $B + JA = A$, then there exists a projective semisimple submodule $P \leq JA$ with $A = P \oplus B$ by [8, Lemma 1.2]. Suppose that $P = \oplus_{\lambda\in\Lambda} P_{\lambda}$, where each P_{λ} is simple and Λ is some index set. For some $\lambda_0 \in \Lambda$, say $X = B \oplus (\oplus_{\lambda\in\Lambda\setminus\{\lambda_0\}} P_{\lambda})$. Then $B \leq X$. Thus $A/X \cong P_{\lambda_0}$, and so $X \leq A$ is a maximal submodule. Now assume that $B + JA \neq A$. Then $(B + JA)/JA \leq A/JA$ is a proper submodule. Since A/JA is semisimple according to Proposition 2.16, there is a maximal submodule $K/JA \leq A/JA$ that contains $(B + JA)/JA$. Therefore $K \leq A$ is a maximal submodule containing B. Hence A is a coatomic module.

($\Longleftarrow$) Since A is a coatomic module, then by [8, Lemma 1.5(4)], $\delta(A)$ is the only biggest δ −small submodule of A. By assumption, we have $A = JA + A$, $JA = JA \cap A \leq JA$ and $JA \leq Soc_{\delta}(A) \leq \delta(A)$. Thus $JA \ll_{\delta} A$ by [8, Lemma 1.3(1)].

Corollary 2.18. Let J be an ideal of S and $A = \oplus_{\lambda\in\Lambda} A_{\lambda}$ be a $J_{\delta_{ss}}$ −supplemented S −module, where each A_{λ} is either a strongly δ −local module or a projective semisimple module. Then JA is δ −small in A.

Proof: It follows from [13, Theorem 2.9] and Corollary 2.17.

Proposition 2.19. Let A be a $J_{\delta_{ss}}$ −supplemented S −module for an ideal J of S. If $JA \leq Rad(A)$, then A is a $\oplus_{ss}$−supplemented module.

Proof: Suppose that $B \leq A$. By the hypothesis, there exists a $C \leq^{\oplus} A$ provided that $A = B + C$, $B \cap C \leq JC$ and $B \cap C \leq Soc_{\delta}(C)$. Note that $B \cap C$ is a semisimple module. Since $JA \leq Rad(A)$, we have $JC = C \cap JA \leq C \cap Rad(A) = Rad(C)$ by [17, Lemma 3.4]. Thus $B \cap C \leq Rad(C)$. Hence A is a $\oplus_{ss}$−supplemented module.

Proposition 2.20. Let S be a Dedekind domain and $A = T(A)$ be an S −module. If A is a $J_{\delta_{ss}}$ −supplemented module, then A is a $\oplus_{ss}$−supplemented module.

Proof: Suppose that $B \leq A$. Then by assumption, there exists $C \leq^{\oplus} A$ provided that $A = B + C$, $B \cap C \leq JC$ and $B \cap C \leq Soc_{\delta}(C)$. Note that $B \cap C \ll_{\delta} C$, and so $B \cap C \ll_{\delta} A$ by [8, Lemma 1.3(2)]. Then $B \cap C \ll A$ by [17, Proposition 2.6]. Therefore $B \cap C \ll C$ by [4, 19.3(5)]. So $B \cap C$ is a semisimple module. Hence A is a $\oplus_{ss}$−supplemented module.

Let $B \leq A$. B is said to be *fully invariant* if for each endomorphism ψ of A, $\psi(B) \leq B$. A module A is said to be *duo* if whole of its submodules are fully invariant as defined in [19].

Proposition 2.21. Let J be an ideal of S and an S −module $A = \oplus_{\lambda\in\Lambda} A_{\lambda}$ be a duo module. If A_{λ} is a $J_{\delta_{ss}}$ −supplemented module for each $\lambda \in \Lambda$, then A is a $J_{\delta_{ss}}$ −supplemented module.

Proof: Suppose that $B \leq A$. Then $B = \oplus_{\lambda\in\Lambda} (B \cap A_{\lambda})$ by [19, Lemma 2.1]. By the hypothesis, there exists a $C_{\lambda} \leq^{\oplus} A_{\lambda}$, where $A_{\lambda} = (B \cap A_{\lambda}) + C_{\lambda}$, $B \cap C_{\lambda} \leq JC_{\lambda}$ and $B \cap C_{\lambda} \leq Soc_{\delta}(C_{\lambda})$ for each $\lambda \in \Lambda$. Put $C = \oplus_{\lambda\in\Lambda} C_{\lambda}$. It is explicit that $C \leq^{\oplus} A$ and $A = B + C$. Moreover, we have $B \cap C = \oplus_{\lambda\in\Lambda} (B \cap C_{\lambda}) \leq JC$, and so by [8, Lemma 1.5(3)] and [4, 21.2(5)], $B \cap C \leq Soc_{\delta}(C)$. Hence A is a $J_{\delta_{ss}}$ −supplemented module.

Proposition 2.22. Let J be an ideal of S and A be a $\oplus_{ss}$−supplemented S −module having the condition $Rad(A) \leq JA$. Then A is a $J_{\delta_{ss}}$ −supplemented module.

Proof: Suppose that $B \leq A$. Then there is a $C \leq^{\oplus} A$, where $A = B + C$, $B \cap C$ is semisimple and $B \cap C \ll C$. Note that $B \cap C \leq Soc_{\delta}(C)$. Moreover, $JC = C \cap JA$ by [17, Lemma 3.4]. Since $Rad(A) \leq JA$, $Rad(C) = Rad(A) \cap C \leq JA \cap C = JC$ by [17, Lemma 3.4]. Hence $B \cap C \leq Rad(C) \leq JC$. This gives the desired.

Corollary 2.23. Let J be an ideal of S and A be an S −module. If A is a $\oplus_{ss}$−supplemented module, where $JA = A$, then A is a $J_{\delta_{ss}}$ −supplemented module.

Corollary 2.24. Let T be a maximal ideal of a commutative ring S, A be an S −module and T' be an ideal of S provided that $TA = T'A$. If A is a $\oplus_{ss}$−supplemented module, then A is a $T'_{\delta_{ss}}$ −supplemented module.

Proof: Since $Rad(A) \leq TA = T'A$, the result follows from Proposition 2.22.

Recall from [4] that an S −module A over a commutative domain S is said to be *divisible* in case $sA = A$ for each nonzero $s \in S$.

Corollary 2.25. Let A be a divisible S −module, where S is a commutative domain. If A is a $\oplus_{ss}$−supplemented module, then A is a $J_{\delta_{ss}}$ −supplemented module for each nonzero ideal J of S.

Proof: The proof follows from Corollary 2.23.

Recall from [4, 23.7] that a ring S is said to be *left good ring* if $Rad(A) = Rad(S)A$ for each S −module A. For instance, semilocal rings are left good rings.

Corollary 2.26. Let A be an S −module. If either A is a projective module, or S is a left good ring, then A is a $Rad(S)_{\delta_{ss}}$ −supplemented module if and only if A is a $\oplus_{ss}$−supplemented module.

Proof: Since $Rad(S)A = Rad(A)$ by [4, 23.7], the result can be obtained from Proposition 2.19 and Proposition 2.22.

Corollary 2.27. Let A be an S −module. If either A is a projective module, or $S/\delta(S)$ is a semisimple ring, then A is a $\delta(S)_{\delta_{ss}}$ −supplemented module if and only if A is a $\oplus_{ss}$−supplemented module.

Proof: Since $\delta(S)A = \delta(A)$ by [8, Theorem 1.8], the result can be derived from Proposition 2.19 and Proposition 2.22.

A module A is said to be *distributive* if $(B + C) \cap K = (B \cap K) + (C \cap K)$ for each $B, C, K \leq A$ (or equivalently, $(B \cap C) + K = (B + K) \cap (C + K)$, for each $B, C, K \leq A$).

Proposition 2.28. Let J be an ideal of S and A be a $J_{\delta_{ss}}$ −supplemented module.

1) If $B \leq A$ having the property $(B + C)/B \leq^{\oplus} A/B$ for each $C \leq^{\oplus} A$, then A/B is a $J_{\delta_{ss}}$ −supplemented module.
2) If $B \leq A$ is a fully invariant submodule, then A/B is a $J_{\delta_{ss}}$ −supplemented module.
3) If A is a distributive module, then A/B is a $J_{\delta_{ss}}$ −supplemented module for each $B \leq A$.

Proof: (1) Suppose that $B \leq K \leq A$. Since A is a $J_{\delta_{ss}}$ −supplemented module, there exists a $C \leq^{\oplus} A$, where $A = K + C$, $K \cap C \leq JC$ and $K \cap C \leq Soc_{\delta}(C)$. Then $A/B = (K/B) + ((C + B)/B)$ and $(K/B) \cap ((C + B)/B) = ((K \cap C) + B)/B \leq (JC + B)/B \leq J((C + B)/B)$. Let $\pi: C \to (C + B)/B$ be the canonical projection. Since $K \cap C \ll_{\delta} C$, we have $\pi(K \cap C) = ((K \cap C) + B)/B \ll_{\delta} (C + B)/B$ by [8, Lemma 1.3(2)]. Also, $\pi(K \cap C)$ is a semisimple module as a factor module of the semisimple module $K \cap C$ by [3, 8.1.5]. Therefore $((K \cap C) + B)/B \leq Soc_{\delta}((C + B)/B)$. Since $(C + B)/B \leq^{\oplus} A/B$ by assumption, then A/B is a $J_{\delta_{ss}}$ −supplemented module.

(2) and (3) are results of (1).

Corollary 2.29. Let J be an ideal of S, A be an S −module and $B \leq^{\oplus} A$ be fully invariant. If A is a $J_{\delta_{ss}}$ −supplemented module, then B is a $J_{\delta_{ss}}$ −supplemented module.

Proof: Suppose that $L \leq B$. By the hypothesis, there are submodules A_1 and A_2 with $A = A_1 \oplus A_2 = A_1 + L$, $A_1 \cap L \leq JA_1$ and $A_1 \cap L \leq Soc_{\delta}(A_1)$. Note that $B = (A_1 \cap B) + L$. Since $B \leq A$ is a fully invariant submodule, then $B = (A_1 \cap B) \oplus (A_2 \cap B)$ by [19, Lemma 2.1]. Thus $A_1 \cap B \leq^{\oplus} A$. However, $J(A_1 \cap B) = (A_1 \cap B) \cap JA$ by [17, Lemma 3.4]. For this reason $(A_1 \cap B) \cap L = A_1 \cap L \leq (A_1 \cap B) \cap JA = J(A_1 \cap B)$. Since $A_1 \cap L \leq Soc_{\delta}(A_1)$ and $A_1 \cap B \leq^{\oplus} A_1$, we have $A_1 \cap L \leq Soc_{\delta}(A_1 \cap B)$ by [8, Lemma 1.3(3)]. Therefore B is a $J_{\delta_{ss}}$ −supplemented module.

Lemma 2.30. Let C be proper submodule of A, where A/C is cyclic. Then the following statements hold:

1) If K is a δ_{ss} −supplement of C in A, then $K = P \oplus Sx$, where $P \leq K \cap C$ is a semisimple projective module and $x \in K$. In this case, Sx is a δ_{ss} −supplement of C in A.
2) If C has a δ_{ss} −supplement that is a direct summand of A, then C has a cyclic δ_{ss} −supplement that is a direct summand of A.

Proof: (1) By assumption, $A = K + C, K \cap C$ is semisimple and $K \cap C \ll_\delta K$. Thus, $A/C \cong K/(K \cap C)$ is cyclic. Assume that x is an element of K with $K = (K \cap C) + Sx$. Since $K \cap C \ll_\delta K$, there is a projective semisimple submodule $P \leq K \cap C$ with $K = P \oplus Sx$ by [8, Lemma 1.2]. Note that $K \cap C = (P \oplus Sx) \cap C = P \oplus (Sx \cap C) \ll_\delta P \oplus Sx$. By [8, Lemma 1.3(3)], we obtain that $P \ll_\delta P$ and $Sx \cap C \ll_\delta Sx$. Therefore P is a projective semisimple module. Also note that $A = Sx + C$ and $Sx \cap C$ is semisimple as a submodule of $K \cap C$ by [3, 8.1.5]. This indicates that Sx is a δ_{ss} −supplement of C in A.
(2) is deduced from (1).

Proposition 2.31. Let S be a non-local Dedekind domain and Q be the quotient field of S. If A is a $J_{\boldsymbol{\delta_{ss}}}$ −supplemented S −module, where J is an ideal of S, then $A/T(A) \cong Q^{(\Lambda)}$ for some index set Λ.

Proof: Suppose that A includes a maximal submodule X containing $T(A)$. Since A is a $J_{\boldsymbol{\delta_{ss}}}$ −supplemented module, there exists a cyclic $Y \leq A$, where $A = X + Y$, $X \cap Y$ is semisimple and $X \cap Y \ll_\delta Y$ by Lemma 2.30. Let S' be an ideal of S with $Y \cong S/S'$. Since Y is not a submodule of X, Y is not a torsion module. Thus $S' = 0$ and $Y \cong {}_SS$. Therefore Y is an indecomposable module. By [17, Proposition 2.3], we have $X \cap Y \ll Y$. Since $Y/(X \cap Y) \cong A/X$, $Y \leq A$ is a strongly local submodule. This is a contradiction, since S is not a local ring. Therefore $Rad(A/T(A)) = A/T(A)$. Hence $A/T(A)$ is an injective module, and so there is an index set Λ with $A/T(A) \cong Q^{(\Lambda)}$.

Proposition 2.32. Let S be a Dedekind domain which is not local and A be a finitely generated S −module. If A is a $J_{\boldsymbol{\delta_{ss}}}$ −supplemented module, where J is an ideal of S, then A is a torsion module.

Proof: Since A is a $J_{\boldsymbol{\delta_{ss}}}$ −supplemented module, there exist $X, Y \leq A$, where $A = X \oplus Y = T(A) + Y$, $T(A) \cap Y \leq JY$ and $T(A) \cap Y \leq Soc_\delta(Y)$. Since $T(A) = T(X) \oplus T(Y)$, we have $A = T(X) \oplus Y$ and $T(A) = T(X) \oplus (T(A) \cap Y)$. Hence $T(X) = X$ and $T(Y) = T(A) \cap Y$. Thus $T(Y) \ll_\delta Y$. So $T(Y) \ll Y$ by [17, Proposition 2.6]. Note that $A/T(A) \cong Y/T(Y)$ is divisible by Proposition 2.31. Then we have $sY + T(Y) = Y$ for each nonzero element s of S. Accordingly, for each nonzero $s \in S, sY = Y$. This means that Y is a divisible module, that is, $Rad(Y) = Y$. But $Rad(Y) \ll Y$ since $Rad(A) \ll A$. Hence $Y = 0$ and $A = X$ is a torsion module.

Corollary 2.33. Let S be a Dedekind domain which is not local and A be a finitely generated S −module. If A is a $J_{\delta_{ss}}$ −supplemented module, where J is an ideal of S, then A is a torsion module, where JA is δ −small in A.

Proof: It is derived from Corollary 2.17 and Proposition 2.32.

3. CONCLUSION

In [13], Nişancı Türkmen and Türkmen described the notion of δ_{ss} −supplemented modules and analyzed all of their algebraic properties. They have established that when the module is projective, necessary and suffient condition for it to be δ_{ss} −supplemented is that each submodule of it has a δ_{ss} −supplement which is also a direct summand of the module itself. Moreover, as mentioned in Zhou's paper, for a projective S −module A, $\delta(A)$ equals $\delta(S)A$. With these observations, we arrive at the fact that we can approach δ_{ss} −supplemented S −modules by considering any ideal J of the ring S, for which we can also make use of direct summands, and thus we define the modules in this study.

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