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A NEW APPROACH TO HYPERBOLIC SPINOR B-DARBOUX EQUATIONS

DOĞAN ÜNAL[1], YASİN ÜNLÜTÜRK[2]

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Abstract. *In this research, spinor descriptions of the curves in surfaces have been given according to the B-Darboux frame in Lorentzian 3-space* $\mathbb{E}_1^3$*. The relations between B-Darboux and Darboux frames have been specified via their spinor conceptions which expressed in both timelike and spacelike surfaces, separately. Additionally, all these spinor representations have been portrayed in the view of the B-Darboux frame (via the curvatures) in Lorentzian 3-space. The findings are supported by some theorems and corollaries.*

Keywords: *Hyperbolic spinors; Bishop-Darboux frame; Darboux frame; Lorentz space.*

1. INTRODUCTION

Clifford algebra is an algebra based on a vector field equipped with a metric, which can be positive or negative definite, and is used for describing the spinors. So, Clifford algebra has a wide and important studying area in differential geometry and theoretical physics [1, 2]. Differential geometry is a crucial tool for physics to understand the real world issues. Thus, spinors are also used to conceive the physical phenomenon. In mathematical areas such as differential geometry, and global analysis, spinors bring extensive usage [3-5].

Motions and flows are characterized by curves which are essential geometric objects. The behavior of curves is considered by raising a trihedron on each point whose vectors are mutually orthogonal to each other. There are main invariants called as curvature and torsion functions describing the nature of the curve. In case that curve lies in a surface, the new invariants which are recalled normal curvature, geodesic curvature, and geodesic torsion, are added to the curvature and torsion functions [6, 7].

Bishop has established a different frame named as Bishop (parallel transport) frame which is well-defined even if the second derivative of a curve vanishes at some points. This frame has been adopted in many research [8, 9]. Additionally, the Darboux frame features curves in surfaces [10]. Recently, an original frame has been offered along curves lying in surfaces, designated as the Bishop Darboux frame in Euclidean and Lorentzian spaces [11-13].

Spinors are specific vectors with two complex components which are figured out via the triad of orthonormal vectors [14, 15]. Concerning various frames such as Serret-Frenet, Bishop, Darboux frames, etc., spinors representing both curves and curves in surfaces, have been portrayed in ambient Euclidean spaces acting as Euclidean and Lorentzian spaces [16-21].

[1] Kırklareli University, Software Engineering, 39100 Kırklareli, Turkey. E-mail: doganunal@klu.edu.tr.
[2] Kırklareli University, Mathematics, 39100 Kırklareli, Turkey. E-mail: yasinunluturk@klu.edu.tr.

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In this study, curves lying in surfaces have been examined by means of Bishop Darboux frame (abbr. B-Darboux frame) by means of spinors in $\mathbb{E}_1^3$. The spinor representations of the pair "curve – surface" have been indicated via B-Darboux frame in Lorentzian 3-space. These spinor terms have been provided with respect to the geodesic and the normal curvatures. The alliances between Darboux and B-Darboux frames have been promoted with the aid of their spinor depictions in timelike and spacelike surfaces, separately.

2. PRELIMINARIES

The Lorentzian space $\mathbb{E}_1^3$ is equipped with the metric

$$g(u,v) = u_1v_1 + u_2v_2 - u_3v_3, \tag{1}$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are two arbitrary vectors in $\mathbb{E}_1^3$. If $g(u,u) > 0$ or $u = 0$, $g(u,u) < 0$ and $g(u,u) = 0$ $(u \neq 0)$, the vector $u \in \mathbb{E}_1^3$ is called spacelike, timelike and lightlike (null) vectors, respectively. From the Eq. (1), the norm of the vector $u \in \mathbb{E}_1^3$ is obtained like as below,

$$\|u\|_L = \sqrt{|g(u,u)|}. \tag{2}$$

The Lorentzian cross product $u \times v$ is defined by

$$u \wedge_L v = \begin{vmatrix} e_1 & e_2 & -e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \tag{3}$$

where $e_1 \wedge e_2 = e_3, e_2 \wedge e_3 = -e_1, e_3 \wedge e_1 = -e_2$ [7]. The tangent vector $\alpha'(s)$ of a space curve $\alpha(s)$ can be spacelike, timelike or null (lightlike) so the curve is called with these names [22]. If $\langle \alpha'(s), \alpha'(s) \rangle = \mp 1$, a non-null curve α is parameterized by arc-length parameter s [23]. A surface in the Lorentz space $\mathbb{E}_1^3$ is called a timelike surface if the induced metric in the surface is a Lorentzian metric and is called a spacelike surface if the induced metric in the surface is a positive definite Riemannian metric, i.e., the normal vector in spacelike (timelike) surface is a timelike (spacelike) vector [7].

The variational equations of the Darboux frame $\{t, g, n\}$ for a spacelike curve $\alpha(s)$ in a spacelike surface M are given as follows:

$$\begin{aligned} \frac{dt}{ds} &= k_g g + k_n n, \\ \frac{dg}{ds} &= -k_g t + \tau_g n, \\ \frac{dn}{ds} &= k_n t + \tau_g n, \end{aligned} \tag{4}$$

where n is the normal of spacelike surface, $g = n \times t$ and

$$\langle t, t \rangle = 1, \langle g, g \rangle = 1, \langle n, n \rangle = -1.$$

The variational equations of the Darboux frame $\{t, g, n\}$ for a spacelike curve $\alpha(s)$ in a timelike surface M are given as follows:

$$\frac{dt}{ds} = k_g g - k_n n, \qquad \frac{dg}{ds} = k_g t + \tau_g n, \qquad \frac{dn}{ds} = k_n t + \tau_g n, \tag{5}$$

where

$$\langle t, t\rangle = 1, \langle g, g\rangle = -1, \langle n, n\rangle = 1.$$

Here ([12]), the geodesic curvature k_g, the normal curvature k_n, and the relative torsion τ_g are given with the following equations:

$$k_g = \kappa cosh\theta, k_n = \kappa sinh\theta, \tau_g = \tau + d\theta. \tag{6}$$

It is well known that the curve $\alpha(s)$ is asymptotic, geodesic or principal curve if and only if $k_n = 0$, $k_g = 0$ or $\tau_g = 0$, respectively [24]. The Bishop–Darboux (abbreviated as B-Darboux) frame is an alternative frame for characterizing curves in surfaces. The B-Darboux frame is set by means of three orthonormal vectors $\{t, b_1, b_2\}$, such that, the tangent vector t is unique and two arbitrary basis vectors b_1 and b_2 lie in the normal plane of the surface. The derivative equations of the B-Darboux frame for a spacelike curve in a spacelike surface are given as follows,

$$\frac{dt}{ds} = n_1 b_1 + n_2 b_2, \qquad \frac{db_1}{ds} = n_1 t, \qquad \frac{db_2}{ds} = -n_2 t, \tag{7}$$

where n_1 and n_2 are B-Darboux curvatures which are expressed with the following equations:

$$n_1 = -k_g \sinh\theta + k_n \cosh\theta, \qquad n_2 = k_n \sinh\theta - k_g \cosh\theta. \tag{8}$$

Here, the hyperbolic angle θ is between the timelike normal vector n and the new timelike normal plane vector b_1. Thus, the rotation matrix between B-Darboux and Darboux frames is presented with the following equation

$$\begin{bmatrix} t \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sinh\theta & \cosh\theta \\ 0 & -\cosh\theta & -\sinh\theta \end{bmatrix} \begin{bmatrix} t \\ g \\ n \end{bmatrix}. \tag{9}$$

On the other hand, the variational equations of the B-Darboux frame $\{t, b_1, b_2\}$ for a spacelike curve in a timelike surface are given as follows:

$$\frac{dt}{ds} = n_1 b_1 - n_2 b_2,$$
$$\frac{db_1}{ds} = n_1 t, \tag{10}$$
$$\frac{db_2}{ds} = n_2 t.$$

Here B-Darboux curvatures n_1 and n_2 are given as follows:

$$n_1 = k_g \cosh\theta + k_n \sinh\theta$$
$$n_2 = k_n \cosh\theta + k_g \sinh\theta. \tag{11}$$

Here, the hyperbolic angle θ is between the timelike vector g and the new timelike normal plane vector b_1. Thus ([12]) the rotation matrix between B-Darboux and Darboux frames, is indicated with the equation

$$\begin{bmatrix} t \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sinh\theta & \cosh\theta \\ 0 & \cosh\theta & \sinh\theta \end{bmatrix} \begin{bmatrix} t \\ g \\ n \end{bmatrix}. \tag{12}$$

The group $U(n, \mathbb{H})$ is said to be hyperbolic unitary group established by the set of Hermitian $n \times n$ matrices. The subgroup $SO(1,3)$ is a special Lorentzian group composed by Lorentzian transformation whose determinant is $+1$ [25]. The relation between the groups $SO(1,3)$ and $SU(2,\mathbb{H})$ is a special one called as homomorphism. By means of this homomorphism, Hermitian matrices of the group $SU(2,\mathbb{H})$ serve as hyperbolic spinors while the elements of the subgroup $SO(1,3)$ express vectors in Lorentz space [26].

A hyperbolic spinor can be defined as

$$\Omega = \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}, \tag{13}$$

by means of three vectors $a, b, c \in \mathbb{E}_1^3$ such that

$$a + jb = \Omega^t \sigma \Omega, \mathrm{c} = -\widehat{\Omega}^t \sigma \Omega, \tag{14}$$

where $j^2 = 1$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a vector whose cartesian components are the hyperbolic symmetric $2x2$ matrices

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} \mathrm{j} & 0 \\ 0 & \mathrm{j} \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{15}$$

which are the products of the matrix

$$K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{16}$$

by the Pauli matrices employed in physics [15].

Let the hyperbolic spinor $\widehat{\Omega}$ be the mate and $\overline{\Omega}$ be the conjugation of hyperbolic spinor Ω. Then the following equations are satisfied:

$$\widehat{\Omega} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\Omega} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\Omega}_1 \\ \overline{\Omega}_2 \end{pmatrix} = \begin{pmatrix} -\overline{\Omega}_2 \\ \overline{\Omega}_1 \end{pmatrix}. \tag{17}$$

If it is chosen as $a + jb = (x_1, x_2, x_3)$, it is obtained from the Eq. (15) and (17) that

$$a + jb = \Omega^t \sigma \Omega = (\Omega_1^2 - \Omega_2^2, i(\Omega_1^2 + \Omega_2^2), -2\Omega_1\Omega_2). \tag{18}$$

Likewise, it can be seen that

$$c = (c_1, c_2, c_3) = (\Omega_1\overline{\Omega}_2 + \overline{\Omega}_1\Omega_2, i(\Omega_1\overline{\Omega}_2 - \overline{\Omega}_1\Omega_2), |\Omega_1|^2 - |\Omega_2|^2).$$

The norms $\|a\|_L = \|b\|_L = \|c\|_L = \overline{\Omega}^t\Omega$ are obtained by using the vector $a + jb$ which is an isotropic one, that is, $g(a + jb, a + jb) = 0$. The equality $\overline{\Omega'}^t\Omega' = \overline{\Omega}^t\Omega$ is satisfied for the equation $\Omega' = U\Omega$ such that a matrix $U \in SU(2, \mathbb{H})$. All transformations from the orthogonal basis of $\mathbb{E}_1^3$ to another orthogonal basis of the same space are the elements of $SU(2, \mathbb{H})$. The elements U, and $-U$ of $SU(2, \mathbb{H})$ match the same ordered set of $\mathbb{E}_1^3$. This occurs because the transformation from $SU(2, \mathbb{H})$ to $SO(1, 3)$ is a two-to-one homomorphism. On the other hand, the sets $\{a, b, c\}$ performs to the spinor Ω. So it can be expressed that the different ordered sets of $\mathbb{E}_1^3$ represent to the different hyperbolic spinors. But the same set can be shown by the spinors Ω, and $-\Omega$. Then, the equations below are satisfied for hyperbolic spinors λ and Ω:

$$\begin{gathered} \overline{\lambda^t\sigma\Omega} = -\hat{\lambda}^t\sigma\widehat{\Omega}, \\ \widehat{a\lambda + b\Omega} = \bar{a}\hat{\lambda} + \bar{b}\widehat{\Omega}, \\ \hat{\hat{\Omega}} = -\Omega, \end{gathered} \tag{19}$$

where a and b are hyperbolic numbers. Moreover, the ordered sets $\{a, b, c\}, \{b, c, a\}, \{c, a, b\}$ represent distinct hyperbolic spinors. The following equation is satisfied for any pair of hyperbolic spinors λ and Ω

$$\lambda^t\sigma\,\Omega = \Omega^t\,\sigma\,\lambda, \tag{20}$$

where the matrices σ are symmetric [27].

3. HYPERBOLIC SPINOR DESCRIPTION OF B-DARBOUX FRAME IN NON-NULL SURFACES

In this section, the causal characters of B-Darboux frame have been revealed via hyperbolic spinor equations for a spacelike curve in spacelike and timelike surfaces, respectively, with the following subsections in Lorentz 3-space.

3.1. Hyperbolic spinor description of B-Darboux frame along a spacelike curve in a spacelike surface

In this subsection of the study, it has been investigated that the hyperbolic spinor representation of B-Darboux frame along a spacelike unit speed curve $\alpha(s)$ lying in a spacelike surface M. Additionally, the relations between B-Darboux and Darboux frames according to hyperbolic spinors have been given for the curve $\alpha(s)$.

Theorem 1. Let λ be a hyperbolic spinor depictions of the B-Darboux frame $\{b_1, b_2, t\}$ of a unit speed spacelike curve $\alpha(s)$ in a spacelike surface $M = M(u,v)$ on $\mathbb{E}_1^3$. Then, a hyperbolic spinor equation is written with the following equation

$$\frac{d\lambda}{ds} = \frac{-n_1 + j\, n_2}{2}\, \hat{\lambda}. \tag{21}$$

Proof: From Eq. (18), it can be written as

$$b_1 + jb_2 = \lambda^t \sigma \lambda, t = -\hat{\lambda}^t \sigma \lambda, \tag{22}$$

where $\bar{\lambda}^t \lambda = 1$. Furthermore, the set $\{\lambda, \hat{\lambda}\}$ asserts a basis for the spinors $(\lambda \neq 0)$ with two components. So, for two arbitrary complex-valued functions f and g, the variations of the set $\{b_1, b_2, t\}$ along the curve lying in a surface M can be addressed as follows:

$$\frac{d\lambda}{ds} = f\lambda + g\hat{\lambda}. \tag{23}$$

Taking derivative of the first equation of Eq. (22) with respect to s, it is attained that

$$\frac{db_1}{ds} + j\frac{db_2}{ds} = \frac{d}{ds}\,(\lambda^t \sigma \lambda) = (\frac{d\lambda}{ds})^t\, \sigma\, \lambda + \lambda^t \sigma \left(\frac{d\lambda}{ds}\right). \tag{24}$$

Substituting the Eqs. (7), (22), (23) into (24), it is identified as

$$(n_1 - jn_2)t = 2\, f\, (b_1 + jb_2) - 2\, g\, t. \tag{25}$$

From the last equation, it is evident that

$$f = 0, g = \frac{-n_1 + j\, n_2}{2}. \tag{26}$$

If Eq. (26) is replaced into Eq. (23), the proof is completed.

Corollary 1. The hyperbolic spinor representation of B-Darboux frame $\{b_1, b_2, t\}$ of a unit speed spacelike curve $\alpha(s)$ lying in a spacelike surface M can be written in the view of Darboux curvatures from Eqs. (8) and (21) in such a way

$$\frac{d\lambda}{ds} = -\frac{e^{-j\theta}}{2}\left(k_n + j\, k_g\right)\hat{\lambda}, \tag{27}$$

where θ is the hyperbolic angle between the timelike normal vector n and the new timelike normal plane vector b_1. If the cases of the spacelike curve in the spacelike surface being asymptotic and geodesic are investigated, the spinor portrayals of B-Darboux frame with reference to Darboux curvatures are allowed by coming corollaries.

Corollary 2. If the unit speed spacelike curve $\alpha(s)$ lying in a spacelike surface M is asymptotic, then the hyperbolic spinor representation of B-Darboux frame is obtained from the Eq. (27) as follows:

$$\frac{d\lambda}{ds} = -j\frac{e^{-j\theta}}{2}\, k_g\, \hat{\lambda}. \tag{28}$$

Corollary 3. If the unit speed spacelike curve $\alpha(s)$ lying in a spacelike surface M is geodesic, then the hyperbolic spinor representation of B-Darboux frame is obtained from the Eq. (27) as follows:

$$\frac{d\lambda}{ds} = -\frac{e^{-j\theta}}{2}\, k_n\, \hat{\lambda}. \tag{29}$$

Theorem 2. The relation between the spinor formulations of B-Darboux and Darboux frames in $\mathbb{E}_1^3$ for a spacelike curve lying in a spacelike surface is presented as

$$\begin{gathered} \lambda^t \sigma\lambda = -je^{-j\theta}(\overline{\Omega^t \sigma\Omega}) \\ t = t, \end{gathered} \tag{30}$$

where the spinors λ and Ω mean the B-Darboux frame $\{b_1, b_2, t\}$ and the Darboux frame $\{g, n, t\}$, respectively.

Proof: From Eq. (9), it can be written

$$\begin{aligned} t &= t, \\ b_1 &= g \sin h\theta + n \cos h\theta, \\ b_2 &= -g \cos h\theta - n \sin h\theta, \end{aligned}$$

and

$$b_1 + jb_2 = -je^{-j\theta}(g - jn). \tag{31}$$

From the Eqs. (17), (18), (22) and (31), the Eq. (31) is obtained and proof is completed.

Lemma 1. Let α be a regular spacelike unit speed curve lying in a spacelike surface M in Lorentz 3-space $\mathbb{E}_1^3$. If the hyperbolic rotation angle between the B-Darboux frame $\{b_1, b_2, t\}$ and the Darboux frame $\{g, n, t\}$ is θ, there is a relation between the hyperbolic spinors λ and Ω that resembles to these frames, respectively, as:

$$\lambda = e^{j\frac{\theta}{2}}\Omega. \tag{32}$$

Proof: Let $g + jn = (1, j, 0)$ be an isotropic vector. In addition, by Eq. (18), the subsequent equation is reached as:

$$g + jn = (x_1, x_2, x_3) = (1, j, 0) = \Omega^t \sigma \Omega = (\Omega_1^2 - \Omega_2^2, j(\Omega_1^2 + \Omega_2^2), -2\Omega_1\Omega_2),$$

then

$$\Omega_1 = \mp\sqrt{\frac{x_1 + jx_2}{2}}, \Omega_2 = \mp\sqrt{\frac{-x_1 + jx_2}{2}}. \tag{33}$$

So, it is obtained as $\Omega = (\Omega_1, \Omega_2) = (\pm 1, 0)$ with substituting the equality $(x_1, x_2, x_3) = (1, j, 0)$ into the Eq. (33). Darboux triad $\{g, n, t\}$ is rotated with the hyperbolic angle θ. Here, the hyperbolic spinor Ω rotates to the hyperbolic spinor λ while the triad $\{g, n, t\}$ rotating to the triad $\{b_1, b_2, t\}$, then from the Eq. (9), it can be written as

$$\begin{aligned} b_1 - jb_2 &= e^{j\theta}(g + jn) = e^{j\theta}(x_1, x_2, x_3) \\ &= \lambda^t \sigma \lambda = (\lambda_1^2 - \lambda_2^2, j(\lambda_1^2 + \lambda_2^2), -2\lambda_1\lambda_2), \end{aligned}$$

and here if it is taken the conjugate of all equalities, then the last equation is obtained as follows:

$$\begin{aligned} b_1 + jb_2 &= e^{-j\theta}(g - jn) = e^{-j\theta}(\overline{x_1, x_2, x_3}) \\ &= \bar{\lambda}^t \sigma \bar{\lambda} = \left(\bar{\lambda}_1^2 - \bar{\lambda}_2^2, -j\left(\bar{\lambda}_1^2 + \lambda_2^2\right), -2\bar{\lambda}_1\bar{\lambda}_2\right). \end{aligned}$$

After the necessary computations,

$$\bar{\lambda}_1 = \mp e^{-j\frac{\theta}{2}}\sqrt{\frac{x_1 - jx_2}{2}}, \bar{\lambda}_2 = \mp e^{-j\frac{\theta}{2}}\sqrt{\frac{-x_1 - jx_2}{2}}$$

are obtained. Thus, from the equality $(\overline{x_1, x_2, x_3}) = (1, -j, 0)$, the equalities are seen like that

$$\bar{\lambda}_1 = \mp e^{-j\frac{\theta}{2}}, \bar{\lambda}_2 = 0.$$

Finally, if it is taken the conjugate of the binary $\bar{\lambda} = \left(\bar{\lambda}_1, \bar{\lambda}_2\right) = \left(\mp e^{-j\frac{\theta}{2}}, 0\right)$, then the following equations are acquired and the proof is completed:

$$\lambda = (\lambda_1, \lambda_2) = \left(\mp e^{j\frac{\theta}{2}}, 0\right) = e^{j\frac{\theta}{2}}(\mp 1, 0) = e^{j\frac{\theta}{2}}\Omega.$$

3.2. Hyperbolic spinor description of B-Darboux frame along a spacelike curve in a timelike surface

In this subsection of the study, the findings in the Section 3.1 have been investigated for a spacelike unit speed curve $\alpha(s)$ lying in a timelike surface M in Lorentz 3-space $\mathbb{E}_1^3$.

Theorem 3. Let λ be a hyperbolic spinor representation of the B-Darboux frame $\{b_1, b_2, t\}$ of a unit speed spacelike curve $\alpha(s)$ lying in a timelike surface $M = M(u, v)$ on $\mathbb{E}_1^3$. Then, a hyperbolic spinor equation is written as

$$\frac{d\lambda}{ds} = \frac{-n_1 - j\, n_2}{2}\, \hat{\lambda}. \tag{34}$$

Proof: If computations are applied for a spacelike curve in a timelike surface by using the Eq. (10), we obtain

$$(n_1 + jn_2)t = 2\, f\, (b_1 + jb_2) - 2\, g\, t. \tag{35}$$

From the last equation, it is obtained that

$$f = 0, g = \frac{-n_1 - j\, n_2}{2}. \tag{36}$$

If Eq. (36) is replaced into Eq. (23), proof is completed.

Corollary 4. The hyperbolic spinor representation of B-Darboux frame $\{b_1, b_2, t\}$ of a unit speed spacelike curve $\alpha(s)$ lying in a timelike surface M can be written in the view of Darboux curvatures from Eqs. (11) and (34) in such a way

$$\frac{d\lambda}{ds} = -\frac{e^{j\theta}}{2}\left(k_g + j\, k_n\right)\hat{\lambda}, \tag{37}$$

where θ is the hyperbolic angle between the timelike vector g and the new timelike normal plane vector b_1. If the cases of the spacelike curve in the timelike surface being asymptotic and geodesic are investigated, the spinor depictions of B-Darboux frame with reference to Darboux curvatures are provided by corollaries;

Corollary 5. If the unit speed spacelike curve $\alpha(s)$ lying in a timelike surface M is asymptotic, then the hyperbolic spinor representation of B-Darboux frame is obtained from the Eq. (37) as follows:

$$\frac{d\lambda}{ds} = -\frac{e^{j\theta}}{2}\, k_g\, \hat{\lambda}. \tag{38}$$

Corollary 6. If the unit speed spacelike curve $\alpha(s)$ lying in a timelike surface M is geodesic, then the hyperbolic spinor representation of B-Darboux frame is obtained from the Eq. (37) as follows:

$$\frac{d\lambda}{ds} = -j\frac{e^{j\theta}}{2}\, k_n\, \hat{\lambda}. \tag{39}$$

Theorem 4. The relation between the spinor formulations of B-Darboux and Darboux frames in $\mathbb{E}_1^3$ for a spacelike curve lying in a timelike surface is expressed as

$$\begin{aligned} \lambda^t \sigma\lambda &= je^{j\theta}(\Omega^t \sigma\Omega), \\ t &= t, \end{aligned} \tag{40}$$

where the spinors λ and Ω mean the B-Darboux frame $\{b_1, b_2, t\}$ and the Darboux frame $\{g, n, t\}$, respectively.

Proof: From Eq. (12), it can be written

$$\begin{aligned} t &= t, \\ b_1 &= g \sin h\theta + n \cos h\theta, \\ b_2 &= g \cos h\theta + n \sin h\theta, \end{aligned}$$

and

$$b_1 + jb_2 = je^{j\theta}(g + jn). \tag{41}$$

From the Eqs. (18) and (41), the Eq. (40) is obtained and proof is completed.

Lemma 2. Let α be a regular spacelike unit speed curve lying in a timelike surface M in Lorentz 3-space $\mathbb{E}_1^3$. If the hyperbolic rotation angle between the B-Darboux frame $\{b_1, b_2, t\}$ and the Darboux frame $\{g, n, t\}$ is θ, there is a relation between the hyperbolic spinors λ and Ω that coincide to these frames respectively as

$$\lambda = e^{j\frac{\theta}{2}}\Omega. \tag{42}$$

Proof: It is known that $\Omega = (\Omega_1, \Omega_2) = (\pm 1, 0)$ from the Lemma 1 and if the similar computations are applied for a spacelike curve in a timelike surface by using the Eq. (12), it is obtained

$$\begin{aligned} b_1 + jb_2 &= e^{j\theta}(g + jn) = e^{j\theta}(x_1, x_2, x_3) \\ &= \lambda^t \sigma \lambda = (\lambda_1^2 - \lambda_2^2, j(\lambda_1^2 + \lambda_2^2), -2\lambda_1\lambda_2). \end{aligned}$$

After the necessary computations, we have

$$\lambda_1 = \mp e^{j\frac{\theta}{2}}\sqrt{\frac{x_1 + jx_2}{2}}, \lambda_2 = \mp e^{j\frac{\theta}{2}}\sqrt{\frac{-x_1 + jx_2}{2}}$$

for hyperbolic spinor λ. Here, if the isotropic vector $(x_1, x_2, x_3) = (1, j, 0)$ is substituted into the last equalities, we find

$$\lambda_1 = \mp e^{j\frac{\theta}{2}}, \lambda_2 = 0.$$

is obtained. Finally, the following equations are acquired:

$$\lambda = (\lambda_1, \lambda_2) = \left(\mp e^{j\frac{\theta}{2}}, 0\right) = e^{j\frac{\theta}{2}}(\mp 1, 0) = e^{j\frac{\theta}{2}}\Omega.$$

Thus the proof is completed.

5. CONCLUSION

In this analyze, the spinor representations of B-Darboux frame $\{t, b_1, b_2\}$ for curves in some surfaces have been investigated in Lorentz 3-space. The relations between B-Darboux and Darboux frames for spacelike curves in spacelike and timelike surfaces, respectively, have been put by adopting spinors. Furthermore, some results have been attained for special cases of the curvatures of B-Darboux and Darboux frames.

In this context, spinor representations of B-Darboux frame have been characterized from the point of the view of Darboux frame at curve-surface pair, that is curves to be geodesic or asymptotic ones. In addition, some findings on the hyperbolic rotation angle between the spinors of the frames have been obtained in $\mathbb{E}_1^3$.

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