ORIGINAL PAPER

ROUGH LACUNARY STATISTICAL ANALYSIS OF ORDER η OF TRIPLE DIFFERENCE SEQUENCE SPACES

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Manuscript received: 27.01.2023; Accepted paper: 12.01.2024;
Published online: 30.03.2024.

Abstract. *We generalized the concepts in the probability of rough lacunary statistical analysis of order η by introducing the difference operator Δ_γ^α of fractional order, where α is a proper fraction and $\gamma = (\gamma_{mnk})$ is any fixed sequence of nonzero real or complex numbers. We study some properties of this operator involving lacunary sequence θ and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers and investigate the topological structures of related with triple difference sequence spaces. The main focus of the present paper is to generalize rough lacunary statistical analysis of order η of triple difference sequence spaces and give the relations between statistical analysis and lacunary statistical analysis of order η.*

Keywords: *Density; analytic sequence; Musielak-Orlicz function; triple sequences; statistical convergence; lacunary statistical convergence.*

1. INTRODUCTION

A triple sequence (real or complex) can be defined as a function $x: \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [1, 2], Esi et al. [3-5], Dutta et al. [6], Subramanian et al. [7], Debnath et al. [8], Esi et al. [9] and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3. A triple sequence $x = (x_{mnk})$ is called triple gai sequence if

$$\left((m+n+k)! \, |x_{mnk}|\right)^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty.$$

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [10] as follows

$$Z(\Delta) = \{x = (x_k) \in w: (\Delta x_k) \in Z\}$$

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https://doi.org/10.46939/J.Sci.Arts-24.1-a05

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

The difference triple sequence space was introduced by Debnath et al. (see [8]) and is defined as

$$\Delta x_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1} \text{ and } \Delta^0 x_{mnk} = \langle x_{mnk} \rangle.$$

2. SOME NEW DIFFERENCE TRIPLE SEQUENCE SPACES WITH FRACTIONAL ORDER

Let $\Gamma(\alpha)$ denote the Euler gamma function of a real number α. Using the definition $\Gamma(\alpha)$ with $\alpha \notin \{0, -1, -2, -3, \cdots\}$ cab be expressed as an improper integral as follows: $\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx$, where α is a positive proper fraction. We have defined the generalized fractional triple sequence spaces of difference operator

$$\Delta_\gamma^\alpha(x_{mnk}) = \sum_{u=0}^\infty \sum_{v=0}^\infty \sum_{w=0}^\infty \frac{(-1)^{u+v+w}\Gamma(\alpha+1)}{(u+v+w)!\,\Gamma(\alpha-(u+v+w)+1)} x_{m+u,n+v,k+w}. \tag{2.1}$$

In particular, we have

(i) $\Delta^{\frac{1}{2}}(x_{mnk}) = x_{mnk} - \frac{1}{16} x_{m+1,n+1,k+1} - \cdots$.

(ii) $\Delta^{-\frac{1}{2}}(x_{mnk}) = x_{mnk} + \frac{5}{16} x_{m+1,n+1,k+1} + \cdots$.

(iii) $\Delta^{\frac{2}{3}}(x_{mnk}) = x_{mnk} - \frac{4}{81} x_{m+1,n+1,k+1} - \cdots$.

Now we determine the new classes of triple difference sequence spaces $\Delta_\gamma^\alpha(x)$ as follows:

$$\Delta_\gamma^\alpha(x) = \left\{x : (x_{mnk}) \in w^3 : \left(\Delta_\gamma^\alpha x\right) \in X\right\}, \tag{2.2}$$

where

$$\Delta_\gamma^\alpha(x_{mnk}) = \sum\nolimits_{u=0}^\infty \sum\nolimits_{v=0}^\infty \sum\nolimits_{w=0}^\infty \frac{(-1)^{u+v+w}\Gamma(\alpha+1)}{(u+v+w)!\,\Gamma(\alpha-(u+v+w)+1)} x_{m+u,n+v,k+w}$$

and

$$X \in \chi_f^{3\Delta}(x) = \chi_f^3\left(\Delta_\gamma^\alpha x_{mnk}\right) = \mu_{mnk}\left(\Delta_\gamma^\alpha x\right) = \left[f_{mnk}\left(\left((m+n+k)!\,\left|\Delta_\gamma^\alpha\right|\right)^{\frac{1}{m+n+k}}, \bar{0}\right)\right].$$

Proposition 2.1. (i) For a proper fraction α, $\Delta^\alpha : W^3 \to W^3$ defined by equation of (2.1) is a linear operator.

(ii) For $\alpha, \beta > 0$, $\Delta^\alpha\left(\Delta^\beta(x_{mnk})\right) = \Delta^{\alpha+\beta}(x_{mnk})$ and $\Delta^\alpha\left(\Delta^{-\alpha}(x_{mnk})\right) = x_{mnk}$.

Proof: Omitted.

Proposition 2.2. For a proper fraction α and f be an Musielak-Orlicz function, if $\chi_f^3(x)$ is a linear space, then $\chi_f^{3\Delta_\gamma^\alpha}(x)$ is also a linear space.

Proof: Omitted.

3. DEFINITIONS AND PRELIMINARIES

Throughout the article $w^3, \chi^3(\Delta), \Lambda^3(\Delta)$ denote the spaces of all, triple gai difference sequence spaces and triple analytic difference sequence spaces respectively. Subramanian et al. (see [7]) introduced by a triple entire sequence spaces, triple analytic sequences spaces and triple gai sequence spaces. The triple sequence spaces of $\chi^3(\Delta), \Lambda^3(\Delta)$ are defined as follows:

$$\chi^3(\Delta) = \left\{x \in w^3 : \left((m+n+k)!\,|\Delta x_{mnk}|\right)^{1/m+n+k} \to 0 \quad as \quad m,n,k \to \infty\right\},$$

$$\Lambda^3(\Delta) = \left\{x \in w^3 : sup_{m,n,k} |\Delta x_{mnk}|^{1/m+n+k} < \infty\right\}.$$

Definition 3.1. An Orlicz function ([see [11]) is a function $f: [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $f(0) = 0$, $f(x) > 0$, for $x > 0$ and $f(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function f is replaced by $f(x+y) \leq f(x) + f(y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri ([12]) used the idea of Orlicz function to construct Orlicz sequence space. A sequence $g = (g_{mnk})$ defined by

$$g_{mnk}(v) = sup\{|v|u - (f_{mnk})(u) : u \geq 0\}, m,n,k = 1,2,\cdots\}$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, [13] the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \to 0 \quad as \quad m,n,k \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right)$$

is an extended real number.

4. ROUGH LACUNARY STATISTICAL ANALYSIS OF ORDER η

In this section by using the operator Δ_γ^α, we introduce some new triple difference sequence spaces involving rough lacunary statistical analysis of order η sequences spaces and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers.

Definition 4.1. The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$m_0 = 0, h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty \text{ and}$$
$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty.$$
$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determine by

$$I_{i,\ell,j} = \{(m,n,k): m_{i-1} < m < m_i \quad and \quad n_{\ell-1} < n \le n_\ell \quad and \quad k_{j-1} < k \le k_j\}, q_i = \frac{m_i}{m_{i-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}.$$

Definition 4.2. Let α be a proper fraction, f be an Musielak-Orlicz function and $\theta = \{m_r n_s k_t\}_{(rst)\in\mathbb{N}\cup\{0\}}$ be the triple difference lacunary of order η sequence spaces of $(\Delta_\gamma^\alpha X_{mnk})$ is said to be $\Delta_\gamma^\alpha -$ lacunary statistically of order η convergent to a number $\bar{0}$ if for any $\epsilon > 0$,

$$lim_{r,s,t\to\infty} \frac{1}{h_{rst}^\eta} \left|\{(m,n,k) \in I_{rst}: f_{mnk}[|\Delta_\gamma^\alpha X_{mnk}, \bar{0}|] \ge \epsilon\}\right| = 0,$$

Where

$$I_{r,s,t} = \{(m,n,k): m_{r-1} < m < m_r \quad and \quad n_{s-1} < n \le n_s \quad and \quad k_{t-1} < k \le k_t\}, q_r = \frac{m_r}{m_{r-1}}, \overline{q_s} = \frac{n_s}{n_{s-1}}, \overline{q_t} = \frac{k_t}{k_{t-1}}.$$

In this case write $\Delta_\gamma^\alpha X \to^{S_\theta} \Delta_\gamma^\alpha x$.

Definition 4.3. If α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and $\theta = \{m_r n_s k_t\}_{(rst)\in\mathbb{N}\cup\{0\}}$ be the triple difference sequence spaces of lacunary of order η. A number X is said to be $\Delta_\gamma^\alpha - N_\theta -$ convergent to a real number $\bar{0}$ if for every $\epsilon > 0$, $lim_{r,s,t\to\infty} \frac{1}{h_{rst}^\eta} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} f_{mnk}[|\Delta_\gamma^\alpha X_{mnk}, \bar{0}|] = 0$. In this case we write $\Delta_\gamma^\alpha X_{mnk} \to^{N_\theta} \bar{0}$.

Definition 4.4. Let α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and arbitary sequence $p = (p_{rst})$ of strictly positive real numbers. A triple difference sequence spaces of random variables is said to be $\Delta_\gamma^\alpha -$ rough lacunary statistically of order η convergent in probability to $\Delta_\gamma^\alpha X: W^3 \to \mathbb{R}^3$ with respect to the roughness of degree

β if for any $\epsilon, \delta > 0$, $lim_{rst\to\infty} \frac{1}{h_{rst}^{\eta}} \left|\left\{(m,n,k) \in I_{rst}: P\left(\left[f_{mnk}\left(\left|\Delta_{\gamma}^{\alpha}(x_{mnk})\right|\right)\right]^{p_{rst}} \geq \beta + \epsilon\right) \geq \delta\right\}\right| = 0$ and we write $\Delta_{\gamma}^{\alpha} X_{mnk} \to_{\beta}^{S^P} \bar{0}$. It will be denoted by βS_{θ}^{P}.

Definition 4.5. Let α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and arbitary sequence $p = (p_{rst})$ of strictly positive real numbers. A triple difference sequence spaces of random variables is said to be $\Delta_{\gamma}^{\alpha} -$ rough $N_{\theta} -$ convergent of order η in probability to $\Delta_{\gamma}^{\alpha} X: W^3 \to \mathbb{R}^3$ with respect to the roughness of degree β if for any $\epsilon > 0$, $lim_{r,s,t\to\infty} \frac{1}{h_{rst}^{\eta}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} \left|\left\{P\left(\left[f_{mnk}\left(\left|\Delta_{\gamma}^{\alpha} X_{mnk}\right|\right)\right]^{p_{rst}} \geq \beta + \epsilon\right)\right\}\right| = 0$, and we write $\Delta_{\gamma}^{\alpha} X_{mnk} \to_{\beta}^{N_{\theta}^{P}} \Delta_{\gamma}^{\alpha} X$. The class of all $\beta - N_{\theta} -$ convergent triple difference sequence spaces of random variables in probability will be denoted by βN_{θ}^{P}.

Definition 4.6. Let α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and arbitary sequence $p = (p_{rst})$ of strictly positive real numbers. A triple difference sequence spaces of random variables is said to be $\Delta_{\gamma}^{\alpha} -$ rough lacunary statistically of order η Cauchy if there exists a number $N = N(\epsilon)$ in probability to $\Delta_{\gamma}^{\alpha} X: W^3 \to \mathbb{R}^3$ with respect to the roughness of degree β if for any $\epsilon, \delta > 0$, $lim_{rst\to\infty} \frac{1}{h_{rst}^{\eta}} \left|\left\{(m,n,k) \in I_{rst}: P\left(\left[f_{mnk}\left(\left|\Delta_{\gamma}^{\alpha}(x_{mnk} - x_N)\right|\right)\right]^{p_{rst}} \geq \beta + \epsilon\right) \geq \delta\right\}\right| = 0$.

Theorem 4.7. Let α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and arbitary sequence $p = (p_{rst})$ of strictly positive real numbers of order η, $0 < p < \infty$. (i) If $(x_{mnk}) \to \left(N\left(\Delta_{\gamma}^{\alpha}, p\right)_{\theta}\right)$ for $p_{rst} = p$ then $(x_{mnk}) \to \left(\Delta_{\gamma}^{\alpha}(S_{\theta})\right)$. (ii) If $x \in \left(\Delta_{\gamma}^{\alpha}(S_{\theta})\right)$, then $(x_{mnk}) \to \left(N\left(\Delta_{\gamma}^{\alpha}, p\right)_{\theta}\right)$.

Proof: Let $x = (x_{mnk}) \in \left(N\left(\Delta_{\gamma}^{\alpha}, p\right)_{\theta}\right)$ and > 0, $\left|\left\{P\left(\left[f_{mnk}\left(\left|\Delta_{\gamma}^{\alpha} X_{mnk}\right|\right)\right]^{p_{rst}} \geq \beta + \epsilon\right)\right\}\right| = 0$. We have

$$\frac{1}{h_{rst}^{\eta}} \sum_{(mnk)\in I_{rst}} \left|\left\{P\left(\left[f_{mnk}\left(\left|\Delta_{\gamma}^{\alpha} X_{mnk}\right|\right)\right]^{p_{rst}} \geq \beta + \epsilon\right)\right\}\right|$$
$$\geq \frac{1}{h_{rst}^{\eta}} \left|\left\{(m,n,k) \in I_{rst}: P\left(\left[f_{mnk}\left(\left|\Delta_{\gamma}^{\alpha}(x_{mnk})\right|\right)\right]^{p_{rst}} \geq \beta + \epsilon\right) \geq \delta\right\}\right| \left(\frac{\beta + \epsilon}{\delta}\right)^{p}.$$

So we observe by passing to limit as $r, s, t \to \infty$,

$$lim_{r,s,t\to\infty} \frac{1}{h_{rst}^{\eta}} \left|\left\{(m,n,k) \in I_{rst}: P\left(\left[f_{mnk}\left(\left|\Delta_{\gamma}^{\alpha}(x_{mnk})\right|\right)\right]^{p_{rst}} \geq \beta + \epsilon\right) \geq \delta\right\}\right|$$
$$\leq \left(\frac{\delta}{\alpha + \epsilon}\right)^{p} P\left(lim_{rst\to\infty} \frac{1}{h_{rst}^{\eta}} \sum_{(m,n,k)\in I_{rst}} \left|\Delta_{\gamma}^{\alpha} x_{mnk}\right|^{p}\right) = 0$$

which implies that $x_{mnk} \to \left(\Delta_{\gamma}^{\alpha}(S_{\theta})\right)$.

Suppose that $x \in \Delta_{\gamma}^{\alpha}(\Lambda^3)$ and $(x_{mnk}) \to \left(\Delta_{\gamma}^{\alpha}(S)\right)$. Then it is obvious that $\left(\Delta_{\gamma}^{\alpha} x\right) \in \Lambda^3$ and $\frac{1}{h_{rst}^{\eta}} \left|\left\{(m,n,k) \in I_{rst}: P\left(\left[f_{mnk}\left(\left|\Delta_{\gamma}^{\alpha}(x_{mnk})\right|\right)\right]^{p_{rst}} \geq \beta + \epsilon\right) \geq \delta\right\}\right| \to 0$ as $r, s, t \to \infty$. Let $\epsilon > 0$ be given and there exists $u_0 v_0 w_0 \in \mathbb{N}$ such that $\left|\left\{(m,n,k) \in I_{rst}: P\left(\left[f_{mnk}\left(\left|\Delta_{\gamma}^{\alpha}(x_{mnk})\right|\right)\right]^{p_{rst}} \geq \beta + \frac{\epsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right| \leq \frac{\epsilon}{2\left(d\left(\Delta_{\gamma}^{\alpha} x, y\right)\right)_{\Lambda^3}} + \frac{\delta}{2}$, where

$\sum_{u=1}^{\infty}\sum_{v=1}^{\infty}\sum_{w=1}^{\infty}|\gamma_{uvw}x_{uvw}|=0$, for all $r\geq u_0, s\geq v_0, t\geq w_0$. Further more, we can write $\left|\Delta_\gamma^\alpha x_{mnk}\right|\leq d\left(\Delta_\gamma^\alpha x_{mnk},y\right)_{\Delta_\gamma^\alpha}\leq d\left(\Delta_\gamma^\alpha x,y\right)_{\Lambda^3}=d(x,y)_{\Delta_\gamma^\alpha x}$. For $r,s,t\geq u_0,v_0,w_0$.

$$\begin{aligned}\frac{1}{h_{rst}^{\eta}}\sum_{(mnk)\in I_{rst}} & P\left(\left[f_{mnk}\left(\left|\Delta_\gamma^\alpha X_{mnk}\right|\right)\right]^p\right)\\ &=\frac{1}{h_{rst}^{\eta}}P\left(\sum_{(mnk)\in I_{rst}}\left[f_{mnk}\left(\left|\Delta_\gamma^\alpha X_{mnk}\right|\right)\right]^p\right)\\ &+\frac{1}{h_{rst}^{\eta}}P\left(\sum_{(mnk)\notin I_{rst}}\left[f_{mnk}\left(\left|\Delta_\gamma^\alpha X_{mnk}\right|\right)\right]^p\right)\\ &<\frac{1}{h_{rst}}P\left(h_{rst}^{\eta}\left(\frac{\epsilon}{2}+\frac{\delta}{2}\right)+h_{rst}^{\eta}\frac{\epsilon}{2}\frac{d(x,y)_{\Delta_\gamma^\alpha X}^p}{d(x,y)_{\Delta_\gamma^\alpha X}^p}+\frac{\delta}{2}\right)=\epsilon+\delta.\end{aligned}$$

Hence $(x_{mnk})\rightarrow\left(N\left(\Delta_\gamma^\alpha,p\right)_\theta\right)$.

Corollary 4.8 If α be a proper fraction, β be nonnegative real number,f be an Musielak-Orlicz function of order η and arbitary sequence $p=(p_{rst})$ of strictly positive real numbers then the following statements are hold:
(i) $S\cap\Lambda^3\subset\Delta_\gamma^\alpha(S_\theta)\cap\Delta_\gamma^\alpha(\Lambda^3)$, (ii) $\Delta_\gamma^\alpha(S_\theta)\cap\Delta_\gamma^\alpha(\Lambda^3)=\Delta_\gamma^\alpha\left(w_p^3\right)$.

Theorem 4.9. Let α be a proper fraction, β be nonnegative real number,f be an Musielak-Orlicz function of order η and arbitary sequence $p=(p_{rst})$ of strictly positive real numbers. If $x=(x_{mnk})$ is a $\Delta_\gamma^\alpha-$ triple difference rough lacunary statistically of order η convergent sequence, then x is a $\Delta_\gamma^\alpha-$ triple difference rough lacunary statistically of order η Cauchy sequence.

Proof: Assume that $(x_{mnk})\rightarrow\left(\Delta_\gamma^\alpha(S_\theta)\right)$ and $\epsilon,\delta>0$. The

$$\text{n}\,\frac{1}{\delta}\left|\left\{(m,n,k)\in I_{rst}: P\left(\left[f_{mnk}\left(\left|\Delta_\gamma^\alpha x_{mnk}\right|\right)\right]^{p_{rst}}\geq\beta+\frac{\epsilon}{2}\right)\right\}\right|$$

for almost all m,n,k and if we select N, then

$$\frac{1}{\delta}\left|\left\{(m,n,k)\in I_{rst}: P\left(\left[f_{mnk}\left(\left|\Delta_\gamma^\alpha x_N\right|\right)\right]^{p_{rst}}\geq\beta+\frac{\epsilon}{2}\right)\right\}\right|$$

holds. Now, we have

$$\begin{aligned}&\left|\left\{(m,n,k)\in I_{rst}: P\left(\left[f_{mnk}\left(\left|\Delta_\gamma^\alpha(x_{mnk}-x_N)\right|\right)\right]^{p_{rst}}\right)\right\}\right|\\ &\leq\frac{1}{\delta}\left|\left\{(m,n,k)\in I_{rst}: P\left(\left[f_{mnk}\left(\left|\Delta_\gamma^\alpha x_{mnk}\right|\right)\right]^{p_{rst}}\geq\beta+\frac{\epsilon}{2}\right)\right\}\right|\\ &+\frac{1}{\delta}\left|\left\{(m,n,k)\in I_{rst}: P\left(\left[f_{mnk}\left(\left|\Delta_\gamma^\alpha x_N\right|\right)\right]^{p_{rst}}\geq\beta+\frac{\epsilon}{2}\right)\right\}\right|<\frac{1}{\delta}(\beta+\epsilon)=\epsilon,\end{aligned}$$

for almost m,n,k. Hence (x_{mnk}) is a $\Delta_\gamma^\alpha-$ rough lacunary statistically of order η Cauchy.

Theorem 4.10. If α be a proper fraction, β be nonnegative real number,f be an Musielak-Orlicz function of order η and arbitary sequence $p = (p_{rst})$ of strictly positive real numbers and $0 < p_{rst} < \infty$ for all r, s, t, then $N\left(\Delta_{\gamma}^{\alpha}, p\right)_{\theta} \subset \Delta_{\gamma}^{\alpha}(S_{\theta})$.

Proof: Suppose that $x = (x_{mnk}) \in N\left(\Delta_{\gamma}^{\alpha}, p\right)_{\theta}$ and

$$\left|\left\{(m,n,k) \in I_{rst}: P\left(\left[f_{mnk}\left(\left|\Delta_{\gamma}^{\alpha} x_{mnk}\right|\right)\right]^{p} \geq \beta + \epsilon\right)\right\}\right|.$$

Therefore we have

$$\frac{1}{h_{rst}^{\eta}} \sum_{(mnk)\in I_{rst}} P\left(\left[f_{mnk}\left(\left|\Delta_{\gamma}^{\alpha} x_{mnk}\right|\right)\right]^{p}\right) \geq \frac{1}{h_{rst}^{\eta}} \sum_{(mnk)\in I_{rst}} (\beta + \epsilon)^{p} \geq \frac{1}{h_{rst}^{\eta}} \left|\left\{(m,n,k) \in I_{rst}: P\left(\left[f_{mnk}\left(\left|\Delta_{\gamma}^{\alpha} x_{mnk}\right|\right)\right]^{p} \geq \beta + \epsilon\right)\right\}\right| (\beta + \epsilon)^{p}.$$

So we observe by passing to limit as $r, s, t \to \infty$,

$$lim_{r,s,t\to\infty} \frac{1}{h_{rst}^{\eta}} \left|\left\{(m,n,k) \in I_{rst}: P\left(\left[f_{mnk}\left(\left|\Delta_{\gamma}^{\alpha}(x_{mnk})\right|\right)\right]^{p} \geq \beta + \epsilon\right) \geq \delta\right\}\right| < \frac{1}{(\beta+\epsilon)^{p}} \left(P\left(lim_{rst\to\infty} \frac{1}{h_{rst}^{\eta}} \sum_{(m,n,k)\in I_{rst}} \left[f_{mnk}\left(\left|\Delta_{\gamma}^{\alpha}(x_{mnk})\right|\right)\right]^{p}\right)\right) = 0$$

implies that $x \in \Delta_{\gamma}^{\alpha}(S_{\theta})$. Hence $N\left(\Delta_{\gamma}^{\alpha}, p\right)_{\theta} \subset \Delta_{\gamma}^{\alpha}(S_{\theta})$.

5. LACUNARY STATISTICALLY ANALYTIC SEQUENCES

Here, we give some new definitions and main results of the paper. In Theorem 5.6 we give the relations between rough triple lacunary statistically analytic sequences of order η and rough triple lacunary statistically analytic sequences of order μ. In Theorem 5.9 we give the relationship between rough triple lacunary statistical boundedness of order η and rough triple lacunary statistical analysis of order μ for different $\theta\,'$s.

Definition 5.1. Let α be a proper fraction, β be nonnegative real number and f be an Musielak-Orlicz function and $\theta = \{m_r n_s k_t\}$ be a rough triple lacunary and $0 < \eta \leq 1$ be given. We define triple lacunary $\eta -$ density of a subset $\mathbb{E}$ of $\mathbb{N}^3$ by

$$\delta_{\theta}^{\eta}(\mathbb{E}) = lim_{r,s,t\to\infty} \frac{1}{h_{rst}^{\eta}} |\{(m,n,k) \in \mathbb{E}: f_{mnk}[|I_{rst}| \quad] \geq \beta + \epsilon\}|,$$

Where

$$I_{r,s,t} = \{(m,n,k): m_{r-1} < m < m_r \quad and \quad n_{s-1} < n \leq n_s \quad and \quad k_{t-1} < k \leq k_t\}, q_r = \frac{m_r}{m_{r-1}}, \overline{q_s} = \frac{n_s}{n_{s-1}}, \overline{q_t} = \frac{k_t}{k_{t-1}}.$$

Provided the limit exists. We remark that triple lacunary $\eta-$ density δ_θ^η reduces to the natural density $\delta(\mathbb{E})$ in the special cases $\eta = 1$ and $\theta = (2^{rst})$.

If $x = (x_{mnk})$ is a sequence such that x_{mnk} satisifies the property $p(mnk)$ for m, n, k except a set of lacunary η −density zero, then we say that x_{mnk} satisifies $p(mnk)$ for " lacunary almost all m, n, k according to η" and we abbreviate this by "$a.a.m_r n_s k_t(\eta)$".

Theorem 5.2. Let α be a proper fraction, β be nonnegative real number and f be an Musielak-Orlicz function and $\theta = \{m_r n_s k_t\}$ be a rough triple lacunary space and $\eta, \mu \in (0,1]$ such that $\eta \le \mu$, then $\delta_\theta^\mu(\mathbb{E}) \le \delta_\theta^\eta(\mathbb{E})$.

Proof: Proof follows from the following inequality

$$\frac{1}{h_{rst}^\mu}|\{(m,n,k) \in \mathbb{E}: f_{mnk}[|I_{rst}| \quad] \ge \beta + \epsilon\}|$$
$$\le \frac{1}{h_{rst}^\eta}|\{(m,n,k) \in \mathbb{E}: f_{mnk}[|I_{rst}| \quad] \ge \beta + \epsilon\}|.$$

Definition 5.3. Let α be a proper fraction, β be nonnegative real number and f be an Musielak-Orlicz function and $\theta = \{m_r n_s k_t\}$ be a rough triple lacunary space and $0 < \eta \le 1$ be given. The triple sequence $x = (x_{mnk}) \in w^3$ is said to be rough lacunary statistically analytic of order η, if there is a $M \ge 0$ such that

$$\frac{1}{h_{rst}^\mu}\left|\left\{(m,n,k) \in I_{rst}: f_{mnk}\left[|x_{mnk}|^{1/m+n+k} > M \quad \right] \ge \beta + \epsilon\right\}\right| = 0,$$

i.e $|x_{mnk}|^{1/m+n+k} \le M$ a.a.$m_r n_s k_t(\eta)$, where

$$I_{r,s,t} = \{(m,n,k): m_{r-1} < m < m_r \quad and \quad n_{s-1} < n \le n_s \quad and \quad k_{t-1} < k \le k_t\}, q_r = \frac{m_r}{m_{r-1}}, \overline{q_s} = \frac{n_s}{n_{s-1}}, \overline{q_t} = \frac{k_t}{k_{t-1}}$$

and

$$h^\eta = h_{rst}^\eta = \left(h_{111}^\eta \cdots h_{rst}^\eta\right).$$

The set of all lacunary statistically analytic sequences of order η and the set of all lacunary statistically analytic sequences will be denoted by $S_\theta^\eta(a)$ and $S_\theta(a)$, respectively.

Theorem 5.4. If α be a proper fraction, β be nonnegative real number and f be an Musielak-Orlicz function then every lacunary statistically convergent sequence of order η is lacunary statistically analytic of order η, but the converse is not true.

Proof: Let $x \in S_\theta^\eta(a)$ and $\epsilon > 0$ be given. Then there exists $L \in \mathbb{C}$ such that

$$lim_{r,s,t\to\infty}\frac{1}{h_{rst}^\mu}\left|\left\{(m,n,k) \in I_{rst}: f_{mnk}\left[|x_{mnk} - L|^{1/m+n+k} \quad \right] \ge \beta + \epsilon\right\}\right| = 0.$$

The result follows from the following inequality

$$lim_{rst\to\infty}\frac{1}{h_{rst}^\mu}\left|\left\{(m,n,k) \in I_{rst}: f_{mnk}\left[|x_{mnk}|^{\frac{1}{m}+n+k} + L \quad \right] \ge \beta + \epsilon\right\}\right|$$

$$\leq lim_{rst\to\infty}\frac{1}{h_{rst}^{\mu}}\left|\left\{(m,n,k)\in I_{rst}: f_{mnk}\left[|x_{mnk}-L|^{1/m+n+k}\quad\right]\geq \beta+\epsilon\right\}\right|.$$

To show the strictness of the inclusion, let $\theta=(2^{rst})$ be given and the triple analytic sequence $x=(x_{mnk})$ defined by

$$f\left[x_{mnk}^{1/m+n+k}\right]=\begin{cases}1, & \mathrm{m,n,k}=6\mathrm{uvw}\\ -1 & \mathrm{m,n,k}\neq 6\mathrm{uvw},\\ m,n,k,u,v,w\in\mathbb{N}\quad . \end{cases}$$

Then $x\in S_{\theta}^{\eta}(a)$, but $x\notin S_{\theta}^{\eta}$.

Theorem 5.5. Let α be a proper fraction, β be nonnegative real number and f be an Musielak-Orlicz function. We have the following:
(i) $S_{\theta}^{\eta}(a)$ is normal and hence monotone,
(ii) $S_{\theta}^{\eta}(a)$ is a sequence algebra.

Proof: Let $x=(x_{mnk})\in S_{\theta}^{\eta}(a)$ and $y=(y_{mnk})$ be a sequence such that $|y_{mnk}|\leq|x_{mnk}|$ for all $m,n,k\in\mathbb{N}^3$. Since $x\in S_{\theta}^{\eta}(a)$ there exists a number M such that $\delta_{\theta}^{\eta}\left(\left|\left\{(m,n,k): f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|\right)=0$. Clearly $y\in S_{\theta}^{\eta}(a)$ as

$$\left|\left\{(m,n,k): f\left[|y_{mnk}|^{\frac{1}{m}+n+k}\right]>M\right\}\geq\beta+\epsilon\right|$$
$$\subset\left|\left\{(m,n,k): f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|.$$

Therefore $S_{\theta}^{\eta}(a)$ is normal. We know that every normal space is monotoe. Hence $S_{\theta}^{\eta}(a)$ is monotone.
(ii) Let $x,y\in S_{\theta}^{\eta}(a)$. Then there exists $K,M>0$ such that

$$\delta_{\theta}^{\eta}\left(\left|\left\{(m,n,k): f\left[|x_{mnk}|^{1/m+n+k}\right]>K\right\}\geq\beta+\epsilon\right|\right)=0$$

and

$$\delta_{\theta}^{\eta}\left(\left|\left\{(m,n,k): f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|\right)=0.$$

The proof follows from the following inclusion

$$\left|\left\{(m,n,k): f\left[|x_{mnk}y_{mnk}|^{1/m+n+k}\right]>KM\right\}\geq\beta+\epsilon\right|$$
$$\subset\left|\left\{(m,n,k): f\left[|x_{mnk}|^{1/m+n+k}\right]>K\right\}\geq\beta+\epsilon\right|$$
$$\cup\left|\left\{(m,n,k): f\left[|y_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|.$$

Theorem 5.6. Let α be a proper fraction, β be nonnegative real number and f be an Musielak-Orlicz function. If $0<\eta\leq\mu\leq 1$ then $S_{\theta}^{\eta}(a)\subseteq S_{\theta}^{\mu}(a)$ and the inclusion is strict.

Proof: The inclusion part of proof is easy. To show the strictness of the inclusion,let θ be given and the triple analytic sequence $x=(x_{mnk})$ defined by

$$f\left[x_{mnk}^{1/m+n+k}\right]=\begin{cases}\left[\sqrt{h_{rst}}\right], & m,n,k=1,2,3,\cdots,\left[\sqrt{h_{rst}}\right]\\ 0, & otherwise.\end{cases}$$

Then $x \in S_\theta^\mu(a)$ for $\frac{1}{2} < \mu \leq 1$ but $x \notin S_\theta^\eta(a)$ for $0 < \eta \leq \frac{1}{2}$.

Corollary 5.7. Let α be a proper fraction, β be nonnegative real number and f be an Musielak-Orlicz function. If a triple sequence is lacunary statistically analytic of order η, then it is lacunary statistically analytic.

Theorem 5.8. Let α be a proper fraction, β be nonnegative real number and f be an Musielak-Orlicz function. If

$$lim_{r,s,t\to\infty} inf \frac{h_{rst}^\eta}{m_r n_s k_t} \tag{5.1}$$

then $S(a) \subset S_\theta^\eta(a)$.

Proof: For $M > 0$, we have

$$\left|\left\{(m,n,k) \leq m_r n_s k_t : f\left[|x_{mnk}|^{1/m+n+k}\right] > M\right\} \geq \beta + \epsilon\right| \supset \left|\left\{(m,n,k) \in I_{rst} : f\left[|x_{mnk}|^{1/m+n+k}\right] > M\right\} \geq \beta + \epsilon\right|.$$

Therefore,

$$\frac{1}{m_r n_s k_t}\left|\left\{(m,n,k) \leq m_r n_s k_t : f\left[|x_{mnk}|^{\frac{1}{m}+n+k}\right] > M\right\} \geq \beta + \epsilon\right|$$
$$\geq \frac{1}{m_r n_s k_t}\left|\left\{(m,n,k) \in I_{rst} : f\left[|x_{mnk}|^{\frac{1}{m}+n+k}\right] > M\right\} \geq \beta + \epsilon\right|$$
$$= \frac{h_{rst}^\eta}{m_r n_s k_t}\frac{1}{h_{rst}^\eta}\left|\left\{(m,n,k) \in I_{rst} : f\left[|x_{mnk}|^{1/m+n+k}\right] > M\right\} \geq \beta + \epsilon\right|.$$

Taking limit as $r,s,t \to \infty$ and using (5.1), we get $S(a) \subset S_\theta^\eta(a)$.

Theorem 5.9 Let α be a proper fraction, β be nonnegative real number and f be an Musielak-Orlicz function. Consider $\theta = (m_r n_s k_t)$ and $\theta' = (u_r v_s w_t)$ be two lacunary sequences such that $I_{rst} \subset J_{rst}$ for all $r,s,t \in \mathbb{N}^3$. Let η and μ be such that $0 < \eta \leq \mu \leq 1$. We have the following:

(i) If

$$lim_{r,s,t\to\infty} inf \frac{h_{rst}^\eta}{\gamma_{rst}^\mu} > 0 \tag{5.2}$$

then $S_{\theta'}^\mu(a) \subset S_\theta^\eta(a)$,

If

$$lim_{r,s,t\to\infty} \frac{\gamma_{rst}}{h_{rst}^\mu} = 1 \tag{5.3}$$

then $S_\theta^\eta(a) \subset S_{\theta'}^\mu(a)$.

Proof: Suppose that $I_{rst} \subset J_{rst}$ for all $r,s,t \in \mathbb{N}^3$ and let (5.2) satisified. For $M > 0$ we have

$$\left|\left\{(m,n,k)\in J_{rst}: f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|\supseteq \left|\left\{(m,n,k)\in I_{rst}: f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|$$

and so

$$\frac{1}{\gamma^{\mu}_{rst}}\left|\left\{(m,n,k)\in J_{rst}: f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|\supseteq\frac{h^{\eta}_{rst}}{\gamma^{\mu}_{rst}}\frac{1}{h^{\eta}_{rst}}\left|\left\{(m,n,k)\in I_{rst}: f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|$$

for all $r,s,t\in\mathbb{N}^3$. Now taking the limit as $r,s,t\to\infty$ and using (5.2) we get

$$S^{\mu}_{\theta'}(a)\subset S^{\eta}_{\theta}(a).$$

(ii) Let $x=(x_{mnk})\in S^{\eta}_{\theta}(a)$ and (5.3) be satisified. Since $I_{rst}\subset J_{rst}$, for $M>0$ we may write

$$\begin{aligned}
&\frac{1}{\gamma^{\mu}_{rst}}\left|\left\{(m,n,k)\in J_{rst}: f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|\\
&=\frac{1}{\gamma^{\mu}_{rst}}\left|\left\{m_{r-1}<m<m_r: f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|\\
&+\frac{1}{\gamma^{\mu}_{rst}}\left|\left\{n_{r-1}<n<n_r: f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|\\
&+\frac{1}{\gamma^{\mu}_{rst}}\left|\left\{k_{r-1}<k<k_r: f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|\\
&\leq\frac{m_r-m_{r-1}}{\gamma^{\mu}_{rst}}+\frac{n_r-n_{r-1}}{\gamma^{\mu}_{rst}}+\frac{k_r-k_{r-1}}{\gamma^{\mu}_{rst}}\\
&+\frac{1}{\gamma^{\mu}_{rst}}\left|\left\{(m,n,k)\in I_{rst}: f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|\\
&=\frac{\gamma_{rst}-h_{rst}}{\gamma^{\mu}_{rst}}+\frac{1}{\gamma^{\mu}_{rst}}\left|\left\{(m,n,k)\in I_{rst}: f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|\\
&\leq\frac{\gamma_{rst}-h^{\mu}_{rst}}{h^{\mu}_{rst}}+\frac{1}{h^{\mu}_{rst}}\left|\left\{(m,n,k)\in I_{rst}: f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|\\
&\leq\left(\frac{\gamma_{rst}}{h^{\mu}_{rst}}-1\right)+\frac{1}{h^{\eta}_{rst}}\left|\left\{(m,n,k)\in I_{rst}: f\left[|x_{mnk}|^{1/m+n+k}\right]>M\right\}\geq\beta+\epsilon\right|
\end{aligned}$$

for all $r,s,t\in\mathbb{N}^3$. Since $lim_{rst\to\infty}\frac{\gamma_{rst}}{h^{\mu}_{rst}}=1$ by (5.3) the first term and since $x\in S^{\eta}_{\theta}(a)$ the second term of right hand side of above inequality tend to 0 as $r,s,t\to\infty$. This implies that $S^{\eta}_{\theta}(a)\subset S^{\mu}_{\theta'}(a)$.

Corollary 5.10. Let α be a proper fraction, β be nonnegative real number and f be an Musielak-Orlicz function. Consider $\theta=(m_r n_s k_t)$ and $\theta'=(u_r v_s w_t)$ be two lacunary sequences such that $I_{rst}\subset J_{rst}$ for all $r,s,t\in\mathbb{N}^3$. If the condition (5.2) is satisfied, then
(i) $S^{\mu}_{\theta'}(a)\subset S^{\eta}_{\theta}(a)$ for each $\eta\in(0,1]$,
(ii) $S_{\theta'}(a)\subset S^{\eta}_{\theta}(a)$ for each $\eta\in(0,1]$,
(iii) $S_{\theta'}(a)\subset S_{\theta}(a)$ for each $\eta\in(0,1]$.

Furthermore, if the condition (5.3) is satisfied, then
(i) $S_\theta^\eta(a) \subset S_{\theta'}^\mu(a)$ for each $\eta \in (0,1]$,
(ii) $S_\theta^\eta(a) \subset S_{\theta'}(a)$ for each $\eta \in (0,1]$,
(iii) $S_\theta(a) \subset S_{\theta'}(a)$ for each $\eta \in (0,1]$.

6. CONCLUSION

In this article, we explored generalized rough lacunary statistical analysis of order η of triple difference sequence spaces and give the relations between statistical analysis and lacunary statistical analysis of order η. We think that this article will be useful for future studies.

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