ORIGINAL PAPER

ON 3-PARAMETER GENERALIZED QUATERNIONS WITH THE LEONARDO *p*-SEQUENCE

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Abstract. *In this article, 3-parameter generalized quaternionic Leonardo p-sequence which is a quaternion generalization of the Leonardo p-sequence is presented. Several properties are examined by means of their homogeneous and non-homogeneous recursive relations. According to the special cases of 3-parameter* $\lambda_{i\in\{1,2,3\}}$ *and* p*, all the results given in paper are reducible to all quaternions mentioned in this paper.*

Keywords: *Leonardo sequence; recurrence relation; generalized quaternion.*

1. INTRODUCTION AND FUNDAMENTALS

In number theory, recurrence sequences are extremely important. For $n \geq 2$, the second order homogeneous recursive relation of the classical Fibonacci sequence is given as

$$F_n = F_{n-1} + F_{n-2},$$

with initial conditions $F_0 = 0$ and $F_1 = 1$, [1]. The Fibonacci sequence is studied extensively. Over the years, there are a lot of generalizations of the Fibonacci sequence. The researchers have generalized the Fibonacci sequence in two ways: first by changing the recursive relation and second by changing the initial conditions. In the literature, there are so many studies that concern about the generalizations of the Fibonacci sequence such as the Lucas, generalized Fibonacci, generalized Lucas, Leonardo, Narayana, Horadam, Pell, Jacobsthal, Pell-Lucas and Jacobsthal–Lucas sequences etc. Sequences are generalized with respect to parameters in many ways. For $n \geq 2$, the second order homogeneous recursive relation of the well-known Lucas sequence is given as

$$L_n = L_{n-1} + L_{n-2},$$

with initial conditions $L_0 = L_1 = 1$, [1]. The Leonardo sequence is defined by the following non-homogeneous recursive relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \ \ n \geq 2$$

or the following the homogeneous recursive relation

$$Le_{n+1} = 2Le_n - Le_{n-2}, \ \ n \geq 3$$

with initial conditions $Le_0 = Le_1 = 1$, and $Le_2 = 3$, [2].

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For any given integer $p > 0$ and $n > p$, the Fibonacci p-sequence, and the Lucas p-sequence are defined recursively by

$$F_{p,n} = F_{p,n-1} + F_{p,n-p-1}$$

and

$$L_{p,n} = L_{p,n-1} + L_{p,n-p-1},$$

with initial conditions $F_{p,0} = 0, F_{p,k} = 1$, and $L_{p,0} = p+1, L_{p,k} = 1$ for $k = 1,2,\dots,p$, respectively, [3-8]. Very recently, for any given integer $p > 0$ and $n > p$, the Leonardo p-sequence is defined recursively by the following non-homogeneous relation

$$Le_{p,n} = Le_{p,n-1} + Le_{p,n-p-1} + p, \tag{1.1}$$

with initial conditions $Le_{p,k} = 1$ for $k = 0,1,2,\dots,p$, [9]. For $n \geq 2p$, the non-homogeneous recursive relation of the the Leonardo p-sequence can also be converted the following homogeneous recursive relation

$$Le_{p,n} = Le_{p,n-1} + Le_{p,n-p} - Le_{p,n-2p-1}. \tag{1.2}$$

The Binet's formulae of the Leonardo p-sequence is

$$Le_{p,n} = (p+1)\sum_{k=1}^{p+1} \frac{\alpha_k^{n+1}}{(p+1)\alpha_k - p} - p. \tag{1.3}$$

It is obvious that if we take $p = 1$, the Fibonacci p-sequence, the Lucas p-sequence, and the Leonardo p-sequence reduce to the Fibonacci sequence, the Lucas sequence, and the Leonardo sequence, respectively.

Hypercomplex number is a well-known mathematical term. Various number systems can be defined by adding imaginary units to real numbers. The complex number system is a classical example of such a system. Hypercomplex numbers, such as quaternions, tessarines, coquaternions, octonions, biquaternions, hybrid numbers and sedenions, are extensions of complex numbers. The quaternions are used in such fields as physics, quantum physics, computer sciences, engineering, mechanism, signal and color image processing via rotation and orientation. In the original works of W. R. Hamilton [10-13], real quaternions (Hamiltonian) are defined in the following general form

$$q = a_0 + a_1 i + a_2 j + a_3 k,$$

where $i, j, k \notin \mathbb{R}$ are quaternionic units and $a_{i \in \{0,1,2,3\}}$ are real numbers. The real quaternionic units $\{i, j, k\}$ satisfy the following multiplication conditions

$$i^2 = -1, \qquad j^2 = -1, \qquad k^2 = -1,$$
$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

This last equations shows that the real quaternion multiplication is non-commutative. This gives a new perspective to the quaternions. Changing conditions on the quaternionic units gives new types of commutative and non-commutative quaternions. In generally, the quaternions fall naturally into two classes: the non-commutative quaternions (real, split, semi, split-semi, quasi, hyperbolic, elliptical, hyperbolic split, 2-parameter generalized, 3-parameter generalized quaternions etc.), and the commutative quaternions (generalized Segre, dual quaternions etc.). By considering conditions of the 2-parameter generalized quaternions given in Table 1, the 2-parameter generalized quaternions are analogous to real quaternions for $\alpha = \beta = 1$, to split quaternions for $\alpha = 1, \beta = -1$, to semi quaternions $\alpha = 1, \beta = 0$, to split-semi quaternions $\alpha = -1, \beta = 0$, and to quasi quaternions $\alpha = \beta = 0$. In addition to these, if we take conditions $\lambda_1 = 1$, $\lambda_2 = \alpha$ and $\lambda_3 = \beta$ on the 3-parameter generalized quaternions in Table 1, then we get the 2-parameter generalized quaternions. It is clear that we can define more specific quaternions according to the quaternionic units. Hence, the 3-parameter generalized quaternion is the generalization of the others.

Table 1. Conditions of the quaternionic units where α, β, $\lambda_{i \in \{1,2,3\}} \in \mathbb{R}$.

2-parameter generalized quaternion [14-17]	$i^2 = -\alpha, \quad j^2 = -\beta, \quad k^2 = -\alpha\beta,$ $ij = -ji = k, \quad jk = -kj = \beta i, \quad ki = -ik = \alpha j.$
3-parameter generalized quaternion [18]	$i^2 = -\lambda_1\lambda_2, \quad j^2 = -\lambda_1\lambda_3, \quad k^2 = -\lambda_2\lambda_3,$ $ij = -ji = \lambda_1 k, \quad jk = -kj = \lambda_3 i, \quad ki = -ik = \lambda_2 j.$

The idea of generalizing quaternions whose components are special integer sequences such as Fibonacci, Lucas, Leonardo and so on has been one of the aims of researchers, [19-23]. In this paper, we answer the question whether it is possible to extend the Leonardo p-sequence via 3-parameter generalized quaternion, which the 3-parameter generalized quaternion units would satisfy the multiplication rules listed in Table 1.

2. MAIN RESULTS

This section starts by mathematical formulations of the generalized quaternions with the Leonardo p-number components depending on 3-parameter.

Definition 2.1. For any given integer $p > 0$ and $n > p$, a general form of the n-th generalized quaternionic Leonardo p-number $\boldsymbol{Le}_{p,n}$ is

$$\boldsymbol{Le}_{p,n} = Le_{p,n} + Le_{p,n+1} i + Le_{p,n+2}\, j + Le_{p,n+3}\, k, \tag{2.1}$$

where $Le_{p,n}$ is the n-th Leonardo p-number and $\{i, j, k\}$ are the 3-parameter generalized quaternionic units (see Table 1). The set of generalized quaternionic Leonardo p-numbers is denoted by $\{\boldsymbol{Le}_{p,n}\}$.

In Table 2, we give a summary of some special cases of the 3-parameter generalized quaternion Leonardo p-sequence briefly. One can derive more specific Leonardo quaternion sequences according to p and $\lambda_{i \in \{1,2,3\}}$.

Table 2. Some special cases of the 3-parameter generalized quaternion Leonardo p-sequence.

λ_1	λ_2	λ_3	p	3-parameter generalized quaternionic Leonardo p-sequence
λ_1	α	β	1	3-parameter generalized quaternionic Leonardo sequence
1	α	β	p	2-parameter generalized quaternionic Leonardo p-sequence
1	α	β	1	2-parameter generalized quaternionic Leonardo sequence

λ_1	λ_2	λ_3	p	**3-parameter generalized quaternionic Leonardo p-sequence**
1	1	1	1	real quaternionic Leonardo sequence
1	1	1	p	real quaternionic Leonardo p-sequence
1	1	-1	1	split quaternionic Leonardo sequence
1	1	-1	p	split quaternionic Leonardo p-sequence
1	1	0	1	semi quaternionic Leonardo sequence
1	1	0	p	semi quaternionic Leonardo p-sequence
1	-1	0	1	split-semi quaternionic Leonardo sequence
1	-1	0	p	split-semi quaternionic Leonardo p-sequence
1	0	0	1	quasi quaternionic Leonardo sequence
1	0	0	p	quasi quaternionic Leonardo p-sequence

The basic algebraic operations on 3-parameter generalized quaternionic Leonardo p-numbers are defined in a standard way. Let $\boldsymbol{Le}_{p,n} = Le_{p,n} + Le_{p,n+1}i + Le_{p,n+2}\,j + Le_{p,n+3}\,k$ and $\boldsymbol{Le}_{p,m} = Le_{p,m} + Le_{p,m+1}i + Le_{p,m+2}\,j + Le_{p,m+3}\,k$ be two 3-parameter generalized quaternionic Leonardo p-numbers. Any $\boldsymbol{Le}_{p,n}$ consists of two parts, the scalar part and the vector part, respectively, $Le_{p,n} = S_{\boldsymbol{Le}_{p,n}} + V_{\boldsymbol{Le}_{p,n}}$, where $S_{\boldsymbol{Le}_{p,n}} = Le_{p,n}$ and $V_{\boldsymbol{Le}_{p,n}} = Le_{p,n+1}i + Le_{p,n+2}\,j + Le_{p,n+3}\,k$. The addition (and hence subtraction), the equality are defined component-wise way whereas the quaternion multiplication is identified by according to the multiplication Table 1. The addition of $\boldsymbol{Le}_{p,n}$ and $\boldsymbol{Le}_{p,m}$ is given by

$$\begin{aligned}\boldsymbol{Le}_{p,n} + \boldsymbol{Le}_{p,m} = {} & \left(Le_{p,n} + Le_{p,m}\right) + \left(Le_{p,n+1} + Le_{p,m+1}\right)i \\ & + \left(Le_{p,n+2} + Le_{p,m+2}\right)j + \left(Le_{p,n+3} + Le_{p,m+3}\right)k.\end{aligned}$$

The quaternion multiplication of $\boldsymbol{Le}_{p,n}$ and $\boldsymbol{Le}_{p,m}$ is calculated by

$$\begin{aligned}\boldsymbol{Le}_{p,n}\boldsymbol{Le}_{p,m} = {} & \left(Le_{p,n}Le_{p,m} - \lambda_1\lambda_2 Le_{p,n+1}Le_{p,m+1} - \lambda_1\lambda_3 Le_{p,n+2}Le_{p,m+2} - \lambda_2\lambda_3 Le_{p,n+3}Le_{p,m+3}\right) \\ & + \left(Le_{p,n}Le_{p,m+1} + Le_{p,n+1}Le_{p,m} + \lambda_3\left(Le_{p,n+2}Le_{p,m+3} - Le_{p,n+3}Le_{p,m+2}\right)\right)i \\ & + \left(Le_{p,n}Le_{p,m+2} + Le_{p,n+2}Le_{p,m} + \lambda_2\left(Le_{p,n+1}Le_{p,m+3} - Le_{p,n+3}Le_{p,m+1}\right)\right)j \\ & + \left(Le_{p,n}Le_{p,m+3} + Le_{p,n+3}Le_{p,m} + \lambda_1\left(Le_{p,n+1}Le_{p,m+2} - Le_{p,n+2}Le_{p,m+1}\right)\right)k.\end{aligned}$$

The scalar multiplication of the 3-parameter generalized quaternionic Leonardo p-number $\boldsymbol{Le}_{p,n}$ by any real scalar c is

$$c\boldsymbol{Le}_{p,n} = cLe_{p,n} + \left(cLe_{p,n+1}\right)i + \left(cLe_{p,n+2}\right)j + \left(cLe_{p,n+3}\right)k.$$

The quaternion conjugate and norm of the 3-parameter generalized quaternionic Leonardo p-number $\boldsymbol{Le}_{p,n}$ is given by

$$\overline{\boldsymbol{Le}}_{p,n} = S_{\boldsymbol{Le}_{p,n}} - V_{\boldsymbol{Le}_{p,n}} = Le_{p,n} - Le_{p,n+1}i - Le_{p,n+2}\,j - Le_{p,n+3}\,k \tag{2.2}$$

and

$$\begin{aligned}N_{\boldsymbol{Le}_{p,n}} & = \boldsymbol{Le}_{p,n}\overline{\boldsymbol{Le}}_{p,n} \\ & = \overline{\boldsymbol{Le}}_{p,n}\boldsymbol{Le}_{p,n} \\ & = (Le_{p,n})^2 + \lambda_1\lambda_2(Le_{p,n+1})^2 + \lambda_1\lambda_3(Le_{p,n+2})^2 + \lambda_2\lambda_3(Le_{p,n+3})^2,\end{aligned}$$

respectively.

Theorem 2.1. For any given integer $p > 0$ and $n > p$, the non-homogeneous recursive relation of the 3-parameter generalized quaternionic Leonardo p-sequence is

$$\boldsymbol{Le}_{p,n} = \boldsymbol{Le}_{p,n-1} + \boldsymbol{Le}_{p,n-p-1} + \boldsymbol{P}, \tag{2.3}$$

where $\boldsymbol{P} = p(1 + i + j + k)$ with initial conditions $\boldsymbol{Le}_{p,k} = 1 + i + j + k$ for $k = 0,1,2, \dots, p-3$, $\boldsymbol{Le}_{p,p-2} = 1 + i + j + Le_{p,p+1}k$,$\boldsymbol{Le}_{p,p-1} = 1 + i + Le_{p,p+1}j + Le_{p,p+2}k$, and $\boldsymbol{Le}_{p,p} = 1 + Le_{p,p+1}i + Le_{p,p+2}j + Le_{p,p+3}k$.

Proof: We concluded from the non-homogeneous recursive relation (1.1) that

$$\begin{aligned}\boldsymbol{Le}_{p,n} = &\ \left(Le_{p,n-1} + Le_{p,n-p-1} + p\right) + \left(Le_{p,n} + Le_{p,n-p} + p\right)i \\ &+ \left(Le_{p,n+1} + Le_{p,n-p+1} + p\right)j + \left(Le_{p,n+2} + Le_{p,n-p+2} + p\right)k \\ = &\ \boldsymbol{Le}_{p,n-1} + \boldsymbol{Le}_{p,n-p-1} + \boldsymbol{P},\end{aligned}$$

where we use $\boldsymbol{P} = p(1 + i + j + k)$ for the sake of brevity.

Throughout this section, $\boldsymbol{P} = p(1 + i + j + k)$ is considered.

Theorem 2.2. For any given integer $p > 0$ and $n > 2p$, the homogeneous recursive relation of the 3-parameter generalized quaternionic Leonardo p-sequence is

$$\boldsymbol{Le}_{p,n} = \boldsymbol{Le}_{p,n-1} + \boldsymbol{Le}_{p,n-p} - \boldsymbol{Le}_{p,n-2p-1}. \tag{2.4}$$

Proof: Applying the homogeneous recursive relation (1.2) yields

$$\begin{aligned}\boldsymbol{Le}_{p,n} = &\ \left(Le_{p,n-1} + Le_{p,n-p} - Le_{p,n-2p-1}\right) + \left(Le_{p,n} + Le_{p,n-p+1} - Le_{p,n-2p}\right)i \\ &+ \left(Le_{p,n+1} + Le_{p,n-p+2} - Le_{p,n-2p+1}\right)j + \left(Le_{p,n+2} + Le_{p,n-p+3} - Le_{p,n-2p+2}\right)k \\ = &\ \boldsymbol{Le}_{p,n-1} + \boldsymbol{Le}_{p,n-p} - \boldsymbol{Le}_{p,n-2p-1}.\end{aligned}$$

Theorem 2.3. The Binet formulae of the 3-parameter generalized quaternionic Leonardo p-number $\boldsymbol{Le}_{p,n}$ is

$$\boldsymbol{Le}_{p,n} = (p+1)\sum_{k=1}^{p+1} \frac{\alpha_k^{n+1}\alpha_k^*}{(p+1)\alpha_k - p} - \boldsymbol{P},$$

where $\alpha_k^* = 1 + \alpha_k i + \alpha_k^2 j + \alpha_k^3 k$.

Proof: By using the identity of the 3-parameter generalized quaternionic Leonardo p-number equation (2.1) and the Binet's formulae of the Leonardo p-sequence (1.3), we obtain

$$\begin{aligned}
\boldsymbol{Le}_{p,n} = & \; Le_{p,n} + Le_{p,n+1} i + Le_{p,n+2}\, j + Le_{p,n+3}\, k \\
= & \left((p+1) \sum_{k=1}^{p+1} \frac{\alpha_k^{n+1}}{(p+1)\alpha_k - p} - p \right) \\
& + \left((p+1) \sum_{k=1}^{p+1} \frac{\alpha_k^{n+2}}{(p+1)\alpha_k - p} - p \right) i \\
& + \left((p+1) \sum_{k=1}^{p+1} \frac{\alpha_k^{n+3}}{(p+1)\alpha_k - p} - p \right) j \\
& + \left((p+1) \sum_{k=1}^{p+1} \frac{\alpha_k^{n+4}}{(p+1)\alpha_k - p} - p \right) k \\
= & \; (p+1) \sum_{k=1}^{p+1} \left(\frac{\alpha_k^{n+1}}{(p+1)\alpha_k - p} (1 + \alpha_k i + \alpha_k^2 j + \alpha_k^3 k) \right) - \boldsymbol{P} \\
= & \; (p+1) \sum_{k=1}^{p+1} \frac{\alpha_k^{n+1} \alpha_k^*}{(p+1)\alpha_k - p} - \boldsymbol{P},
\end{aligned}$$

where $\alpha_k^* = 1 + \alpha_k i + \alpha_k^2 j + \alpha_k^3 k$.

Theorem 2.4. Let $\boldsymbol{Le}_{p,n}$ be the n-th 3-parameter generalized quaternionic Leonardo p-number. Then the following relations hold:

i. $\boldsymbol{Le}_{p,n} + \overline{\boldsymbol{Le}}_{p,n} = 2Le_{p,n}$,
ii. $\left(\boldsymbol{Le}_{p,n}\right)^2 = 2Le_{p,n}\boldsymbol{Le}_{p,n} - \boldsymbol{Le}_{p,n}\overline{\boldsymbol{Le}}_{p,n}$,
iii. $\boldsymbol{Le}_{p,n} - \boldsymbol{Le}_{p,n+1} i - \boldsymbol{Le}_{p,n+2} j - \boldsymbol{Le}_{p,n+3} k = Le_{p,n} + \lambda_1\lambda_2 Le_{p,n+2} + \lambda_1\lambda_3 Le_{p,n+4} + \lambda_2\lambda_3 Le_{p,n+6}$.

Proof:
i. The proof is clear from the identity of the 3-parameter generalized quaternionic Leonardo p-number (see equation (2.1)) and the equality of quaternion conjugate (see equation (2.2)).
ii. The equality $\boldsymbol{Le}_{p,n} + \overline{\boldsymbol{Le}}_{p,n} = 2Le_{p,n}$, which is the part of Theorem 2.4. item i, implies that

$$\begin{aligned}
\left(\boldsymbol{Le}_{p,n}\right)^2 = & \; \boldsymbol{Le}_{p,n}\left(2Le_{p,n} - \overline{\boldsymbol{Le}}_{p,n}\right) \\
= & \; 2Le_{p,n}\boldsymbol{Le}_{p,n} - \boldsymbol{Le}_{p,n}\overline{\boldsymbol{Le}}_{p,n}.
\end{aligned}$$

iii. Considering the conditions for the 3-parameter generalized quaternionic units $\{i, j, k\}$ in Table 1 gives

$$\begin{aligned}\boldsymbol{Le}_{p,n}-\boldsymbol{Le}_{p,n+1}i-\boldsymbol{Le}_{p,n+2}j-\boldsymbol{Le}_{p,n+3}k = {} & Le_{p,n}+Le_{p,n+1}i+Le_{p,n+2}\,j+Le_{p,n+3}\,k\\ & -Le_{p,n+1}i-Le_{p,n+2}i^2-Le_{p,n+3}\,ji-Le_{p,n+4}\,ki\\ & -Le_{p,n+2}j-Le_{p,n+3}ij-Le_{p,n+4}\,j^2-Le_{p,n+5}\,kj\\ & -Le_{p,n+3}k-Le_{p,n+4}ik-Le_{p,n+5}\,jk-Le_{p,n+6}\,k^2\\ = {} & Le_{p,n}+\lambda_1\lambda_2Le_{p,n+2}+\lambda_1\lambda_3Le_{p,n+4}\;+\lambda_2\lambda_3Le_{p,n+6}.\end{aligned}$$

Next, we state the relations between the 3-parameter generalized quaternionic Fibonacci p-sequence, the 3-parameter generalized quaternionic Lucas p-sequence, and the 3-parameter generalized quaternionic Leonardo p-sequence and so derive some well-known mathematical properties.

Theorem 2.5. For $n \geq 0$, the following relations hold:

i. $\boldsymbol{Le}_{p,n}=(p+1)\boldsymbol{F}_{p,n+1}-\boldsymbol{P}$,
ii. $\boldsymbol{Le}_{p,n}=\boldsymbol{L}_{p,n+p+1}-\boldsymbol{F}_{p,n+p+1}-\boldsymbol{P}$,

where $\boldsymbol{F}_{p,n}$ is the n-th 3-parameter generalized quaternionic Fibonacci p-number, and $\boldsymbol{L}_{p,n}$ is the n-th 3-parameter generalized quaternionic Lucas p-number.

Proof:
i. Taking $Le_{p,n}=(p+1)F_{p,n+1}-p$ (see [9]), we can assert that

$$\begin{aligned}\boldsymbol{Le}_{p,n} = {} & Le_{p,n}+Le_{p,n+1}i+Le_{p,n+2}\,j+Le_{p,n+3}\,k\\ = {} & \big((p+1)F_{p,n+1}-p\big)+\big((p+1)F_{p,n+2}-p\big)i\\ & \big((p+1)F_{p,n+3}-p\big)j+\big((p+1)F_{p,n+4}-p\big)k\\ = {} & (p+1)(F_{p,n+1}+F_{p,n+2}i+F_{p,n+3}j+F_{p,n+4}k)-\boldsymbol{P}\\ = {} & (p+1)\boldsymbol{F}_{p,n+1}-\boldsymbol{P}.\end{aligned}$$

ii. Writing $Le_{p,n}=L_{p,n+p+1}-F_{p,n+p+1}-p$ (see [9]), we can see that

$$\begin{aligned}\boldsymbol{Le}_{p,n} = {} & Le_{p,n}+Le_{p,n+1}i+Le_{p,n+2}\,j+Le_{p,n+3}\,k\\ = {} & \big(L_{p,n+p+1}-F_{p,n+p+1}-p\big)+\big(L_{p,n+p+2}-F_{p,n+p+2}-p\big)i\\ & \big(L_{p,n+p+3}-F_{p,n+p+3}-p\big)j+\big(L_{p,n+p+4}-F_{p,n+p+4}-p\big)k\\ = {} & \big(L_{p,n+p+1}+L_{p,n+p+2}i+L_{p,n+p+3}j+L_{p,n+p+4}k\big)\\ & -\big(F_{p,n+p+1}+F_{p,n+p+2}i+F_{p,n+p+3}j+F_{p,n+p+4}k\big)-\boldsymbol{P}\\ = {} & \boldsymbol{L}_{p,n+p+1}-\boldsymbol{F}_{p,n+p+1}-\boldsymbol{P}.\end{aligned}$$

Theorem 2.6. For $n \geq 0$, we have the following summation formulae

$$\begin{aligned}\sum_{k=0}^{n}\boldsymbol{Le}_{p,k} = {} & \boldsymbol{Le}_{p,n+p+1}-(n+1)\boldsymbol{P}-1-\big(1+p+Le_{p,0}\big)i-\big(1+2p+Le_{p,0}+Le_{p,1}\big)j\\ & -\big(1+3p+Le_{p,0}+Le_{p,1}+Le_{p,2}\big)k.\end{aligned}$$

Proof: We first compute

$$\begin{aligned}\sum_{k=0}^{n} \boldsymbol{Le}_{p,k} = &\sum_{k=0}^{n} Le_{p,k} + \sum_{k=1}^{n+1} Le_{p,k}\, i + \sum_{k=2}^{n+2} Le_{p,k}\, j + \sum_{k=3}^{n+3} Le_{p,k} k \\ = &\sum_{k=0}^{n} Le_{p,k} + \left(\sum_{k=0}^{n+1} Le_{p,k} - Le_{p,0}\right) i \\ &+ \left(\sum_{k=0}^{n+2} Le_{p,k} - Le_{p,0} - Le_{p,1}\right) j \\ &+ \left(\sum_{k=0}^{n+3} Le_{p,k} - Le_{p,0} - Le_{p,1} - Le_{p,2}\right) k.\end{aligned}$$

Using the summation formulae $\sum_{k=0}^{n} Le_{p,k} = Le_{p,n+p+1} - (n+1)p - 1$ (see [9]), we see that

$$\begin{aligned}\sum_{k=0}^{n} \boldsymbol{Le}_{p,k} = &\left(Le_{p,n+p+1} - (n+1)p - 1\right) \\ &+\left(Le_{p,n+p+2} - (n+2)p - 1 - Le_{p,0}\right) i \\ &+\left(Le_{p,n+p+3} - (n+3)p - 1 - Le_{p,0} - Le_{p,1}\right) j \\ &+\left(Le_{p,n+p+4} - (n+4)p - 1 - Le_{p,0} - Le_{p,1} - Le_{p,2}\right) k.\end{aligned}$$

The proof can be easily seen and left to the reader.

Example 2.1 Let us consider the 3-parameter generalized quaternionic Leonardo 3-sequence, the 3-parameter generalized quaternionic Leonardo 4-sequence, the 3-parameter generalized quaternionic Leonardo 5-sequence, and the 3-parameter generalized quaternionic Leonardo 6-sequence (see several terms in Table 3).

Table 3. For *p = 3, p = 4,p = 5* and *p = 6*, several terms of the 3-parameter generalized quaternionic Leonardo p-sequence

n	**Case** $p = 3$	**Case** $p = 4$	**Case** $p = 5$	**Case** $p = 6$
0	$1 + i + j + k$	$1 + i + j + k$	$1 + i + j + k$	$1 + i + j + k$
1	$1 + i + j + 5k$	$1 + i + j + k$	$1 + i + j + k$	$1 + i + j + k$
2	$1 + i + 5j + 9k$	$1 + i + j + 6k$	$1 + i + j + k$	$1 + i + j + k$
3	$1 + 5i + 9j + 13k$	$1 + i + 6j + 11k$	$1 + i + j + 7k$	$1 + i + j + k$
4	$5 + 9i + 13j + 17k$	$1 + 6i + 11j + 16k$	$1 + i + 7j + 13k$	$1 + i + j + 8k$
5	$9 + 13i + 17j + 25k$	$6 + 11i + 16j + 21k$	$1 + 7i + 13j + 19k$	$1 + i + 8j + 15k$
6	$13 + 17i + 25j + 37k$	$11 + 16i + 21j + 26k$	$7 + 13i + 19j + 25k$	$1 + 8i + 15j + 22k$
7	$17 + 25i + 37j + 53k$	$16 + 21i + 26j + 36k$	$13 + 19i + 25j + 31k$	$8 + 15i + 22j + 29k$
8	$25 + 37i + 53j + 73k$	$21 + 26i + 36j + 51k$	$19 + 25i + 31j + 37k$	$15 + 22i + 29j + 36k$
9	$37 + 53i + 73j + 101k$	$26 + 36i + 51j + 71k$	$25 + 31i + 37j + 49k$	$22 + 29i + 36j + 43k$
10	$53 + 73i + 101j + 141k$	$36 + 51i + 71j + 96k$	$31 + 37i + 49j + 67k$	$29 + 36i + 43j + 50k$

By using (2.4) and Table 3, one can see the following relations:

$$\begin{aligned}
\boldsymbol{Le}_{3,7} &= \boldsymbol{Le}_{3,6} + \boldsymbol{Le}_{3,4} - \boldsymbol{Le}_{3,0}, \\
\boldsymbol{Le}_{3,8} &= \boldsymbol{Le}_{3,7} + \boldsymbol{Le}_{3,5} - \boldsymbol{Le}_{3,1}, \\
\boldsymbol{Le}_{3,9} &= \boldsymbol{Le}_{3,8} + \boldsymbol{Le}_{3,6} - \boldsymbol{Le}_{3,2}, \\
\boldsymbol{Le}_{4,9} &= \boldsymbol{Le}_{4,8} + \boldsymbol{Le}_{4,5} - \boldsymbol{Le}_{4,0}, \\
\boldsymbol{Le}_{4,10} &= \boldsymbol{Le}_{4,9} + \boldsymbol{Le}_{4,6} - \boldsymbol{Le}_{4,1}.
\end{aligned}$$

For $n = 5$ and $p = 4$, we obtain

$$\sum_{k=0}^{5} \boldsymbol{Le}_{4,5} = 11 + 21i + 26j + 46k = \boldsymbol{Le}_{4,10} - 6\boldsymbol{P} - 1 - 6i - 11j - 16k,$$

for $n = p = 5$, we have

$$\sum_{k=0}^{5} \boldsymbol{Le}_{5,5} = 6 + 12i + 24j + 42k = \boldsymbol{Le}_{5,11} - 6\boldsymbol{P} - 1 - 7i - 13j - 19k,$$

for $n = p = 6$, we get

$$\sum_{k=0}^{5} \boldsymbol{Le}_{6,5} = 6 + 6i + 13j + 27k = \boldsymbol{Le}_{6,12} - 6\boldsymbol{P} - 1 - 8i - 15j - 22k.$$

4. CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

In this study, based on the definitions of 3-parameter generalized quaternions and the Leonardo p-sequence, we give a new generalization. It is not difficult to see that 2-parameter generalized, real, split, semi, split-semi, quasi quaternionic Leonardo p-sequences as well as 2-parameter generalized, real, split, semi, split-semi, quasi quaternionic Leonardo sequences are special cases of 3-parameter generalized quaternionic Leonardo p-sequence. One can see non-homogeneous recursive relation, homogeneous recursive relation, the Binet's and summation formulas of the 3-parameter generalized quaternionic Leonardo p-sequence. Now, dual quaternions with the Leonardo p-sequence is open problem for researchers.

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