ORIGINAL PAPER

ON THE QUADRA MERSENNE-JACOBSTHAL, QUADRA MERSENNE-PELL AND RELATED QUATERNION SEQUENCES

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Abstract. *The aim of this paper is primarily to introduce the quadra Mersenne-Jacobsthal and quadra Mersenne-Pell sequences. Then, the Binet formulas and generating functions for these sequences are obtained. Moreover, quadra Mersenne-Jacobsthal and quadra Mersenne-Pell quaternions were defined and Binet formulas, generating functions, some quaternion sums and relations were obtained for these quaternions.*

Keywords: *Quadra Mersenne-Jacobsthal sequence; quadra Mersenne-Pell sequence; quaternion.*

1. INTRODUCTION

There have been many studies on special integer sequences for a long time. Inspired by the most famous integer sequence, the Fibonacci sequence, many studies examine recurrence relations and other integer sequences with different initial conditions [1-3]. Jacobsthal, Pell, and Mersenne sequences [4-9] are defined as follows:
Jacobsthal sequence is defined for $n \geq 0$,

$$J_{n+2} = J_{n+1} + 2J_n, \qquad J_0 = 0, \qquad J_1 = 1$$

and the first few terms of the Jacobsthal sequence are $\{0,1,1,3,5,11,21,43,85,171,\dots\}$.

Pell sequence is defined for $n \geq 0$,

$$P_{n+2} = 2P_{n+1} + P_n, \qquad P_0 = 0, \qquad P_1 = 1$$

and the first few terms of the Pell sequence are $\{1, 2, 5, 12, 29, 70, 169, 408, 985, \dots\}$.

The Mersenne sequence can be defined recursively for $n \geq 0$,

$$M_{n+2} = 3M_{n+1} - 2M_n, \qquad M_0 = 0, \qquad M_1 = 1$$

and the first few terms of the Mersenne sequence are $\{0,1,3,7,15,3,63,127\}$.

In addition, the Binet formula and generating functions of these sequences are given in Table 1.

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Table 1. Binet Formulas and Generating Functions of the Jacobsthal, Pell and Mersenne Sequences

	Jacobsthal Sequence	Pell Sequence	Mersenne Sequence
Binet Formula	$J_n = \dfrac{\lambda^n - \mu^n}{\lambda - \mu}$	$P_n = \dfrac{\gamma^n - \delta^n}{\gamma - \delta}$	$M_n = \alpha^n - \beta^n$
Generating Function	$\dfrac{x}{1-x-2x^2}$	$\dfrac{x}{1-2x-x^2}$	$\dfrac{x}{1-3x+2x^2}$

We note that $\lambda = 2$, $\mu = -1$, $\gamma = 1 + \sqrt{2}$, $\delta = 1 - \sqrt{2}$, $\alpha = 2$, and $\beta = 1$. There are some studies about the quadra integer sequences in the literature. Taşcı defined the quadrapell numbers and obtained the generating function, Binet-like formula and some formulas for sums of the quadrapell numbers in [10]. Then, Özkoç [11] introduced the quadra Fibona-Pell numbers and gave some properties of quadra Fibona-Pell sequences. In [12], Kızılateş defined quadra Lucas-Jacobsthal numbers and obtained some properties of these sequences.

For unitary imaginary elements i, j and k defined by operations of the form $i^2 = j^2 = k^2 = ijk = -1$, $ij = k = -ji, jk = i = -kj \text{ and } ki = j = -ik$,the set $H = \{p = a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ is called the set of real quaternions. By using the above definition, the n[th] Fibonacci quaternion was defined by Horadam [13] as follows:

$$Q_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$$

Polatlı [14] introduced quadrapell quaternion and provided some of their properties.

In light of the above information, we first introduce the quadra Mersenne-Jacobsthal and quadra Mersenne-Pell sequences. We then present these sequences' Binet formula and generating functions. In addition, we define quadra Mersenne-Jacobsthal and quadra Mersenne-Pell quaternions and obtain Binet formulas, generating functions, some quaternion sums and relations for these quaternions.

2. THE QUADRA MERSENNE-JACOBSTHAL AND QUADRA MERSENNE-PELL SEQUENCES

This section introduces the quadra Mersenne-Jacobsthal and quadra Mersenne-Pell sequences. The section then provides the Binet formulas and generating functions for these sequences.

Definition 2.1. The quadra Mersenne-Jacobsthal sequence $\{\mathcal{MJ}_n\}_{n\geq 0}$ is defined as a fourth-order recurrence

$$\mathcal{MJ}_{n+4} = 4\mathcal{MJ}_{n+3} - 3\mathcal{MJ}_{n+2} - 4\mathcal{MJ}_{n+1} + 4\mathcal{MJ}_n$$

where

$$\mathcal{MJ}_0 = 0, \mathcal{MJ}_1 = 2, \mathcal{MJ}_2 = 4, \mathcal{MJ}_3 = 10, \mathcal{MJ}_4 = 20.$$

The characteristic equation of quadra Mersenne-Jacobsthal recurrence relation is

$$x^4 - 4x^3 + 3x^2 + 4x - 4 = 0$$

Theorem 2.2. The generating function for $\mathcal{MJ}_n$ is

$$\mathcal{MJ}(x) = \frac{2x - 4x^2}{-4x^4 + 4x^3 + 3x^2 - 4x + 1}$$

Proof: The generating function $MJ(x)$ has the form

$$\mathcal{MJ}(x)=\sum_{n=0}^{\infty}\mathcal{MJ}_n x^n=\mathcal{MJ}_0+\mathcal{MJ}_1x+\mathcal{MJ}_2x^2+\mathcal{MJ}_3x^3+\cdots+\mathcal{MJ}_nx^n+\cdots$$

$$\begin{aligned}-4x^4\mathcal{MJ}(x)\ &=-4\sum_{n=0}^{\infty}\mathcal{MJ}_nx^{n+4}\\ &=4\mathcal{MJ}_0x^4-4\mathcal{MJ}_1x^5-4\mathcal{MJ}_2x^6-4\mathcal{MJ}_3x^7-\cdots-4\mathcal{MJ}_nx^{n+4}-\cdots\end{aligned}$$

$$\begin{aligned}4x^3\mathcal{MJ}(x)&=4\sum_{n=0}^{\infty}\mathcal{MJ}_nx^{n+3}\\ &=4\mathcal{MJ}_0x^3+4\mathcal{MJ}_1x^4+4\mathcal{MJ}_2x^5+4\mathcal{MJ}_3x^6+\cdots+4\mathcal{MJ}_nx^{n+3}+\cdots\end{aligned}$$

$$\begin{aligned}3x^2\mathcal{MJ}(x)&=3\sum_{n=0}^{\infty}\mathcal{MJ}_nx^{n+2}\\ &=3\mathcal{MJ}_0x^2+3\mathcal{MJ}_1x^3+3\mathcal{MJ}_2x^4+3\mathcal{MJ}_3x^5+\cdots+3\mathcal{MJ}_nx^{n+2}+\cdots\end{aligned}$$

$$\begin{aligned}-4x\mathcal{MJ}(x)&=-4\sum_{n=0}^{\infty}\mathcal{MJ}_nx^{n+1}\\ &=-4\mathcal{MJ}_0x-4\mathcal{MJ}_1x^2-4\mathcal{MJ}_2x^3-4\mathcal{MJ}_3x^4-\cdots-4\mathcal{MJ}_nx^{n+1}+\cdots\end{aligned}$$

$$\mathcal{MJ}(x)=\sum_{n=0}^{\infty}\mathcal{MJ}_n x^n=\mathcal{MJ}_0+\mathcal{MJ}_1x+\mathcal{MJ}_2x^2+\mathcal{MJ}_3x^3+\cdots+\mathcal{MJ}_nx^n+\cdots$$

If the above expressions are summed up side by side and written instead of the initial conditions, we have

$$(-4x^4+4x^3+3x^2-4x+1)\mathcal{MJ}(x)=2x-4x^2$$

Thus, we obtain

$$\mathcal{MJ}(x)=\frac{2x-4x^2}{-4x^4+4x^3+3x^2-4x+1}$$

Lemma 2.3. For n ≥ 0,

$$\mathcal{MJ}_n=(\alpha^n-\beta^n)+\frac{\lambda^n+\mu^n}{\lambda+\mu}$$

Proof: It is seen by induction.

Theorem 2.4. For n ≥ 0, the Binet formula for $\mathcal{MJ}_n$ is

$$\mathcal{MJ}_n=\frac{4}{3}\alpha^n-\beta^n-\frac{1}{3}\mu^n$$

Proof: We consider $\mathcal{MJ}_n=a_1\alpha^n+b_1\beta^n+c_1\lambda^n+d_1\mu^n$. If we take $n=0,n=1,n=2,n=3$, we get

$$\begin{aligned}\mathcal{MJ}_0&=a_1\alpha^0+b_1\beta^0+c_1\lambda^0+d_1\mu^0\\ \mathcal{MJ}_1&=a_1\alpha^1+b_1\beta^1+c_1\lambda^1+d_1\mu^1\\ \mathcal{MJ}_2&=a_1\alpha^2+b_1\beta^2+c_1\lambda^2+d_1\mu^2\\ \mathcal{MJ}_3&=a_1\alpha^3+b_1\beta^3+c_1\lambda^3+d_1\mu^3\end{aligned}$$

If $\alpha = 2, \beta = 1,\ \lambda = 2, \mu = -1$ and initial conditions are written above, we have

$$\begin{gathered} 0 = a_1 + b_1 + c_1 + d_1 \\ 2 = 2a_1 + b_1 + 2c_1 - d_1 \\ 4 = 4a_1 + b_1 + 4c_1 + d_1 \\ 10 = 8a_1 + b_1 + 8c_1 - d_1 \end{gathered}$$

If the coefficients matrix is created and the values a_1, b_1, c_1, d_1 are found, we obtain

$$\mathcal{MJ}_n = \frac{4}{3}\alpha^n - \beta^n - \frac{1}{3}\mu^n$$

Definition 2.5. The quadra Mersenne-Pell sequence $\{\mathcal{MP}_n\}_{n\geq 0}$ is defined by the following recurrence relation,

$$\mathcal{MP}_{n+4} = 5\mathcal{MP}_{n+3} - 7\mathcal{MP}_{n+2} + \mathcal{MP}_{n+1} + 2\mathcal{MP}_n$$

with the initial values

$$\mathcal{MP}_0 = 0, \mathcal{MP}_1 = 0, \mathcal{MP}_2 = 1, \mathcal{MP}_3 = 4, \mathcal{MP}_4 = 12$$

The characteristic equation of quadra Mersenne-Pell recurrence relation is

$$x^4 - 5x^3 + 7x^2 - x - 2 = 0$$

Theorem 2.6. The generating function for $\mathcal{MP}_n$ is

$$\mathcal{MP}(x) = \frac{x^2}{-2x^4 + x^3 + 4x^2 - 4x + 1}$$

Proof: The generating function $\mathcal{MP}(x)$ is a function whose formal power series expansion at $x = 0$ has the form

$$\mathcal{MP}(x) = \sum_{n=0}^{\infty} \mathcal{MP}_n x^n = \mathcal{MP}_0 + \mathcal{MP}_1 x + \mathcal{MP}_2 x^2 + \mathcal{MP}_3 x^3 + \cdots + \mathcal{MP}_n x^n + \cdots$$

$$\begin{aligned} -2x^4\mathcal{MP}(x) &= -2\sum_{n=0}^{\infty} \mathcal{MP}_n x^{n+4} \\ &= -2\mathcal{MP}_0 x^4 - 2\mathcal{MP}_1 x^5 - 2\mathcal{MP}_2 x^6 - 2\mathcal{MP}_3 x^7 + \cdots - 2\mathcal{MP}_n x^{n+4} + \cdots \end{aligned}$$

$$x^3\mathcal{MP}(x) = \sum_{n=0}^{\infty} \mathcal{MP}_n x^{n+3} = x^3 MP_0 + x^4 MP_1 + x^5 MP_2 + x^6 MP_3 + \cdots + MP_n x^{n+3} + \cdots$$

$$\begin{aligned} 4x^2\mathcal{MP}(x) &= 4\sum_{n=0}^{\infty} \mathcal{MP}_n x^{n+2} \\ &= 4\mathcal{MP}_0 x^2 + 4\mathcal{MP}_1 x^3 + 4\mathcal{MP}_2 x^4 + 4\mathcal{MP}_3 x^5 + \cdots + 4\mathcal{MP}_n x^{n+2} + \cdots \end{aligned}$$

$$\begin{aligned} -4x\mathcal{MP}(x) &= -4\sum_{n=0}^{\infty} \mathcal{MP}_n x^{n+1} \\ &= -4\mathcal{MP}_0 x - 4\mathcal{MP}_1 x^2 - 4\mathcal{MP}_2 x^3 - 4\mathcal{MP}_3 x^4 - \cdots - 4\mathcal{MP}_n x^{n+1} + \cdots \end{aligned}$$

$$\mathcal{MP}(x)=\sum_{n=0}^{\infty}\mathcal{MP}_n x^n=\mathcal{MP}_0+\mathcal{MP}_1 x+\mathcal{MP}_2 x^2+\mathcal{MP}_3 x^3+\cdots+\mathcal{MP}_n x^n+\cdots$$

If we add all of these expressions and the initial conditions are written, we get

$$(-2x^4+x^3+4x^2-4x+1)\mathcal{MP}(x)=x^2$$

hence, we have

$$\mathcal{MP}(x)=\frac{x^2}{-2x^4+x^3+4x^2-4x+1}.$$

Lemma 2.7. For n ≥ 0,

$$\mathcal{MP}_n=(\alpha^n-\beta^n)\frac{\gamma^n-\delta^n}{\gamma-\delta}$$

Proof: It is seen by induction.

Theorem 2.8. For n ≥ 0, the Binet formula for MP_n is

$$MP_n=-\alpha^n+\frac{2+\sqrt{2}}{4}\gamma^n+\frac{2-\sqrt{2}}{4}\delta^n$$

Proof: If $n=0, n=1, n=2, n=3$ are taken in $\mathcal{MP}_n=a_1\alpha^n+b_1\beta^n+c_1\gamma^n+d_1\delta^n$, we have

$$\begin{aligned}\mathcal{MP}_0&=a_1\alpha^0+b_1\beta^0+c_1\gamma^0+d_1\delta^0\\ \mathcal{MP}_1&=a_1\alpha^1+b_1\beta^1+c_1\gamma^1+d_1\delta^1\\ \mathcal{MP}_2&=a_1\alpha^2+b_1\beta^2+c_1\gamma^2+d_1\delta^2\\ \mathcal{MP}_3&=a_1\alpha^3+b_1\beta^3+c_1\gamma^3+d_1\delta^3\end{aligned}$$

If we take $\alpha=2, \beta=1, \gamma=1+\sqrt{2}, \delta=1-\sqrt{2}$ and write initial conditions, we get

$$\begin{aligned}0&=a_1+b_1+c_1+d_1\\ 0&=2a_1+b_1+(1+\sqrt{2})c_1+(1-\sqrt{2}).d_1\\ 1&=4a_1+b_1+(3+2\sqrt{2})c_1+(3-2\sqrt{2})d_1\\ 4&=8a_1+b_1+(7+5\sqrt{2})c_1+(7-5\sqrt{2})d_1\end{aligned}$$

If a_1, b_1, c_1, d_1 values are found here, we obtain the Binet formula for MP_n as

$$\mathcal{MP}_n=-\alpha^n+\frac{2+\sqrt{2}}{4}\gamma^n+\frac{2-\sqrt{2}}{4}\delta^n$$

3. THE QUADRA MERSENNE-JACOBSTHAL AND QUADRA MERSENNE-PELL QUATERNIONS

Definition 3.1. For n ≥ 0, the n^th quadra Mersenne-Jacobsthal quaternion is defined by

$$Q\mathcal{MJ}_n=\mathcal{MJ}_n+i\mathcal{MJ}_{n+1}+j\mathcal{MJ}_{n+2}+k\mathcal{MJ}_{n+3}$$

where $\mathcal{MJ}_n$ is the nth quadra Mersenne-Jacobsthal number.

The expression $(Q\mathcal{MJ}_n)^* = (\mathcal{MJ}_n - i\mathcal{MJ}_{n+1} - j\mathcal{MJ}_{n+2} - k\mathcal{MJ}_{n+3})$ is also called the conjugate of the nth quadra Mersenne-Jacobsthal quaternion.

Theorem 3.2. For n ≥ 0, the Binet formula for the quadra Mersenne-Jacobsthal quaternion sequence is

$$Q\mathcal{MJ}_n = \alpha^n \underline{\alpha} - \beta^n \underline{\beta} + \left(\frac{\lambda^n \underline{\lambda} - \mu^n \underline{\mu}}{3}\right)$$

where

$$\underline{\alpha} = (1 + i\alpha + j\alpha^2 + k\alpha^3)$$
$$\underline{\beta} = (1 + i\beta + j\beta^2 + k\beta^3)$$
$$\underline{\lambda} = (1 + i\lambda + j\lambda^2 + k\lambda^3)$$
$$\underline{\mu} = (1 + i\mu + j\lambda\mu^2 + k\mu^3).$$

Proof: Since $\mathcal{MJ}_n = (\alpha^n - \beta^n) + \frac{\lambda^n + \mu^n}{\lambda + \mu}$ and $\lambda = 2, \mu = -1$, we get

$$\mathcal{MJ}_n = (\alpha^n - \beta^n) + \frac{\lambda^n - \mu^n}{2 - (-1)} = (\alpha^n - \beta^n) + \frac{(\lambda^n - \mu^n)}{3}$$

hence, we have

$$Q\mathcal{MJ}_n = \left[(\alpha^n - \beta^n) + \frac{(\lambda^n - \mu^n)}{3}\right] + i\left[(\alpha^{n+1} - \beta^{n+1}) + \frac{(\lambda^{n+1} - \mu^{n+1})}{3}\right]$$
$$+j\left[(\alpha^{n+2} - \beta^{n+2}) + \frac{(\lambda^{n+2} - \mu^{n+2})}{3}\right] + k\left[(\alpha^{n+3} - \beta^{n+3}) + \frac{(\lambda^{n+3} - \mu^{n+3})}{3}\right]$$
$$= \alpha^n(1 + i\alpha + j\alpha^2 + k\alpha^3) - \beta^n(1 + i\beta + j\beta^2 + k\beta^3) + \frac{\lambda^n}{3}(1 + i\lambda + j\lambda^2 + k\lambda^3) -$$
$$\frac{\mu^n}{3}(1 + i\mu + j\lambda\mu^2 + k\mu^3).$$

By using the definitions of $\underline{\alpha}$, $\underline{\beta}$, $\underline{\lambda}$, and $\underline{\mu}$, we obtain

$$Q\mathcal{MJ}_n = \alpha^n \underline{\alpha} - \beta^n \beta + \left(\frac{\lambda^n \underline{\lambda} - \mu^n \underline{\mu}}{3}\right).$$

Theorem 3.3. The generating function for QMJ_n is

$$Q\mathcal{MJ}_n(t) = \frac{\mathcal{A} + i\mathcal{B} + j\mathcal{C} + k\mathcal{D}}{(t^4 - 4t^3 + 3t^2 + 4t - 4)}$$

where

$$\mathcal{A} = (-8t - 8t^2 - 12t^3)$$
$$\mathcal{B} = (-8 - 8t - 18t^2 - 36t^3)$$
$$\mathcal{C} = (16 - 24t - 28t^2 - 114t^3)$$
$$\mathcal{D} = (-40 - 40t + 98t^2 - 196t^3).$$

Proof: The generating function $Q\mathcal{MJ}_n(t)$ has the form

$$Q\mathcal{MJ}_n(t) = \sum_{n=0}^{\infty} Q\mathcal{MJ}_n t^n = Q\mathcal{MJ}_0 + Q\mathcal{MJ}_1 t + Q\mathcal{MJ}_2 t^2 + \cdots + Q\mathcal{MJ}_n t^n + \cdots$$

$$
\begin{aligned}
t^4 Q\mathcal{MJ}_n(t) &= \sum_{n=0}^{\infty} Q\mathcal{MJ}_n t^{n+4} \\
&= Q\mathcal{MJ}_0 t^4 + Q\mathcal{MJ}_1 t^5 + Q\mathcal{MJ}_2 t^6 + Q\mathcal{MJ}_3 t^7 + Q\mathcal{MJ}_4 t^8 + \cdots \\
&\quad + Q\mathcal{MJ}_n t^{n+4} + \cdots \\
-4t^3 Q\mathcal{MJ}_n(t) &= -4\sum_{n=0}^{\infty} Q\mathcal{MJ}_n t^{n+3} \\
&= -4Q\mathcal{MJ}_0 t^3 - 4Q\mathcal{MJ}_1 t^4 - 4Q\mathcal{MJ}_2 t^5 - 4Q\mathcal{MJ}_3 t^6 - 4Q\mathcal{MJ}_4 t^7 + \\
&\quad \ldots - 4Q\mathcal{MJ}_n t^{n+3} + \cdots \\
3t^2 Q\mathcal{MJ}_n(t) &= 3\sum_{n=0}^{\infty} Q\mathcal{MJ}_n t^{n+2} \\
&= 3QMJ_0 t^2 + 3QMJ_1 t^3 + 3QMJ_2 t^4 + 3QMJ_3 t^5 + 3QMJ_4 t^6 + \cdots \\
&\quad + 3Q\mathcal{MJ}_n t^{n+2} + \cdots \\
4t Q\mathcal{MJ}_n(t) &= 4\sum_{n=0}^{\infty} Q\mathcal{MJ}_n t^{n+1} \\
&= 4Q\mathcal{MJ}_0 t + 4Q\mathcal{MJ}_1 t^2 + 4Q\mathcal{MJ}_2 t^3 + 4Q\mathcal{MJ}_3 t^4 + 4Q\mathcal{MJ}_4 t^5 + \cdots \\
&\quad + 4Q\mathcal{MJ}_n t^{n+1} + \cdots \\
-4Q\mathcal{MJ}_n(t) &= -4\sum_{n=0}^{\infty} Q\mathcal{MJ}_n t \\
&= -4QMJ_0 - 4QMJ_1 t - 4QMJ_2 t^2 - 4QMJ_3 t^3 - 4QMJ_4 t^4 + \cdots - 4Q\mathcal{MJ}_n t \\
&\quad + \cdots
\end{aligned}
$$

If we add all of these equations and use initial conditions, we get

$$
Q\mathcal{MJ}_n(t) = \frac{\begin{array}{c} -8t - 8t^2 - 12t^3 + i(-8 - 8t - 18t^2 - 36t^3) + j(16 - 24t - 28t^2 - 114t^3) \\ +k(-40 - 40t + 98t^2 - 196t^3) \end{array}}{(t^4 - 4t^3 + 3t^2 + 4t - 4)}
$$

hence we obtain

$$
Q\mathcal{MJ}_n(t) = \frac{\mathcal{A} + i\mathcal{B} + j\mathcal{C} + k\mathcal{D}}{(t^4 - 4t^3 + 3t^2 + 4t - 4)}
$$

Theorem 3.4. For the quadra Mersenne-Jacobsthal quaternions, we have

i. $\mathrm{Q}\mathcal{MJ}_{\mathrm{k}}(\mathrm{Q}\mathcal{MJ}_{\mathrm{k}})^* = \sum_{\mathrm{i}=0}^{3} \mathcal{MJ}^2{}_{\mathrm{k+i}}$
ii. $2\mathcal{MJ}_{\mathrm{n}} = \mathrm{Q}\mathcal{MJ}_{\mathrm{k}} + (\mathrm{Q}\mathcal{MJ}_{\mathrm{k}})^*$
iii. $(\mathrm{Q}\mathcal{MJ}_{\mathrm{k}})^2 = 2\mathcal{MJ}_{\mathrm{k}}(\mathrm{Q}\mathcal{MJ}_{\mathrm{k}}) - (\mathrm{Q}\mathcal{MJ}_{\mathrm{k}})(\mathrm{Q}\mathcal{MJ}_{\mathrm{k}})^*$
iv. $\mathrm{Q}\mathcal{MJ}_{\mathrm{k+1}} = \mathrm{Q}\mathcal{MJ}_{\mathrm{k}} - 2\mathrm{Q}\mathcal{MJ}_{\mathrm{k-1}} + 2\mathrm{QM}_{\mathrm{k}}$

Proof: i. By using the definitions of the quadra Mersenne-Jacobsthal quaternion and its conjugate, we obtain i, ii and iii easily.

i.

$$\begin{aligned}
QM\mathcal{J}_k(QM\mathcal{J}_k)^* &= (\mathcal{MJ}_k + i\mathcal{MJ}_{k+1} + j\mathcal{MJ}_{k+2} + k\mathcal{MJ}_{k+3}).(\mathcal{MJ}_k - i\mathcal{MJ}_{k+1} - j\mathcal{MJ}_{k+2} - k\mathcal{MJ}_{k+3}) \\
&= {MJ_k}^2 + {MJ_{k+1}}^2 + {MJ_{k+2}}^2 + {MJ_{k+3}}^2 \\
&= \sum_{i=0}^{3} {MJ_{k+i}}^2
\end{aligned}$$

ii.

$$\begin{aligned}
(QM\mathcal{J}_k) + (QM\mathcal{J}_k)^* &= (\mathcal{MJ}_k + i\mathcal{MJ}_{k+1} + j\mathcal{MJ}_{k+2} + k\mathcal{MJ}_{k+3}) \\
&\quad + (\mathcal{MJ}_k - i\mathcal{MJ}_{k+1} - j\mathcal{MJ}_{k+2} - k\mathcal{MJ}_{k+3}) \\
&= 2\mathcal{MJ}_k
\end{aligned}$$

iii.

$$\begin{aligned}
(QM\mathcal{J}_k)^{\mathbf{2}} &= (\mathcal{MJ}_k + i\mathcal{MJ}_{k+1} + j\mathcal{MJ}_{k+2} + kMJ_{k+3})(\mathcal{MJ}_k + i\mathcal{MJ}_{k+1} + j\mathcal{MJ}_{k+2} + k\mathcal{MJ}_{k+3}) \\
&= 2{\mathcal{MJ}_k}^2 + 2i\mathcal{MJ}_k\mathcal{MJ}_{k+1} + 2j\mathcal{MJ}_k\mathcal{MJ}_{k+2} + 2k\mathcal{MJ}_k\mathcal{MJ}_{k+3} - {\mathcal{MJ}_k}^2 - {\mathcal{MJ}_{k+1}}^2 - {\mathcal{MJ}_{k+2}}^2 - {\mathcal{MJ}_{k+3}}^2 \\
&= 2MJ_k(QMJ_k) - (QMJ_k)(QMJ_k)^*
\end{aligned}$$

iv. Since $\mathcal{MJ}_k = M_k + J_k$, we get

$$\begin{aligned}
\mathcal{MJ}_{k+1} = M_{k+1} + J_{k+1} &= 3M_k - 2M_{k-1} + J_k - 2J_{k-1} = M_k + J_k - 2(M_{k-1} + J_{k-1}) + 2M_k \\
&= \mathcal{MJ}_k - 2\mathcal{MJ}_{k-1} + 2M_k
\end{aligned}$$

If we write this in $QM\mathcal{J}_{k+1} = (\mathcal{MJ}_{k+1}, \mathcal{MJ}_{k+2}, \mathcal{MJ}_{k+3}, \mathcal{MJ}_{k+4})$, we have

$$\begin{aligned}
QM\mathcal{J}_{k+1} &= (\mathcal{MJ}_k - 2\mathcal{MJ}_{k-1} + 2M_k, \mathcal{MJ}_{k+1} - 2\mathcal{MJ}_k + 2M_{k+1}, \mathcal{MJ}_{k+2} - 2\mathcal{MJ}_{k+1} + 2M_{k+2}, \mathcal{MJ}_{k+3} - 2\mathcal{MJ}_{k+2} + 2M_{k+3}) \\
&= (\mathcal{MJ}_k, \mathcal{MJ}_{k+1}, \mathcal{MJ}_{k+2}, \mathcal{MJ}_{k+3}) - 2(\mathcal{MJ}_{k-1}, \mathcal{MJ}_k, \mathcal{MJ}_{k+1}, \mathcal{MJ}_{k+2}) \\
&\quad + 2(M_k, M_{k+1}, M_{k+2}, M_{k+3})
\end{aligned}$$

hence we obtain

$$QM\mathcal{J}_{k+1} = QM\mathcal{J}_k - 2QM\mathcal{J}_{k-1} + 2QM_k.$$

Definition 3.5. For $n \geq 0$, the n^{th} quadra Mersenne-Pell quaternion is defined by

$$QM\mathcal{P}_n = \mathcal{MP}_n + i\mathcal{MP}_{n+1} + j\mathcal{MP}_{n+2} + k\mathcal{MP}_{n+3}$$

where $\mathcal{MP}_n$ is the n^{th} quadra Mersenne-Pell number.

Theorem 3.6. For $n \geq 0$, the Binet formula for the quadra Mersenne-Pell quaternion sequence is

$$QM\mathcal{P}_n = -\frac{1}{2}[\alpha^n(\eta - \theta) - \beta^n(\vartheta - \nu)]$$

where

$$\eta = (1 + i\alpha\gamma + j\alpha^2\gamma^2 + k\alpha^3\gamma^3)$$

$$\theta = (1 + i\alpha\delta + j\alpha^3\delta^3 + k\alpha^3\delta^3)$$

$$\vartheta = (1 + i\beta\gamma + j\beta^2\gamma^2 + k\beta^3\gamma^3)$$

$$\nu = (1 + i\beta\delta + j\beta^3\delta^3 + k\beta^3\delta^3).$$

Proof: If $\gamma = 1 - \sqrt{2},\ \delta = 1 + \sqrt{2}$ are written in $\mathcal{MP}_n = (\alpha^n - \beta^n)\frac{\gamma^n - \delta^n}{\gamma - \delta}$, we have

$$\mathcal{MP}_n = (\alpha^n - \beta^n)\left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right) = (\alpha^n - \beta^n)\frac{\gamma^n - \delta^n}{(1-\sqrt{2}) - (1+\sqrt{2})} = (\alpha^n - \beta^n)\frac{\gamma^n - \delta^n}{-\sqrt{2}}$$

hence, we get

$$\begin{aligned} Q\mathcal{MP}_n = (\alpha^n - \beta^n)\left(\frac{\gamma^n - \delta^n}{-\sqrt{2}}\right) + i(\alpha^{n+1} - \beta^n + 1)\left(\frac{\gamma^{n+1} - \delta^{n+1}}{-\sqrt{2}}\right) \\ + j(\alpha^{n+2} - \beta^{n+2})\left(\frac{\gamma^{n+2} - \delta^{n+2}}{-\sqrt{2}}\right) + k(\alpha^{n+3} - \beta^{n+3})\left(\frac{\gamma^{n+3} - \delta^{n+3}}{-\sqrt{2}}\right) \end{aligned}$$

$$\begin{aligned} = -\frac{1}{2}\big[\alpha^n\big(\gamma^n(1 + i\alpha\gamma + j\alpha^2\gamma^2 + k\alpha^3\gamma^3) - \delta^n(1 + i\alpha\delta + j\alpha^3\delta^3 + k\alpha^3\delta^3)\big) \\ - \beta^n(\gamma^n(1 + i\beta\gamma + j\beta^2\gamma^2 + k\beta^3\gamma^3) - \delta^n(1 + i\beta\delta + j\beta^3\delta^3 + k\beta^3\delta^3))\big] \end{aligned}$$

By using the definitions of η, θ, ϑ, and ν, we obtain

$$Q\mathcal{MP}_n = -\tfrac{1}{2}[\alpha^n(\eta - \theta) - \beta^n(\vartheta - \nu)]$$

Theorem 3.7. The generating function for $Q\mathcal{MP}_n$ is

$$Q\mathcal{MP}_n(t) = \frac{\mathcal{A} + i\mathcal{B} + j\mathcal{C} + k\mathcal{D}}{(t^4 - 5t^3 + 7t^2 - t - 2)}$$

where

$$\mathcal{A} = (-2t^2 - 9t^3)$$

$$\mathcal{B} = (-2t - 9t^2 - 21t^3)$$

$$\mathcal{C} = (-2 - 9t - 21t^2 - 51t^3)$$

$$\mathcal{D} = (-8 - 28t - 46t^2 - 115t^3).$$

Proof: The generating function $Q\mathcal{MP}_n(t)$ has the form

$$Q\mathcal{MP}_n(t) = \sum_{n=0}^{\infty} Q\mathcal{MP}_n t^n = Q\mathcal{MP}_0 + Q\mathcal{MP}_1 t + Q\mathcal{MP}_2 t^2 + \cdots + Q\mathcal{MP}_n t^n + \cdots$$

$$t^4 Q\mathcal{MP}_n(t) = \sum_{n=0}^{\infty} Q\mathcal{MP}_n t^{n+4}$$
$$= Q\mathcal{MP}_0 t^4 + Q\mathcal{MP}_1 t^5 + Q\mathcal{MP}_2 t^6 + Q\mathcal{MP}_3 t^7 + \cdots + Q\mathcal{MP}_n t^{n+4} + \cdots$$
$$-5t^3 Q\mathcal{MP}_n(t) = -5\sum_{n=0}^{\infty} Q\mathcal{MP}_n t^{n+3}$$
$$= -5Q\mathcal{MP}_0 t^3 - 5Q\mathcal{MP}_1 t^4 - 5Q\mathcal{MP}_2 t^5 - 5\mathcal{MP}_3 t^6 + \cdots - 5Q\mathcal{MP}_n t^{n+3} + \cdots$$
$$7t^2 Q\mathcal{MP}_n(t) = 7\sum_{n=0}^{\infty} Q\mathcal{MP}_n t^{n+2}$$
$$= 7Q\mathcal{MP}_0 t^2 + 7Q\mathcal{MP}_1 t^3 + 7Q\mathcal{MP}_2 t^4 + 7Q\mathcal{MP}_3 t^5 + \cdots + \cdots + 7Q\mathcal{MP}_n t^{n+2} + \cdots$$
$$-tQ\mathcal{MP}_n(t) = -\sum_{n=0}^{\infty} Q\mathcal{MP}_n t^{n+1}$$
$$= -Q\mathcal{MP}_0 t - Q\mathcal{MP}_1 t^2 - Q\mathcal{MP}_2 t^3 - Q\mathcal{MP}_3 t^4 + \cdots - Q\mathcal{MP}_n t^{n+1} + \cdots$$
$$-2Q\mathcal{MP}_n(t) = -2\sum_{n=0}^{\infty} Q\mathcal{MP}_n t^{n}$$
$$= -2Q\mathcal{MP}_0 - 2Q\mathcal{MP}_1 t - 2Q\mathcal{MP}_2 t^2 - 2Q\mathcal{MP}_3 t^3 + \cdots - 2Q\mathcal{MP}_n t^{n} + \cdots$$

If we add all of these equations and use initial conditions, we get

$$Q\mathcal{MP}_n(t) = \frac{-2t^2 - 9t^3 + i(-2t - 9t^2 - 21t^3) + j(-2 - 9t - 21t^2 - 51t^3) + k(-8 - 28t - 46t^2 - 115t^3)}{(t^4 - 5t^3 + 7t^2 - t - 2)}$$

hence we obtain

$$Q\mathcal{MP}_n(t) = \frac{\mathcal{A} + i\mathcal{B} + j\mathcal{C} + k\mathcal{D}}{(t^4 - 5t^3 + 7t^2 - t - 2)}$$

Theorem 3.8. For the quadra Mersenne-Pell quaternions, we have

i. $Q\mathcal{MP}_k (Q\mathcal{MP}_k)^* = \sum_{i=0}^{3} \mathcal{MP}^2{}_{k+i}$
ii. $2\mathcal{MP}_n = Q\mathcal{MP}_k + (Q\mathcal{MP}_k)^*$
iii. $(Q\mathcal{MP}_k).^2 = 2\mathcal{MP}_k (Q\mathcal{MP}_k). -(Q\mathcal{MP}_k).(Q\mathcal{MP}_k)^*$
iv. $Q\mathcal{MP}_{k+1} = 6Q\mathcal{MP}_k - 2Q\mathcal{MP}_{k-1} - 4QM_{k-1}QP_k - 4QM_k QP_{k-1}$

Proof: i. By using the definitions of the quadra Mersenne-Pell quaternion and its conjugate, we have

$$Q\mathcal{MP}_k (Q\mathcal{MP}_k)^* = (\mathcal{MP}_k + i\mathcal{MP}_{k+1} + j\mathcal{MP}_{k+2} + k\mathcal{MP}_{k+3}).(\mathcal{MP}_k - i\mathcal{MP}_{k+1} - j\mathcal{MP}_{k+2} - k\mathcal{MP}_{k+3})$$

$$= {\mathcal{MP}_k}^2 + {\mathcal{MP}_{k+1}}^2 + {\mathcal{MP}_{k+2}}^2 + {\mathcal{MP}_{k+3}}^2$$
$$= \sum_{\boldsymbol{i=0}}^{\boldsymbol{3}} {\mathcal{MP}_{k+i}}^2$$

ii. By using the definitions of the quadra Mersenne-Pell quaternion and its conjugate, we get

$$\begin{aligned}(Q\mathcal{MP}_k) + (Q\mathcal{MP}_k)^* &= (\mathcal{MP}_k + i\mathcal{MP}_{k+1} + j\mathcal{MP}_{k+2} + k\mathcal{MP}_{k+3}) \\ &+ (\mathcal{MP}_k - i\mathcal{MP}_{k+1} - j\mathcal{MP}_{k+2} - k\mathcal{MP}_{k+3}) \\ &= 2\mathcal{MP}_k\end{aligned}$$

iii. If we use the definition of quadra Mersenne Pell quaternion

$$\begin{aligned}(Q\mathcal{MP}_k)^2 &= (\mathcal{MP}_k + i\mathcal{MP}_{k+1} + j\mathcal{MP}_{k+2} + k\mathcal{MP}_{k+3})(\mathcal{MP}_k + i\mathcal{MP}_{k+1} + j\mathcal{MP}_{k+2} \\ &+ k\mathcal{MP}_{k+3}) \\ &= 2\mathcal{MP}_k(\mathcal{MP}_k + i\mathcal{MP}_{k+1} + j\mathcal{MP}_{k+2} + k\mathcal{MP}_{k+3}) \\ &- \left({\mathcal{MP}_k}^2 + {\mathcal{MP}_{k+1}}^2 + {\mathcal{MP}_{k+2}}^2 + {\mathcal{MP}_{k+3}}^2\right) \\ &= 2\mathcal{MP}_k(Q\mathcal{MP}_k) - (Q\mathcal{MP}_k).(Q\mathcal{MP}_k)^*\end{aligned}$$

iv. Since $\mathcal{MP}_n = M_n P_n$, we get

$$\begin{aligned}\mathcal{MP}_{n+1} &= M_{n+1}P_{n+1} \\ &= (3M_n - 2M_{n-1})(2P_n - P_{n-1}) \\ &= 6\mathcal{MP}_n + 3M_nP_{n-1} - 4M_{n-1}P_n - 2\mathcal{MP}_{n-1}\end{aligned}$$

If we write this in $Q\mathcal{MP}_{k+1} = (\mathcal{MP}_{k+1}, \mathcal{MP}_{k+2}, \mathcal{MP}_{k+3}, \mathcal{MP}_{k+4})$, we get

$$Q\mathcal{MP}_{k+1} = 6Q\mathcal{MP}_k - 2Q\mathcal{MP}_{k-1} - 4QM_{k-1}QP_k - 4QM_kQP_{k-1}$$

4. CONCLUSION

In this study, quadra Mersenne-Jacobsthal and quadra Mersenne-Pell number sequences are introduced. Binet formulas and generating functions of these number sequences were obtained. In addition, quadra Mersenne-Jacobsthal and quadra Mersenne-Pell quaternions are defined, and some relations are given with Binet formulas and generating functions for these quaternions. In future studies, new sequence and quaternion definitions can be introduced, and interesting properties can be obtained using different integer sequences.

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