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WEIGHTED VARIABLE EXPONENT LEBESGUE SPACES ON A PROBABILITY SPACE

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Abstract. *In this paper, we introduce the weighted variable exponent Lebesgue spaces defined on a probability space and give some information about the martingale theory of these spaces. We first prove several basic inequalities for expectation operators and obtain several norm convergence conditions for martingales in weighted variable exponent Lebesgue spaces. We discuss the Hölder inequality and embedding properties in these spaces. Finally, under some conditions we investigate Doob's maximal function.*

Keywords: *weighted variable; Lebesque spaces; expectation operator; Hölder inequality; probability spaces.*

1. INTRODUCTION

In recent years, martingale function spaces have attracted more attention. For example, we know the martingale Lorentz spaces from [1-3]. Its extension on the martingale function spaces built on Lorentz-Karamata is studied in [4-5]. The study of martingale Hardy spaces generated by Orlicz spaces is given in [6]. The family of martingale function spaces had been generalized to Lebesgue spaces with variable exponents [7-9] and Morrey spaces [10-11]. For martingales in variable exponent spaces we use more some different methods than classical martingale spaces. A lot of effective techniques for the classical case may not be used in the variable exponent case.

Inspired by the above results, in the present paper, our aim is to define the weighted variable exponent Lebesgue spaces $L_{\vartheta}^{p(\cdot)}(\Omega)$ defined on a probability space. In addition, we show some basic properties of $L_{\vartheta}^{p(\cdot)}(\Omega)$ by definition of expectation operator. Especially, we obtain more general results than [9]. Finally, we study Doob's maximal function in martingale Lebesgue space with a variable exponent.

2. DEFINITION AND PRELIMINARY RESULTS

Let (Ω, F, P) be a probability space. Then a function $X: \Omega \to \mathbb{R}$ is called a random variable, if the event

$$X^{-1}\big((-\infty, a]\big) = \{w \in \Omega: X(w) \leq a\} \in F$$

for $a \in \mathbb{R}$, i.e., a random variable is a real valued F-measurable function on a probability space (Ω, F, P) (see [12]).

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Remark 1. Let Y be an $\mathbb{R}$-valued random variable in (Ω, F, P) with $Y \in L^1(\Omega)$. We denote its expected value by

$$E(Y) = \int_\Omega Y(w)dP(w) = \int_\mathbb{R} yd\mu_Y(y),$$

where μ_Y is the distribution measure for Y, defined on the Borel set B of $\mathbb{R}$; given by $\mu_Y(B) = P\left(X^{-1}(B)\right)$ [13]. If μ_Y is absolutely continuous with respect to the Lebesgue measure then there exists a density function $\sigma: \mathbb{R} \to [0, \infty)$ such that

$$E(Y) = \int_\mathbb{R} y\sigma(y)dy.$$

Definition 2. For a measurable function $p: \Omega \to [1, \infty)$ (called a variable exponent or random variable on Ω), we put

$$p^- = essinf p_{w\in\Omega}(w),\ p^+ = esssupp_{w\in\Omega}(w).$$

The variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ consist of all random variables X such that $\rho_{p(\cdot)}(\lambda X) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|X\|_{p(\cdot)} = inf\left\{\lambda > 0: \rho_{p(\cdot)}\left(\frac{X}{\lambda}\right) \le 1\right\},$$

where

$$\rho_{p(\cdot)}(X) = E\left[|X|^{p(\cdot)}\right] = \int_\Omega |X(w)|^{p(w)}dP(w).$$

If $p^+ < \infty$, then $X \in L^{p(\cdot)}(\Omega)$ if and only if $\rho_{p(\cdot)}(X) = E\left[|X|^{p(\cdot)}\right] < \infty$. The space $\left(L^{p(\cdot)}(\Omega),\ \|X\|_{p(\cdot)}\right)$ is a Banach space. If $p(\cdot) = p$ is a constant function, then the norm $\|X\|_{p(\cdot)}$ coincides with the usual Lebesgue norm $\|X\|_p$ [14-16]. In this paper we assume that $1 < p^- \le p(\cdot) \le p^+ < \infty$.

Let $(A,\ \|X\|_A)$ and $(B,\ \|X\|_B)$ be normed spaces. We write $A \hookrightarrow B$ if A is continuously embedded in B, i.e., if $A \subset B$ and the inclusion map I is continuous.

Let p and q be random variables on Ω. Then it is well known that $p(w) \le q(w)$ for a.e. $w \in \Omega$ if and only if $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ due to $P(\Omega) = 1$ (see Theorem 2.8 in [16]). So we can write $L^{p(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$.

A positive, measurable and locally integrable function $\vartheta: \Omega \to (0, \infty)$ is called a weight function. The weighted modular is defined by

$$\rho_{p(\cdot),\vartheta}(X) = E\left[|X|^{p(\cdot)}\vartheta\right] = \int_\Omega |X(w)|^{p(w)}\vartheta(w)dP(w).$$

The weighted variable exponent Lebesgue space $L_\vartheta^{p(\cdot)}(\Omega)$ consists of all random variables Y on Ω for which $\|Y\|_{p(\cdot),\vartheta} = \left\|Y\vartheta^{\frac{1}{p(\cdot)}}\right\|_{p(\cdot)} < \infty$ if and only if $E\left[\left|Y\vartheta^{\frac{1}{p(\cdot)}}\right|^{p(\cdot)}\right] < \infty$. It is well known that the property $\rho_{p(\cdot),\vartheta}\left(\frac{X}{\|X\|_{p(\cdot),\vartheta}}\right) \le 1$ is satisfied [9-13]. Moreover, if $0 < C \le \vartheta$, then we have $L_\vartheta^{p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$, since one easily sees that

$$C\int_\Omega |Y(w)|^{p(w)}dP(w) \le \int_\Omega |Y(w)|^{p(w)}\vartheta(w)dP(w)$$

and $C\|Y\|_{p(\cdot)} \le \|Y\|_{p(\cdot),\vartheta}$ [17-18].

Theorem 3. Let $\frac{1}{p(\cdot)}+\frac{1}{q(\cdot)}=1$. Then we have $XY\in L^1(\Omega)$ and the inequality

$$E[|XY|]\le C\|X\|_{p(\cdot),\vartheta}\|Y\|_{q(\cdot),\vartheta^*}$$

holds for every $X\in L_\vartheta^{p(\cdot)}(\Omega)$ and $Y\in L_{\vartheta^*}^{q(\cdot)}(\Omega)$, where $\vartheta^*=\vartheta^{1-q(\cdot)}=\vartheta^{\frac{1}{1-p(\cdot)}}$.

Proof 1: By the Hölder inequality for variable exponent Lebesgue spaces (see [16]), we get

$$\begin{aligned}E[|XY|]=\int_\Omega|X(w)Y(w)|dP(w)&=\int_\Omega|X(w)Y(w)|\vartheta(w)^{\frac{1}{p(w)}-\frac{1}{p(w)}}dP(w)\\&\le\left(1+\frac{1}{p^-}-\frac{1}{p^+}\right)\left\|X\vartheta^{\frac{1}{p(\cdot)}}\right\|_{p(\cdot)}\left\|Y\vartheta^{-\frac{1}{p(\cdot)}}\right\|_{q(\cdot)}\\&=C\|X\|_{p(\cdot),\vartheta}\|Y\|_{q(\cdot),\vartheta^*},\end{aligned}$$

where $C=1+\frac{1}{p^-}-\frac{1}{p^+}>0$. That is the desired result.

Proof 2: By the Young inequality, we have

$$\frac{|XY|}{\|X\|_{p(\cdot),\vartheta}\|Y\|_{q(\cdot),\vartheta^*}}=\frac{\vartheta^{\frac{1}{p(\cdot)}}|X|\vartheta^{-\frac{1}{p(\cdot)}}|Y|}{\|X\|_{p(\cdot),\vartheta}\|Y\|_{q(\cdot),\vartheta^*}}\le\frac{1}{p(\cdot)}\left(\frac{\vartheta^{\frac{1}{p(\cdot)}}|X|}{\|X\|_{p(\cdot),\vartheta}}\right)^{p(\cdot)}+\frac{1}{q(\cdot)}\left(\frac{\vartheta^{-\frac{1}{p(\cdot)}}|Y|}{\|Y\|_{q(\cdot),\vartheta^*}}\right)^{q(\cdot)}$$

If we take the integral of the above inequality, we obtain

$$\begin{aligned}E\left[\frac{|XY|}{\|X\|_{p(\cdot),\vartheta}\|Y\|_{q(\cdot),\vartheta^*}}\right]&\le E\left[\frac{1}{p(\cdot)}\left(\frac{\vartheta^{\frac{1}{p(\cdot)}}|X|}{\|X\|_{p(\cdot),\vartheta}}\right)^{p(\cdot)}+\frac{1}{q(\cdot)}\left(\frac{\vartheta^{-\frac{1}{p(\cdot)}}|Y|}{\|Y\|_{q(\cdot),\vartheta^*}}\right)^{q(\cdot)}\right]\\&=E\left[\frac{1}{p(\cdot)}\left(\frac{\vartheta^{\frac{1}{p(\cdot)}}|X|}{\|X\|_{p(\cdot),\vartheta}}\right)^{p(\cdot)}\right]+E\left[\frac{1}{q(\cdot)}\left(\frac{\vartheta^{-\frac{1}{p(\cdot)}}|Y|}{\|Y\|_{q(\cdot),\vartheta^*}}\right)^{q(\cdot)}\right]\\&\le\frac{1}{p^-}E\left[\left(\frac{\vartheta^{\frac{1}{p(\cdot)}}|X|}{\|X\|_{p(\cdot),\vartheta}}\right)^{p(\cdot)}\right]+\left(1-\frac{1}{p^+}\right)E\left[\left(\frac{\vartheta^{-\frac{1}{p(\cdot)}}|Y|}{\|Y\|_{q(\cdot),\vartheta^*}}\right)^{q(\cdot)}\right]\\&\le1+\frac{1}{p^-}-\frac{1}{p^+}\end{aligned}$$

for $X\vartheta^{\frac{1}{p(\cdot)}}\in L^{p(\cdot)}(\Omega)$ and $Y\vartheta^{-\frac{1}{p(\cdot)}}\in L^{q(\cdot)}(\Omega)$. This completes the proof.

Theorem 4. Let $\vartheta^{\frac{1}{1-p(\cdot)}}\in L^1(\Omega)$. Then $L_\vartheta^{p(\cdot)}(\Omega)$ is a Banach space with respect to $\|\cdot\|_{p(\cdot),\vartheta}$.

Proof: Let (X_n) be a Cauchy sequence in $L_\vartheta^{p(\cdot)}(\Omega)$. Then, by Theorem 3,

$$\begin{aligned}\int_\Omega|X_n(w)-X_m(w)|dP(w)&=\int_\Omega|X_n(w)-X_m(w)|\vartheta(w)^{\frac{1}{p(w)}-\frac{1}{p(w)}}dP(w)\\&\le C\left\|(X_n-X_m)\vartheta^{\frac{1}{p(\cdot)}}\right\|_{p(\cdot)}\left\|\vartheta^{-\frac{1}{p(\cdot)}}\right\|_{q(\cdot)}.\end{aligned}$$

It is clear that

$$\left\|(X_n - X_m)\vartheta^{\frac{1}{p(\cdot)}}\right\|_{p(\cdot)} = \|X_n - X_m\|_{p(\cdot),\vartheta}$$

and

$$\left\|\vartheta^{-\frac{1}{p(\cdot)}}\right\|_{q(\cdot)} = inf\left\{\lambda > 0: \int_\Omega \left(\frac{\left|\vartheta^{-\frac{1}{p(w)}}\right|}{\lambda}\right)^{\frac{P(w)}{p(w)-1}} dP(w) \le 1\right\}$$

$$= inf\left\{\lambda > 0: \int_\Omega \frac{\vartheta^{\frac{1}{1-p(w)}}}{\lambda^{\frac{P(w)}{p(w)-1}}} dP(w) \le 1\right\}$$

$$\le max\left\{\left(\int_\Omega \vartheta^{\frac{1}{1-p(w)}} dP(w)\right)^{\frac{p^-}{p^- -1}}, \left(\int_\Omega \vartheta^{\frac{1}{1-p(w)}} dP(w)\right)^{\frac{p^+}{p^+ -1}}\right\} < \infty.$$

Hence (X_n) is a Cauchy sequence in $L^1(\Omega)$. Due to the fact that $L^1(\Omega)$ is a Banach space, then (X_n) converges in $L^1(\Omega)$. Assume that $X_n \to X$ in $L^1(\Omega)$. So we have $X_n(w) \to X(w)$ a.e. in $L^1(\Omega)$ (subtracting a subsequence if necessary). For any $\varepsilon > 0$, there exists n_0 such that $\|X_n - X_m\|_{p(\cdot),\vartheta} < \varepsilon$ for $n, m > n_0$. By using Fatou Lemma, we have

$$E\left[\left(\frac{|X_n - X|}{\varepsilon}\right)^{p(\cdot)} \vartheta\right] \le \liminf_{m\to\infty} E\left[\left(\frac{|X_n - X|}{\varepsilon}\right)^{p(\cdot)} \vartheta\right] \le 1.$$

Thus, $\|X_n - X\|_{p(\cdot),\vartheta} < \varepsilon$ for $n > n_0$ and $X \in L_\vartheta^{p(\cdot)}(\Omega)$.

Remark 5. The space $L_\vartheta^{p(\cdot)}(\Omega)$ is a reflexive Banach space with respect to $\|\cdot\|_{p(\cdot),\vartheta}$. Moreover, the dual space of $L_\vartheta^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L_{\vartheta^*}^{q(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$ and $\vartheta^* = \vartheta^{1-q(\cdot)}$ (see [18-20]). The relations between the modular $\rho_{p(\cdot),\vartheta}(\cdot)$ and $\|\cdot\|_{p(\cdot),\vartheta}$ are given in the following theorem (see [19, 21-22]).

Theorem 6. Suppose that $p(\cdot)$ satisfies $1 < p^- \le p(\cdot) \le p^+$. If $X_k, X \in L_\vartheta^{p(\cdot)}(\Omega)$, then

(1) If $\|X\|_{p(\cdot),\vartheta} \ge 1$, then $\|X\|_{p(\cdot),\vartheta}^{p^-} \le \rho_{p(\cdot),\vartheta}(X) \le \|X\|_{p(\cdot),\vartheta}^{p^+}$,

(2) If $\|X\|_{p(\cdot),\vartheta} \le 1$, then $\|X\|_{p(\cdot),\vartheta}^{p^+} \le \rho_{p(\cdot),\vartheta}(X) \le \|X\|_{p(\cdot),\vartheta}^{p^-}$,

(3) $\lim_{k\to\infty}\|X_k\|_{p(\cdot),\vartheta} = 0$ if and only if $\lim_{k\to\infty} \rho_{p(\cdot),\vartheta}(X_k) = 0$; i.e., the norm topology is equivalent modular topology,

(4) $\lim_{k\to\infty}\|X_k\|_{p(\cdot),\vartheta} = \infty$ if and only if $\lim_{k\to\infty} \rho_{p(\cdot),\vartheta}(X_k) = \infty$.

Proof:

(1) By $\|X\|_{p(\cdot),\vartheta} \ge 1$ and the definition of the norm,

$$E\left[\frac{|X|^{p(\cdot)}\vartheta}{\|X\|_{p(\cdot),\vartheta}^{p^+}}\right] \le E\left[\left(\frac{|X|}{\|X\|_{p(\cdot),\vartheta}}\right)^{p(\cdot)} \vartheta\right] \le 1.$$

Hence, $\rho_{p(\cdot),\vartheta}(X) \le \|X\|_{p(\cdot),\vartheta}^{p^+}$. Since $\|X\|_{p(\cdot),\vartheta}^{\frac{p^-}{p(\cdot)}} \le \|X\|_{p(\cdot),\vartheta}$, then we obtain

$$E\left[\left(\frac{|X|}{\|X\|_{p(\cdot),\vartheta}^{\frac{p^-}{p(\cdot)}}}\right)^{p(\cdot)}\vartheta\right] \geq 1$$

and $\|X\|_{p(\cdot),\vartheta}^{p^-} \leq \rho_{p(\cdot),\vartheta}(X)$. This completes the proof. (2)-(4) can be proved by using similar methods.

By the Theorem 6, it is easy to see that a subset B in $L_\vartheta^{p(\cdot)}(\Omega)$ is bounded in the norm, i.e., $\sup_{u\in B}\|u\|_{p(\cdot),\vartheta} < \infty$ iff it is bounded in the modular, i.e., $\sup_{u\in B}E\left[|u|^{p(\cdot)}\vartheta\right] < \infty$.

Now we discuss the embedding properties of the spaces $L_\vartheta^{p(\cdot)}(\Omega)$ with respect to ϑ and $p(\cdot)$ under suitable assumptions. We say that $\vartheta_1 \prec \vartheta_2$ if and only if there exists a $C > 0$ such that $\vartheta_1(w) \leq C\vartheta_2(w)$ for all $w \in \Omega$. Two weight functions are called equivalent and written $\vartheta_1 \approx \vartheta_2$, if $\vartheta_1 \prec \vartheta_2$ and $\vartheta_2 \prec \vartheta_1$.

Proposition 7. Let ϑ_1 and ϑ_2 be weight functions on Ω. If $\vartheta_1 \prec \vartheta_2$, then the embedding $L_{\vartheta_2}^{p(\cdot)}(\Omega) \hookrightarrow L_{\vartheta_1}^{p(\cdot)}(\Omega)$ holds.

Proof: Since $\vartheta_1 \prec \vartheta_2$, then there exists a $C > 0$ such that $\vartheta_1(w) \leq C\vartheta_2(w)$ for all $w \in \Omega$. Hence we have

$$E\left[|X|^{p(\cdot)}\vartheta_1\right] \leq CE\left[|X|^{p(\cdot)}\vartheta_2\right]$$

and $L_{\vartheta_2}^{p(\cdot)}(\Omega) \hookrightarrow L_{\vartheta_1}^{p(\cdot)}(\Omega)$ for $X \in L_{\vartheta_2}^{p(\cdot)}(\Omega)$.

Corollary 8. If $\vartheta_1 \approx \vartheta_2$, then $L_{\vartheta_2}^{p(\cdot)}(\Omega) = L_{\vartheta_1}^{p(\cdot)}(\Omega)$.

Theorem 9. Let $p_1(\cdot)$, $p_2(\cdot)$ be variable exponents satisfying $1 < p_2^- \leq p_2(\cdot) \leq p_1(\cdot) \leq p_1^+ < \infty$ and $\left\|\frac{\vartheta_2}{\vartheta_1}\right\|_{\frac{p_1(\cdot)}{p_1(\cdot)-p_2(\cdot)},\vartheta_1} < \infty$. Then the embedding $L_{\vartheta_1}^{p_1(\cdot)}(\Omega) \hookrightarrow L_{\vartheta_2}^{p_2(\cdot)}(\Omega)$ holds.

Proof: Suppose that $f \in L_{\vartheta_1}^{p_1(\cdot)}(\Omega)$. It is known that $L_{\vartheta_1}^{p_1(\cdot)}(\Omega) \hookrightarrow L_{\vartheta_2}^{p_2(\cdot)}(\Omega)$ with $\left\|\frac{\vartheta_2}{\vartheta_1}\right\|_{\frac{p_1(\cdot)}{p_1(\cdot)-p_2(\cdot)},\vartheta_1} < \infty$ (Theorem 5.1 in [23]).

Theorem 10. Let $p(\cdot)$, $q(\cdot)$ be random variables on Ω. If the inclusion $L_{\vartheta_1}^{p(\cdot)}(\Omega) \subset L_{\vartheta_2}^{q(\cdot)}(\Omega)$ holds for the weights ϑ_1 and ϑ_2, if and only if the embedding $L_{\vartheta_1}^{p(\cdot)}(\Omega) \hookrightarrow L_{\vartheta_2}^{q(\cdot)}(\Omega)$ is satisfied.

Proof: The sufficient condition of the theorem is clear by the definition of continuous embedding. Now, assume that the inclusion $L_{\vartheta_1}^{p(\cdot)}(\Omega) \subset L_{\vartheta_2}^{q(\cdot)}(\Omega)$ is valid. Moreover, we define the sum norm $\|\cdot\| = \|\cdot\|_{p(\cdot),\vartheta_1} + \|\cdot\|_{q(\cdot),\vartheta_2}$. It is easy to see that $\left(L_{\vartheta_1}^{p(\cdot)}(\Omega), \|\cdot\|\right)$ is a Banach space. If we define the unit function I from $\left(L_{\vartheta_1}^{p(\cdot)}(\Omega), \|\cdot\|\right)$ into $\left(L_{\vartheta_1}^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot),\vartheta_1}\right)$, then the function I is continuous. Because we can obtain the inequality $\|I(X)\|_{p(\cdot),\vartheta_1} = \|X\|_{p(\cdot),\vartheta_1} \leq \|X\|$ for any $X \in L_{\vartheta_1}^{p(\cdot)}(\Omega)$. By Banach's theorem I is a homeomorphism, see [24]. So the norms $\|\cdot\|$ and $\|\cdot\|_{p(\cdot),\vartheta_1}$ are equivalent. Thus, for every $X \in L_{\vartheta_1}^{p(\cdot)}(\Omega)$ there exists a $k > 0$ such that

$$\|X\| \leq k\|X\|_{p(\cdot),\vartheta_1}.$$

By the definition of the norm $\|\cdot\|$ we have

$$\|X\|_{q(\cdot),\vartheta_2} \leq \|X\| \leq k\|X\|_{p(\cdot),\vartheta_1}.$$

Hence the continuous embedding $L^{p(\cdot)}_{\vartheta_1}(\Omega) \hookrightarrow L^{q(\cdot)}_{\vartheta_2}(\Omega)$ is obtained.

Theorem 11. Assume that $\vartheta \in L^1(\Omega)$. Let $q(w) \leq p(w)$ for a.e. $w \in \Omega$. Then we have $L^{p(\cdot)}_{\vartheta}(\Omega) \hookrightarrow L^{q(\cdot)}_{\vartheta}(\Omega)$.

Proof 1: Let $r(\cdot) = \frac{P(\cdot)}{q(\cdot)}$ and $s(\cdot) = \frac{r(\cdot)}{r(\cdot)-1}$. Then $\frac{1}{r(\cdot)} + \frac{1}{s(\cdot)} = 1$. By the Hölder inequality we have

$$\begin{aligned} E\left[|X|^{q(\cdot)}\vartheta\right] &= \int_\Omega |X(w)|^{q(w)}\vartheta(w)dP(w) \\ &= \int_\Omega |X(w)|^{q(w)}\vartheta(w)^{\frac{1}{r(\cdot)}+\frac{1}{s(\cdot)}}dP(w) \\ &\leq C\left\| |X(\cdot)|^{q(\cdot)}\vartheta^{\frac{1}{r(\cdot)}}\right\|_{r(\cdot)} \left\|\vartheta^{\frac{1}{s(\cdot)}}\right\|_{s(\cdot)} < \infty \end{aligned}$$

for $X \in L^{p(\cdot)}_{\vartheta}(\Omega)$. Because it is well known that $\left\|\vartheta^{\frac{1}{s(\cdot)}}\right\|_{s(\cdot)} < \infty$ if and only if $E\left[\left(\vartheta^{\frac{1}{s(\cdot)}}\right)^{s(\cdot)}\right] < \infty$. Hence, we write

$$\begin{aligned} E\left[\left(\vartheta^{\frac{1}{s(\cdot)}}\right)^{s(\cdot)}\right] &= \int_\Omega \left(\vartheta(w)^{\frac{1}{s(w)}}\right)^{s(w)} dP(w) \\ &= \int_\Omega \vartheta(w)dP(w) = \|\vartheta\|_1 < \infty. \end{aligned}$$

Proof 2: If $X \in L^{p(\cdot)}_{\vartheta}(\Omega)$, then there exists a $\gamma > 0$ such that

$$E\left[\frac{|X|^{p(\cdot)}\vartheta}{\gamma}\right] \leq 1.$$

Moreover, we get

$$\begin{aligned} E\left[\frac{|X|^{q(\cdot)}\vartheta}{\gamma}\right] &= \int_{\{w\in\Omega:|X(w)|\leq\gamma\}} \left|\frac{X(w)}{\gamma}\right|^{q(w)} \vartheta(w)dP(w) + \int_{\{w\in\Omega:|X(w)|>\gamma\}} \left|\frac{X(w)}{\gamma}\right|^{q(w)} \vartheta(w)dP(w) \\ &\leq \int_\Omega \vartheta(w)dP(w) + \int_{\{w\in\Omega:|X(w)|>\gamma\}} \left|\frac{X(w)}{\gamma}\right|^{p(w)} \vartheta(w)dP(w) \\ &\leq \|\vartheta\|_1 + E\left[\frac{|X|^{p(\cdot)}\vartheta}{\gamma}\right] \leq \|\vartheta\|_1 + 1 < \infty \end{aligned}$$

which implies that

$$E\left[\frac{|X|^{q(\cdot)}\vartheta}{\gamma}\right] < \infty \text{ and } X \in L^{q(\cdot)}_{\vartheta}(\Omega).$$

Therefore, by Banach's Theorem we have $L^{p(\cdot)}_{\vartheta}(\Omega) \subset L^{q(\cdot)}_{\vartheta}(\Omega)$ and $L^{p(\cdot)}_{\vartheta}(\Omega) \hookrightarrow L^{q(\cdot)}_{\vartheta}(\Omega)$.

Theorem 12. Let $\vartheta^{\frac{1}{1-p(\cdot)}} \in L^1(\Omega)$. Then we have $L^{p(\cdot)}_{\vartheta}(\Omega) \hookrightarrow L^1(\Omega)$.

Proof: Using Hölder's inequality in Theorem 3, we have

$$\begin{aligned}\|X\|_1 = E[|X|] &= \int_\Omega |X| dP(w) = \int_\Omega |X| \vartheta(w)^{\frac{1}{p(w)}} \vartheta(w)^{-\frac{1}{p(w)}} dP(w) \\ &\leq \left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right) \left\| X \vartheta^{\frac{1}{p(\cdot)}} \right\|_{p(\cdot)} \left\| \vartheta^{-\frac{1}{p(\cdot)}} \right\|_{q(\cdot)} \\ &\leq \left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right) \|X\|_{p(\cdot),\vartheta} \left(\int_\Omega \vartheta^{\frac{1}{1-p(w)}} dP(w) + 1\right)^{\frac{1}{q^-}} < \infty\end{aligned}$$

for $X \in L_\vartheta^{p(\cdot)}(\Omega)$.

Theorem 13. Let $\vartheta(w) > C$ for a.e. $w \in \Omega$ and some $C > 0$. If $X_n, X \in L_\vartheta^{p(\cdot)}(\Omega)$ and $X_n \to X$ in $L_\vartheta^{p(\cdot)}(\Omega)$, then $X_n \overset{P}{\to} X$ in probability (measure).

Proof: Suppose that there exists $\beta > 0$ and $\theta \in (0,1)$ such that

$$P\{w \in \Omega : |X_n(w) - X(w)| \geq \theta\} > \beta$$

for any $n \in \mathbb{N}$ and $X_n \to X$ in $L_\vartheta^{p(\cdot)}(\Omega)$. There exists n_0 such that $\|X_n - X\|_{p(\cdot),\vartheta} \leq 1$ for any $n \geq n_0$. Hence we have

$$\frac{1}{\|X_n - X\|_{p(\cdot),\vartheta}} E\left[|X_n - X|^{p(\cdot)} \vartheta\right] \leq \int_\Omega \left| \frac{X_n - X}{\|X_n - X\|_{p(\cdot),\vartheta}} \right|^{p(w)} \vartheta(w) dP(w) \leq 1$$

and

$$\begin{aligned}\|X_n - X\|_{p(\cdot),\vartheta} &\geq E\left[|X_n - X|^{p(\cdot)} \vartheta\right] \\ &\geq \int_{\{w \in \Omega : |X_n(w) - X(w)| \geq \theta\}} |X_n(w) - X(w)|^{p(\cdot)} \vartheta(w) dP(w) \\ &\geq \beta C \theta^{p^+}.\end{aligned}$$

We observe that $\{X_n\}$ can not converge to X in $L_\vartheta^{p(\cdot)}(\Omega)$.

Theorem 14. Suppose that $X_n, X \in L_\vartheta^{p(\cdot)}(\Omega)$ and $X_n \to X$ in $L_\vartheta^{p(\cdot)}(\Omega)$.
(1) Let $\vartheta \in L^1(\Omega)$. If $q(w) \leq p(w)$ for a.e. $w \in \Omega$, then $X_n \to X$ in $L_\vartheta^{q(\cdot)}(\Omega)$ for $n \to \infty$.
(2) If $X_n \to X$ and $Y_n \to Y$ in $L_\vartheta^{p(\cdot)}(\Omega)$, then $X_n + Y_n \to X + Y$ in $L_\vartheta^{p(\cdot)}(\Omega)$.
(3) Let $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$ and $\vartheta^* = \vartheta^{1-q(\cdot)}$. If $X_n \to X$ in $L_\vartheta^{p(\cdot)}(\Omega)$ and $Y_n \to Y$ in $L_{\vartheta^*}^{q(\cdot)}(\Omega)$, then $X_n Y_n \to XY$ in $L^1(\Omega)$ for $n \to \infty$.
(4) Let $\vartheta \in L^1(\Omega)$. If $X_n \to X$ uniformly in Ω, then $X_n \to X$ in $L_\vartheta^{p(\cdot)}(\Omega)$.

Proof:

(1) By Theorem 11, there exists a $C > 0$ such that

$$E\left[|X_n - X|^{q(\cdot)} \vartheta\right] \leq C E\left[|X_n - X|^{p(\cdot)} \vartheta\right] \to 0.$$

This completes the proof.

(2) Using the following inequality

$$(a + b)^{p(\cdot)} \leq 2^{p^+ - 1}\left(a^{p(\cdot)} + b^{p(\cdot)}\right)$$

for $a, b \geq 0$, we have

$$E\left[|(X_n+Y_n)-(X+Y)|^{p(\cdot)}\vartheta\right] \leq 2^{p^+-1}\left(E\left[|X_n-X|^{p(\cdot)}\vartheta\right]+E\left[|Y_n-Y|^{p(\cdot)}\vartheta\right]\right) \to 0$$

as $n \to \infty$.

(3) By Theorem 3, we obtain

$$\begin{aligned} E[|X_nY_n-XY|] &= E[|(X_nY_n-X_nY)+(X_nY-XY)|] \\ &\leq E[|X_n||Y_n-Y|]+E[|X_n-X||Y|] \\ &\leq C_1\|X_n\|_{p(\cdot),\vartheta}\|Y_n-Y\|_{q(\cdot),\vartheta^*}+C_2\|X_n-X\|_{p(\cdot),\vartheta}\|Y\|_{q(\cdot),\vartheta^*} \to 0 \end{aligned}$$

as $n \to \infty$ for some $C_1, C_2 > 0$.

(4) For each $1 \geq \varepsilon > 0$,

$$\begin{aligned} E\left[|X_n-X|^{p(\cdot)}\vartheta\right] &\leq \int_\Omega \varepsilon^{p(\cdot)}\vartheta(w)dP(w) \\ &\leq \varepsilon^{p^-}\|\vartheta\|_1 \to 0. \end{aligned}$$

Let (Ω, F, P) be a complete probability space and $(F_n)_{n\geq 0}$ be a non-decreasing sequence of sub-σ-algebras of F with $F=\sigma(U_{n\geq 0}F_n)$. The conditional expectation operator related to F_n is denoted by E_n; that is, $E(f|F_n)=E_n(f)$. A sequence of measurable functions $f=(f_n)_{n\geq 0}\subset L^1(\Omega,F,P)$ is called a martingale with respect to $(F_n)_{n\geq 0}$ if $E_n(f_{n+1})=f_n$ for every $n\geq 0$. Let M be the set of all martingale $f=(f_n)_{n\geq 0}$ relative to $(F_n)_{n\geq 0}$ such that $f_0=0$. If in addition $f_n\in L_\vartheta^{p(\cdot)}(\Omega)$, f is called an $L_\vartheta^{p(\cdot)}(\Omega)$-martingale with respect to $(F_n)_{n\geq 0}$. In this case we set

$$\|f\|_{p(\cdot),\vartheta} = \sup_{n\geq 0}\|f_n\|_{p(\cdot),\vartheta}.$$

If $\|f\|_{p(\cdot),\vartheta}<\infty$, f is called a bounded $L_\vartheta^{p(\cdot)}(\Omega)$-martingale and denoted by $f\in L_\vartheta^{p(\cdot)}(\Omega)$. As usual, we denote f's pointwise limit and Doob's maximal function as follows:

$$f_\infty = \lim_{n\to\infty} f_n \text{ a.e.}, \quad Mf = f^* = \sup_{n\geq 0}|f_n|.$$

A non-empty family B of random variables is said to be uniformly integrable (UI) if

$$\lim_{K\to\infty}\left(\sup_{X\in B}E\left[|X|_{\chi_{\{|X|\geq K\}}}\right]\right) = \lim_{K\to\infty}\left(\sup_{X\in B}\int_{\{|X|\geq K\}}|X(w)|dP(w)\right)=0,$$

and that B is uniformly integrable iff B is $L^1(\Omega)$ bounded and their integrals have equi-absolute continuity [5, 9, 25].

Lemma 15. Let $X=(X_n)_{n\geq 0}$ be a martingale and $\vartheta^{\frac{1}{1-p(\cdot)}}\in L^1(\Omega)$. Then;

(1) If $\sup_{n\geq 0}\|X_n\|_{p(\cdot),\vartheta}<\infty$, then (X_n) converges a.e. to a function $X_\infty\in L_\vartheta^{p(\cdot)}(\Omega)$.

(2) Let $k>0$ and $\vartheta(w)>k$ for a.e. $w\in\Omega$. If $\sup_{n\geq 0}\|X_n\|_{p(\cdot),\vartheta}<\infty$, then (X_n) is uniformly integrable and

$$X_n = E[X_\infty|F_n], \quad n\geq 0.$$

(3) If $X^*\in L_\vartheta^{p(\cdot)}(\Omega)$, then $\|X_n-X^*\|_{p(\cdot),\vartheta}\to 0$ as $n\to\infty$.

Proof:

(1) By Theorem 12, we obtain that

$$\sup_{n\geq 0}\|X_n\|_1 \leq C\sup_{n\geq 0}\|X_n\|_{p(\cdot),\vartheta} < \infty$$

for some $C > 0$. From Doob's convergence theorem for martingales, $X_n \to X_\infty$ a.e., and $|X_n|^{p(\cdot)} \to |X_\infty|^{p(\cdot)}$. Now, we obtain

$$\begin{aligned} E\left[|X_\infty|^{p(\cdot)}\vartheta\right] &= E\left[\liminf_n |X_n|^{p(\cdot)}\vartheta\right] \\ &\leq \liminf_n E\left[|X_n|^{p(\cdot)}\vartheta\right] \\ &\leq \sup_{n\geq 0} E\left[|X_n|^{p(\cdot)}\vartheta\right] < \infty \end{aligned}$$

by Fatou's Lemma.

(2) Since $p^- > 1$, then

$$\sup_{n\geq 0}\|X_n\|_{p^-} \leq k\sup_{n\geq 0}\|X_n\|_{p(\cdot)} \leq \sup_{n\geq 0}\|X_n\|_{p(\cdot),\vartheta} < \infty$$

Thus (X_n) is uniformly integrable. The conclusion is a classical result.

(3) From $\sup_{n\geq 0}\|X_n\|_{p(\cdot),\vartheta} \leq \|X^*\|_{p(\cdot),\vartheta} < \infty$ and (1) above, we have that $|X_n|^{p(\cdot)} \to |X_\infty|^{p(\cdot)}$ a.e. Since $|X_n|^{p(\cdot)}\vartheta(\cdot) \leq |X_\infty|^{p(\cdot)}\vartheta(\cdot) \in L^1(\Omega)$, by Lebesgue's dominated convergence theorem, we have $\rho_{p(\cdot),\vartheta}(X_n) \to \rho_{p(\cdot),\vartheta}(X_\infty)$. Using Theorem 1.3 in [26], we obtain that $\|X_n - X_\infty\|_{p(\cdot),\vartheta} \to 0$ as $n \to \infty$.

The proof of following the theorem is similar to Theorem 4.2 in [9].

Theorem 16. Let $X = (X_n)_{n\geq 0}$ be a martingale and $\vartheta^{\frac{1}{1-p(\cdot)}} \in L^1(\Omega)$. Then the following statements are equivalent:

(1) $\left\{|X_n|^{p(\cdot)}\vartheta(\cdot), n \geq 0\right\}$ is uniformly integrable;

(2) $\rho_{p(\cdot),\vartheta}(X_n) \to \rho_{p(\cdot),\vartheta}(X_\infty)$;

(3) $\|X_n\|_{p(\cdot),\vartheta} \to \|X_\infty\|_{p(\cdot),\vartheta}$;

(4) $\|X_n - X_\infty\|_{p(\cdot),\vartheta} \to 0$ as $n \to \infty$.

4. CONCLUSION

In this study, we mention several basic properties of the weighted variable exponent Lebesgue spaces defined on a probability. We give also some applications of the Hölder inequality for expectation operators, martingales, convergences and embeddings. We discuss the Hölder inequality and embedding properties in these spaces. Finally, we talk about Doob's maximal function in $L_\vartheta^{p(\cdot)}(\Omega)$.

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