ORIGINAL PAPER

FUGLEDE'S THEOREM FOR (*n*, *m*)-POWER-NORMAL OPERATORS

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Abstract. *In this paper we introduce the FugledePutnum theorem for (n,m) power normal operators on a complex Hilbert space H. We give some basic properties.All operators are assumed to be bounded.*
Keywords: *Hilbert space; (n,m)-power-normal operators; n-normal operators.*

1. INTRODUCTION

A bounded linear operator T on a complex Hilbert space is said to be normal if $TT^* = T^*T$ is n-normal if $T^nT^* = T^*T^n$. We said that T is unitary if $TT^* = T^*T = I$, isometry if $T^*T = I$, co-isometry if $TT^* = I$, hyponormal if $TT^* \leq T^*T$, n-hyponormal if $T^nT^* \leq T^*T^n$. An operator T is said to be (n, m) power hyponormal if $T^n(T^m)^* \leq (T^m)^*T^n$.

An operator T is said to be (A; n)-power-hyponormal operator for a positive integer n, if

$$T^n(T^m)^* \leq_A (T^m)^*T^n.$$

2. BASIC DEFINITIONS

Definition 2.1. [1] A bounded linear operator T on a complex Hilbert space is said to be

- normal if $TT^* = T^*T$
- n-normal if $T^nT^* = T^*T^n$
- (n; m)-power-normal if $(T^m)^*T^n = T^n(T^m)^*$

Theorem2.2.[2] If T is an (n, m) power hyponormal operator then:

(1) T is (m, n) power normal operator;
(2) T^k is (n, m) power normal operator for $k \in N$;
(3) αT is (n, m) power normal operator for $\alpha \in R$;
(4) If S, T are unitarily equivalent and if T is (n, m) power-normal operators then so is S.

Proof:

- We have$T^n(T^m)^* = (T^m)^*T^n$,

  In the same time we have:$T^m(T^n)^* = (T^n(T^m)^*)^*$.
  So$T^m(T^n)^* = ((T^m)^*T^n)^* = (T^n)^*T^m$
  Hence T is (m, n) power –normal

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$$(T^k)^n((T^k)^m)^* = \underbrace{(T^nT^nT^n........)}_{k times}\underbrace{((T^mT^mT^m....))^*}_{k times}$$

$$= \underbrace{(T^nT^nT^n \dots\dots..)}_{(k-1)\, times} T^n(T^m)^* \underbrace{((T^mT^mT^m....))^*}_{(k-1)\, times}$$

$$= \underbrace{(T^nT^nT^n \dots\dots..)}_{(k-1)\, times} (T^m)^* T^n \underbrace{((T^mT^mT^m....))^*}_{(k-1)\, times}$$

$$= ((T^m)^*T^n)^k = ((T^k)^m)^*(T^k)^n$$

Hence αT is (n,m) power-normal operator for $k \in N$.

- We have

$$(\alpha T)^n((\alpha T)^m)^* = \alpha^n T^n(\alpha^m(T^m))^*$$

$$= \alpha^n T^n(\alpha^m)^*(T^m)^* = \alpha^n(\alpha^m)^*(T^m)^*T^n$$

$$= ((\alpha T)^m)^\#(\alpha T)^n$$

Hence αT is (A,n) power hyponormal

- Let Tbe an (n,m) power-normal operator, since$S$isunitaryequivalent of Tthenthereexists a unitaryoperator$U$suchthat $S = UTU^*$, itiseasly to chekthat$S^* = UTU^*$

We have

$$S^n(S^*)^m = (UT^nU^*)(U(T^*)^mU^*) = UT^nU^*U(T^*)^mU^* = UT^n(T^*)^mU^*$$

$$UT^n(T^*)^mU^* = UT^{*m}T^nU^* = (UT^{*m}U^*)(UT^nU^*) = (S^*)^mS^n$$

Hence, $S^n(S^*)^m = (S^*)^mS^n$ then S is (n,m) power-normal operator. Hence, S is (n,m) −power normal.

2. THE SUM AND THE PRODUCT OF TWO (*n*, *m*) POWER NORMAL OPERATORS

The following discusses the conditions for product and sum of two (n,m) power normal operators to be (n,m) power normal.

Proposition 2.1. If T, S are commuting (n,m) powernormal operators such that $ST^* = T^*S$ then TS is (n,m) power normal operator.

Proof: Since $ST = TS$and$T^* = T^*S$, then$(T^*)^mS^n = S^n(T^*)^m$

$$(ST)^n((ST)^*)^m = (S^nT^n)(S^mT^m)^* = S^nT^n(T^*)^m(S^*)^m$$
$$= S^n(T^*)^mT^n(S^*)^m$$

$$\boldsymbol{S^n(T^*)^mT^n(S^*)^m = (T^*)^m S^nT^n(S^*)^m}$$

$$= (T^*)^m S^n(S^*)^mT^n = (T^*)^m(S^*)^mS^nT^n$$

$$= ((ST)^*)^m(ST)^n$$

Proposition 2.2. Let T, S are (n, m) power normal operators for some positive integers n and m such that $TS^* = S^*T$, $ST^* = T^*S$ and $TS = ST = 0$. Then $(S+T)$ are (n, m) power normal operators for some positive integers n and m.

Proof:

$$(S+T)^{*m}(S+T)^n = (S^{*m} + T^{*m})(S^n + T^n)$$
$$= S^{*m}S^n + S^{*m}T^n + T^{*m}S^n + T^{*m}T^n$$
$$= S^{*m}S^n + T^nS^{*m} + S^nT^{*m} + T^{*m}T^n$$
$$= (S^n + T^n)(S^{*m} + T^{*m}) = (S+T)^n(S+T)^{*m}$$

3. DIRECT SUM AND TENSOR PRODUCT

In the following theorem we will prove the stability of the class of (n, m) power-normal operators under the direct sum and tensor product.

Theorem 3.1. Let $T_1, T_2, ..., T_k$ be (n, m) power normal operators, then

- $(T_1 \oplus T_2 \oplus .... \oplus T_k)$ is (n, m) power -normal operator
- $(T_1 \otimes T_2 \otimes .... \otimes T_k)$ is (n, m) power-normal operator.

Proof:

- *The direct sum*

$$(T_1 \oplus T_2 \oplus ... \oplus T_k)^n (T_1 \oplus T_2 \oplus ... \oplus T_k)^{*m}$$
$$= (T_1^n \oplus T_2^n \ldots \oplus T_k^n)(T_1^{*m} \oplus T_2^{*m} \ldots \oplus \ T_k^{*m})$$
$$= T_1^n T_1^{*m} \oplus T_2^n T_2^{*m} \oplus \ldots \oplus T_k^n T_k^{*m}$$
$$= T_1^{*m} T_1^n \oplus T_2^{*m} T_2^n \oplus \ldots \oplus T_k^{*m} T_k^n$$
$$= (T_1^{*m} \oplus T_2^{*m} \oplus \ldots \oplus T_k^{*m})(T_1^n \oplus T_2^n \oplus \ldots \oplus T_k^n)$$
$$= (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^{*m}(T_1 \oplus T_2 \oplus \ldots \oplus T_k)^n.$$

Then $(T_1 \oplus T_2 \oplus ... \oplus T_m)$ is (n, m) power -normal operator.

- *The tensorproduct*

$$(T_1 \otimes T_2 \otimes \ldots \otimes T_k)^n (T_1 \otimes T_2 \otimes \ldots \otimes T_k)^{*m}(x_1 \otimes x_2 \otimes ... \otimes x_k)$$
$$= (T_1^n \otimes T_2^n \otimes \ldots \otimes T_k^n)(T_1^{*m} \otimes T_2^{*m} \otimes \ldots \otimes T_k^{*m})(x_1 \otimes x_2 \otimes ... \otimes x_k)$$
$$= (T_1^n T_1^{*m} x_1 \otimes T_2^n T_2^{*m} x_2 \otimes \ldots \otimes T_k^n T_k^{*m} x_k)$$
$$\leq (T_1^{*m} T_1^n x_1 \otimes T_2^{*m} T_2^n x_2 \otimes \ldots \otimes T_k^{*m} T_k^n x_k)$$
$$= (T_1^{*m} \otimes T_2^{*m} \otimes \ldots \otimes T_k^{*m})\big((T_1^n \otimes T_2^n \otimes \ldots \otimes T_k^n)(x_1 \otimes x_2 \otimes ... \otimes x_k\big)$$

$=(T_1 \otimes T_2 \otimes \ldots \otimes T_k)^{*m}(T_1 \otimes T_2 \otimes \ldots \otimes T_k)^n(x_1 \otimes x_2 \otimes \ldots \otimes x_k)$

4. MAIN RESULT

Theorem 4.1.[3] (FugledeTheorem)Let T, S two bounded linear operators on a complex Hilbert space, assume that S is normal. If $S\mathrm{T} = \mathrm{T}S$, then $TS^* = S^*T$. The theorem claims that commutativity between operators is transitive under the given assumptions. The claim does not hold in general if *S*is not normal.

Proposition 4.2. If T is (n, m) power normal operators,and S is a normal operator such that $S\mathrm{T} = \mathrm{T}S$ then $TS^* = S^*T$

Proof: If T commutes with S, it has to commute with $P_i$where P_i are pairwise orthogonal projections(e.g., using thethe spectral theorem$S = \sum \alpha_i P_i$). We used the same proof of Fuglede theorem where one of the two operators is normal [3].

5. CONCLUSION

It may be concluded that, ifan operator (*n*,*m*)power-normal is isometricallyequivalent to an operator *S*, then *S* is (*n*,*m*) power normal operator. We gave also the conditions for product and sum of two (*n*,*m*) power normal operators to be (*n*,*m*) power normal. Finaly, we proved the stability of the Fuglede theorem if two commuting operators where one is (*n*,*m*) power-normal operator and the second is normal .

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