*ORIGINAL PAPER*

# LIFTS ON THE SUPERSTRUCTURE $\mathbf{F(\pm a^2, \pm b^2)}$ OBEYING $\mathbf{(F^2 + a^2)(F^2 - a^2)(F^2 + b^2)(F^2 - b^2) = 0}$

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***Abstract.*** *The purpose of the present paper is to analyze the concept of the horizontal and complete lifts on the superstructure $F(\pm a^2, \pm b^2)$, which is defined as $(F^2 + a^2)(F^2 - a^2)(F^2 + b^2)(F^2 - b^2) = 0$, over the tangent bundles and establish its integrability conditions using the horizontal and complete lifts. Finally, some properties of the third-order tangent bundle are investigated.*

***Keywords:*** *Complete lift; horizontal lift; projection tensor; tangent bundle.*

## 1. INTRODUCTION

Over a differentiable manifold, lift functions are the prominent feature of any structure and by using the lift functions we can easily generalize the differentiable structures on any differentiable manifold M to its tangent bundle T(M). Yano and Ishihara [1] studied the properties of complete, horizontal, and vertical lifts on a differentiable manifold subjected to its tangent bundle T(M). Das et. al. [2] have studied the properties of almost contact structures using all types of lifts. Khan [3] defined and studied the tangent bundle's properties on the non-metric quarter-symmetric connection on the specific manifold. Nivas et. al. [4] studied and defined superstructure $F(\pm a^2, \pm b^2)$ and studied several properties of it. Bogoyavlenskij [5] studied some algebraic identities for the Nijenhuis tensors. The specific purpose of this paper is to investigate superstructure $F(\pm a^2, \pm b^2)$ on the tangent bundle and establish integrability conditions for it.

The work of several scholars on various geometric structures and connections has been extremely beneficial, for instance, Saxena [6, 7], Nivas et. Al [8, 9], Mishra [10-11] Dida et. al. [12-14], Khan et.al. [15-21], Mesland et. al [22-23] Georgiou et.al. [24-25], Brzezinski et. Al. [26], Miebach [27] and Peyghan et. al. [28]. Let M (dim n) be a differentiable manifold. A tensor field F ($\neq 0$) of type (1, 1) is called the superstructure, $F(\pm a^2, \pm b^2)$, if,

$$(F^2 + a^2)(F^2 - a^2)(F^2 + b^2)(F^2 - b^2) = 0, \tag{1.1}$$

therefore the manifold M over which (1.1) is defined is said to be super manifold.

Let m and *l* be operators defined as,

$$m = \frac{(F^2+b^2)(F^2-b^2)}{(a^4-b^4)} \tag{1.2}$$

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$$l = \frac{(F^2 + a^2)(F^2 - a^2)}{(a^4 - b^4)} \tag{1.3}$$

using (1.2) and (1.3), the following relations hold:

$$\begin{gathered} m + l = 0 \\ m^2 = m, \quad l^2 = l, \quad ml = lm = 0 \\ Fm = mF = 0, \quad Fl = lF = F. \end{gathered} \tag{1.4}$$

Now we define complementary distributions over M as $D_m$ and $D_l$ corresponding to the projection operator m and *l* respectively. If the rank of F is r, then the rank of $D_m$ is also r and rank of $D_l$ is $(n - r)$, so that the dimension of M is equal to n.

## 2. THE COMPLETE LIFT OF THE SUPERSTRUCTURE $F(\pm a^2, \pm b^2)$

Let $T_p(M)$ be a tangent space of M, defined over a point p on M, then the set $\cup_{p \in M} T_p(M)$ is said to be the tangent bundle over M with dimension 2n. Let us define $T_s^r$ and $T_s^r(M)$ be the set of the tensor fields of type (r, s) over M and T(M) respectively. Let $G, H \in T_1^1(M)$, then we have, Tekkoyun [29]

$$(GK)^2 = G^2K^2. \tag{2.1}$$

Putting $G = H$ in in the above equation, we get

$$(G^2)^C = (G^C)^2, \tag{2.2}$$

and

$$(G + K)^C = (G)^C + (K)^C. \tag{2.3}$$

Now in equation (1.1) apply the property of complete lift, we have

$$[(F^2 + a^2)(F^2 - a^2)(F^2 + b^2)(F^2 - b^2)]^C$$
$$= [(F^2)^C + a^2][(F^2)^C - a^2][(F^2)^C + b^2][(F^2)^C - b^2] = 0.$$

Given equation (2.2), we get

$$[(F^C)^2 + a^2][(F^C)^2 - a^2][(F^C)^2 + b^2][(F^C)^2 - b^2] = 0. \tag{2.4}$$

In the view of equations (1.1) and (2.4), it is evident that the rank of $F^C$ is of dimension 2r subjected to the condition that rank (F) = r. Hence, we have the below result in form of theorems:

**Theorem 2.1.** If F is defined ais also formingupesuperstructurea$^2, \pm b^2$), on M and $F \in T_1^1(M)$, then the complete lift of F defined as $F^C$ is also forms a superstructure $F(\pm a^2, \pm b^2)$ in the tangent bundle T(M).

**Theorem 2.2.** The superstructure $F(\pm a^2, \pm b^2)$ is of rank r in M iff the complete lift of superstructure $F^C(\pm a^2, \pm b^2)$ is of rank 2r in T(M).

Let F defined as a superstructure, $F(\pm a^2, \pm b^2)$, which is of the rank r. Then the complete lift defined over the projection operators are $m^C$ of m and $l^C$ of $l$, where m and

$l$ are projection operator tensors in $T(M)$. The complete lift of projection operators $m^C$ and $l^C$ define distribution which are complementary and noted as $D_m^C$ and $D_l^C$ respectively in the tangent manifold. $D_m^C$ and $D_l^C$ are the complete lifts of $D_m$ and $D_l$ respectively [2].

## 3. THE INTEGRABILITY CONDITIONS FOR THE SUPERSTRUCTURE $\mathbf{F(\pm a^2, \pm b^2)}$ IN T(M)

Let F be the superstructure, $F(\pm a^2, \pm b^2)$, defined over the differential manifold M. Let N be the Nijenhuis tensor of type (1, 2) of F and given by [10].

$$N(\xi, \psi) = F^2(\xi, \psi) + (F\xi, F\psi) - F(\xi, F\psi) - F(F\xi, \psi), \tag{3.1}$$

where, $\xi, \psi \in T_0^1(M)$.

Let $N^C$ be the complete lift of Nijenhuis tensor N in $T(M)$, then we have

$$N^C(\xi^C, \psi^C) = (F^2)^C(\xi^C, \psi^C) + (F^C\xi^C, F^C\psi^C) - F^C(\xi^C, F^C\psi^C) \\ -F^C(F^C\xi^C, \psi^C). \tag{3.2}$$

Let $\xi, \psi \in T_0^1(M)$ and $F \in T_1^1(M)$, then we have

$$(\xi^C, \psi^C) = (\xi, \psi)^C,$$
$$(\xi^C + \psi^C) = \xi^C + \psi^C,$$

and

$$F^C\xi^C = (F\xi)^C. \tag{3.3}$$

In the view of equations (1.2), (1.3) and (3.3), we get

$$F^C m^C = (Fm)^C = 0,$$

and

$$F^C l^C = (Fl)^C = (F)^C. \tag{3.4}$$

**Theorem 3.1.** For the Nijenhuise tensor N, the following identities holds:

$$N^C(m^C\xi^C, m^C\psi^C) = (F^C)^C(m^C\xi^C, m^C\psi^C) \tag{3.5}$$

$$m^C N^C(\xi^C, \psi^C) = m^C(F^C\xi^C, F^C\psi^C) \tag{3.6}$$

$$m^C(l^C\xi^C, l^C\psi^C) = m^C(F^C\xi^C, F^C\psi^C) \tag{3.7}$$

$$m^C N^C[(F^2 - a^2)^C(F^2 + a^2)^C\xi^C, (F^2 - a^2)^C(F^2 + a^2)^C\psi^C \\ = m^C N^C(l^C\xi^C, l^C\psi^C). \tag{3.8}$$

*Proof:* To prove the above identities of the theorem, we use the properties of projection tensors $l$ and m, defined in the section one. Also using (3.1) – (3.4) we can easily prove (3.5) – (3.8).

**Theorem 3.2.** If $\xi, \psi \in T_0^1(M)$, then the following conditions are identical

(i). $mN^C(\xi^C, \psi^C) = 0$.

(ii). $\mathrm{mN^C}\left(l^C\xi^C, l^C\psi^C\right) = 0.$
(iii). $\mathrm{m^C N^C}[(\mathrm{F}^2 - \mathrm{a}^2)^C(\mathrm{F}^2 + \mathrm{a}^2)^C\xi^C, (\mathrm{F}^2 - \mathrm{a}^2)^C(\mathrm{F}^2 + \mathrm{a}^2)^C\psi^C = 0.$

*Proof:* Using theorem (3.1) and its identities, we have

$$\mathrm{N^C}\left(l^C\xi^C, l^C\psi^C\right) = 0,$$
$$\mathrm{N^C}[(\mathrm{F}^2 - \mathrm{a}^2)^C(\mathrm{F}^2 + \mathrm{a}^2)^C\xi^C, (\mathrm{F}^2 - \mathrm{a}^2)^C(\mathrm{F}^2 + \mathrm{a}^2)^C\psi^C = 0.$$

Second and third identities of theorem (3.1) have same terms in the right hand side which along with the last identity of the above theorem shows that conditions (i) – (iii) of the theorem are identical.

**Theorem 3.3.** The distribution $\mathrm{D_m}$ is integrable over the manifold M, then complete lift $\mathrm{D^C}$ on the tangent bundle of a distribution $\mathrm{D_m}$ is also integrable over the manifold M.

*Proof:* The distribution $\mathrm{D_m}$ is integrable if the following property of the projection operator satisfy,

$$l(\mathrm{m}\xi, \mathrm{m}\psi) = 0, \tag{3.9}$$

for all $\xi, \psi \in \mathrm{T(M)}$, where $l$ is the projection operator and $l = \mathrm{I} - \mathrm{m}$.

Applying the complete lift on the equation (3.9), we have

$$l^C\left(\mathrm{m^C}\xi^C, \mathrm{m^C}\psi^C\right) = 0, \tag{3.10}$$

where, $l^C = (\mathrm{I} - \mathrm{m})^C = \mathrm{I} - \mathrm{m^C}$,is the projection tensor complementary to m. This implies that if (3.9) holds, then (3.10) also holds.

**Theorem 3.4.** If $l^C N^C(m^C\xi^C, m^C\psi^C) = 0$, or equivalently $N^C(m^C\xi^C, m^C\psi^C) = 0$, for all $\xi, \psi \in \mathrm{T(M)}$, then the complete lift $\mathrm{D_m^C}$ in T(M) of a distribution $\mathrm{D_m}$ in M is integrable.

*Proof:* The given distribution $\mathrm{D_m}$ is integrable over the manifold M if it satisfies,

$$\mathrm{N}(\mathrm{m}\xi, \mathrm{m}\psi) = 0,$$

for all $\xi, \psi \in \mathrm{T(M)}$. By virtue of condition (3.5), we have

$$\mathrm{N^C}\left(\mathrm{m^C}\xi^C, \mathrm{m^C}\psi^C\right) = (\mathrm{F}^2)^C\left(\mathrm{m^C}\xi^C, \mathrm{m^C}\psi^C\right) = 0.$$

Now operate $l^C$ on the above equation, we have

$$l^C\mathrm{N^C}\left(\mathrm{m^C}\xi^C, \mathrm{m^C}\psi^C\right) = (\mathrm{F}^2)^C l^C\left(\mathrm{m^C}\xi^C, \mathrm{m^C}\psi^C\right) = 0,$$

considering (3.10), the above equation can be expressed as

$$l^C\mathrm{N^C}\left(\mathrm{m^C}\xi^C, \mathrm{m^C}\psi^C\right) = 0. \tag{3.11}$$

Also, we have

$$\mathrm{m^C N^C}\left(\mathrm{m^C}\xi^C, \mathrm{m^C}\psi^C\right) = 0, \tag{3.12}$$

combining above two equations, (3.11) and (3.12), we get

$$(m^C + l^C)N^C(m^C\xi^C, m^C\psi^C) = 0.$$

Since,

$$(m^C + l^C) = I^C = I.$$

Therfore we have,

$$N^C(m^C\xi^C, m^C\psi^C) = 0.$$

**Theorem 3.5.** If any of the condition of theorem (3.2) satisfies on the differentiable manifold M and the distribution $D_l$ be integrable on it, also $mN(\xi, \psi) = 0$ for all $\xi, \psi \in T(M)$, then the distribution $D_l^C$ is also integrable over the tangent bundle $T(M)$.

*Proof:* The distribution $D_l$ is integrable over M iff

$$mN(l\xi, l\psi) = 0.$$

$\Rightarrow D_l^C$ is integrable in the tangent bundle $T(M)$ iff

$$m^C N^C(I^C\xi^C, I^C\psi^C) = 0,$$

hence the theorem is established by using (3.8).

**Theorem 3.6** For a differentiable manifold M, if F, which forms a superstructure, $F(\pm a^2, \pm b^2)$, is partially integrable on its tangent bundle $T(M)$, then $F^C$, the complete lift of F, is also partially integrable.

*Proof:* The essential condition for the existence of partially integrability of superstructure $F(\pm a^2, \pm b^2)$, defined over M is

$$N(l\xi, l\psi) = 0, \forall\ \xi, \psi \in T_0^1(M). \tag{3.13}$$

In view of the equations (1.3) and (3.1), we can have

$$N^C(l^C\xi^C, l^C\psi^C) = [N(l\xi, l\psi)]^C,$$

which implies

$$N^C(l^C\xi^C, l^C\psi^C) = 0 \Leftrightarrow [N(l\xi, l\psi)] = 0.$$

with the help of theorem (3.2),

$$N^C(l^C\xi^C, l^C\psi^C) = 0,$$
$$N^C[(F^2 - a^2)^C(F^2 + a^2)^C\xi^C, (F^2 - a^2)^C(F^2 + a^2)^C\psi^C = 0.$$

**Theorem 3.7.** For a differentiable manifold M, if F, which also forms a superstructure $F(\pm a^2, \pm b^2)$, is integrable on its tangent bundle $T(M)$, then $F^C$, the complete lift of F, is also integrable in the tangent bundle $T(M)$.

*Proof:* A necessary and sufficient condition for a superstructure on the manifold M to be integrable is

$$N(\xi, \psi) = 0, \tag{3.14}$$

$\forall\ \xi, \psi \in T_0^1(M)$.

Reference to equation (3.1), we have

$$N^C(\xi^C, \psi^C) = [N(\xi, \psi)]^C,$$

so, along with the equation (3.14), we get the result.

Now, we shall establish certain important theorems on horizontal lift of the superstructure $F(\pm a^2, \pm b^2)$. Let S be a tensor field along with $\nabla_\psi S$ which are defined over differential manifold M and tangent bundle $T(M)$ respectively, then we have,

$$S = S_{k \dots j}^{i \dots h} \frac{\partial}{\partial x^i} \otimes \dots\dots \otimes \frac{\partial}{\partial x^h} \otimes dx^k \otimes \dots\dots \otimes dx^j,$$
$$\nabla_\beta S = \beta^l \nabla_\beta S_{k \dots j}^{i \dots h} \frac{\partial}{\partial x^i} \otimes \dots\dots \otimes \frac{\partial}{\partial x^h} \otimes dx^k \otimes \dots\dots \otimes dx^j,$$

for all $(\xi^H, \psi^H) \in \pi^{-1}(U)$, where $\nabla$ be the affine connection.

Over the differentiable manifold M, $S^H$ be the horizontal lift of S on M to $T(M)$ by

$$S^H = S - \nabla_\psi S.$$

**Theorem 3.8.** Over a differential manifold M, if F forms a superstructure $F(\pm a^2, \pm b^2)$, where $F \in T_1^1(M)$ then its horizontal lift $F^H$ also form corresponding superstructure $F(\pm a^2, \pm b^2)$ on the tangent bundle $T(M)$.

*Proof:* To prove this theorem we define a polynomial $P(t)$, where t is variable then we have

$$[P(F)]^H = P(F^H), \tag{3.15}$$

$\forall\ F \in T_1^1(M)$.

Consider equation (1.1) and apply horizontal lift on it, we have

$$(F^2 + a^2)^H (F^2 - a^2)^H (F^2 + b^2)^H (F^2 - b^2)^H = 0,$$
$$[(F^2)^H + a^2][(F^2)^H - a^2][(F^2)^H + b^2][(F^2)^H - b^2] = 0.$$

Applying property of (3.15), then above equation become

$$[(F^H)^2 + a^2][(F^H)^2 - a^2][(F^H)^2 + b^2][(F^H)^2 - b^2] = 0, \tag{3.16}$$

equation (3.16) clearly indicates that $F^H$ is a superstructure on tangent bundle T(M), further it is also clear about the rank of F and $F^H$. Rank of $F^H$ is two times the rank of F which deduce in the following theorem:

**Theorem 3.9.** Superstructure F defined as $F(\pm a^2, \pm b^2)$, is of rank r on M iff its horizontal lift $F^H$ is of rank 2r on $T(M)$.

*Proof:* Prove of this theorem is obvious. From (1.3), m is defined as a projection tensor field of type (1,1) on differentiable manifold M. Also

$$m^2 = m.$$

In view of (3.15) we have

$$(m^H)^2 = m^H, \tag{3.17}$$

equation (3.17) deduce that $m^H$ is also a projection tensor on the tangent bundle. Horizontal distribution $D^H$ exist on the tangent manifold T(M) reference to $m^H$, where $m^H$ is horizontal lift on the defined distribution D.

## 4. SOME CALCULATIONS OF SUPERSTRUCTURE $F(\pm a^2, \pm b^2)$ ON THIRD ORDER BUNDLE

Let $T_3(M)$ be the third order tangent bundle over M and $F^{III}$ be the third lift of F in $T_3(M)$. For $\forall$ $E, F \in T_1^1(M)$, Forsyth et. al [30] we have

$$\begin{aligned}(E^{III}F^{III})\xi^{III} &= (E^{III}(F^{III}\xi^{\,III})) \\ &= (E^{III}(F\xi)^{III}) \\ &= [E(F\xi)]^{III} \\ &= (EF)^{III}\xi^{III}\end{aligned} \tag{4.1}$$

for all $\xi \in T_1^1(M)$. Thus we have

$$E^{III}F^{III} = (EF)^{III} \tag{4.2}$$

Now let us define one variable polynomial P(t), then we have

$$[P(F)]^{III} = P(F)^{III}, \tag{4.2}$$

for all $F \in T_1^1(M)$

**Theorem 4.1.** If $T_3(M)$ be a third order tangent bundle and $F \in T_1^1(M)$ forms a superstructure $F(\pm a^2, \pm b^2)$ in the differentiable manifold M, then $F^{III}$ is also a superstructure in $T_3(M)$.

*Proof:* Reference to equation (4.2), one variable polynomial P(t), we have

$$[P(F)]^{III} = P(F)^{III},$$

where $F \in T_1^1(M)$.

Now using (1.1) and applying third lift on it, we have

$$[(F^2 + a^2)(F^2 - a^2)(F^2 + b^2)(F^2 - b^2)]^{III} = 0$$

and

$$[(F^2 + a^2)^{III}(F^2 - a^2)^{III}(F^2 + b^2)^{III}(F^2 - b^2)^{III}] = 0,$$

in view of (4.2), we get

$$[(F^{III})^2 + a^2][(F^{III})^2 - a^2][(F^{III})^2 + b^2][(F^{III})^2 - b^2] = 0. \tag{4.3}$$

Equation (4.3) reveals that $F^{III}$ is a superstructure over $T_3(M)$.

**Theorem 4.2.** For a differentiable manifold M, $F^{III}$ is integrable in $T_3(M)$, iff F is integrable in M.

*Proof:* Let Nijenhuis tensors of $F^{III}$ and F are $N^{III}$ and N respectively. Then we have

$$N^{III}(\xi, \psi) = [N(\xi, \psi)]^{III}. \tag{4.4}$$

The superstructure $F(\pm a^2, \pm b^2)$ is integrable in M iff $N(\xi, \psi) = 0$. Then from (4.4), we get,

$$N^{III}(\xi, \psi) = 0, \tag{4.5}$$

Hence, $F^{III}$ and F both are integrable in M.

## CONCLUSION

The author along with others defined the superstructure in 2007 [4]. In this paper we study the properties of superstructure $(F^2 + a^2)(F^2 - a^2)(F^2 + b^2)(F^2 - b^2) = 0$, we analyze the concept of the horizontal and complete lifts over the tangent bundles and establish its integrability conditions using the horizontal and complete lifts. We have investigated some properties of the third-order tangent bundle. In the future superstructure can be studied further using the properties defined in this paper.

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