ORIGINAL PAPER

BELL POLYNOMIAL SOLUTION OF LINEAR FREDHOLM-VOLTERRA INTEGRO DIFFERENTIAL EQUATION SYSTEMS WITH HYBRID DELAYS

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Abstract. *In this study, the Bell collocation method is applied to solve system of high order linear delay Fredholm-Volterra integro differential under initial conditions. The numerical method is substantially dependent on the truncated Bell series, their derivatives, and collocation points. By using this method, solutions of the integral system are obtained from the Bell series. Additionally, the error analysis and residual functions are performed, and some examples are provided to indicate the availability and applicability of the method.*
Keywords: *Bell polynomials and series; collocation points; system of high-order linear hybrid delay Fredholm–Volterra integro-differential equations; residual error estimation.*

1. INTRODUCTION

The Fredholm-Volterra system of integro differential equations appears in many fields of science and engineering. Microsecond delays have an important place in control systems, population dynamics, chemical reactions, and biological models. This type of modeling is encountered as differential equations, integral and integro differential equations, and systems. In addition, methods of solving these and other equations play a central role in such applications. Since these types of systems are difficult to solve analytical methods, numerical methods are required. Numerical methods for the difference-differential equation system and Fredholm-Volterra integral system such as Adomian decomposition method [1], Differential transformation method [2], Haar functions method [3], homotopy analysis method [4], via Laplace Transformation [5], Taylor collocation method [6], Euler–Chebyshev [7] and Runge–Kutta [8] methods, the homotopy perturbation method [9, 10], the variational iteration method [11], used the hybrid Legendre functions, the Chebyshev polynomial method [12], the Bessel collocation method [13, 14] etc.

In this study, the numerical method is improved based on Bell polynomials and collocation points to solve the system of high order linear hybrid delay Fredholm Volterra integro differential equations in given form

$$\sum_{k=0}^{m_1}\sum_{j=1}^{J} P_{ij}^{k}(x)y_j^{(k)}(x)+\sum_{r=0}^{m_2}\sum_{j=1}^{J} Q_{ij}^{r}(x)\,y_j^{(r)}\left(\alpha_{jr}x+\beta_{jr}\right)=g_i(x)+ \tag{1}$$

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$$+\int_a^b \sum_{j=1}^{J} K_{ij}^{f}(x,t) y_j(t)dt + \int_a^x \sum_{j=1}^{J} K_{ij}^{v}(x,t) y_j(t)dt\,;\ \ a \le x,t \le b\,, m = max\{m_1, m_2\}, \qquad i = 1,2,\dots,J$$

under the initial conditions

$$y_j^{(k)}(x) = \lambda_{jk}, j = 1,2,\dots,J; k = 0,1,\dots,m-1 \tag{2}$$

where $y_j^{(0)} = y_j(x)$ are unknown functions; $P_{ij}^{(k)}(x)$, $g_i(x)$ are continuous functions in $[a,b]$,λ_{jk}, α_{jr} *and* β_{jr} is real constant coefficients.

The aim is to obtain an approximate solution for (1) in the form of Bell polynomials

$$y_j(x) = \sum_{n=0}^{N} a_{jn} B_n(x) \tag{3}$$

where a_{Jn} , $n = 0,1,\cdots N$ are unknown Bell coefficients and the Bell polynomials $B_n(x)$ where $n = 0,1,\cdots,N$ are defined by

$$B_n(x) = \sum_{k=0}^{n} S(n,k)\, x^k \tag{4}$$

where

$$S(n,k) = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k!} \binom{k}{j} j^n$$

are stirling numbers of the second kind [15-18].

2. FUNDAMENTAL MATRIX RELATION

The Bell polynomials defined in Eq. (4) is written in terms of matrix form as

$$\mathbf{B}(x) = \mathbf{X}(x)\mathbf{S} \tag{5}$$

where

$$\mathbf{B}(x) = [B_0(x)\ \ B_1(x) \dots\ B_N(x)]\ \ ,\ \ \mathbf{X}(x) = [1\ \ x\ \ x^2 \dots\ x^N]$$

and

$$\mathbf{S} = \begin{bmatrix} S(0,0) & S(1,0) & S(2,0) & \cdots & S(N,0) \\ 0 & S(1,1) & S(2,1) & \cdots & S(N,1) \\ 0 & 0 & S(2,2) & \cdots & S(N,2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & S(N,N) \end{bmatrix}.$$

Eq. (1) is written in the form:

$$D(x) = g(x) + I(x) \tag{6}$$

where the system of differential part is

$$D(x) = \sum_{k=0}^{m_1}\sum_{j=1}^{J} P_{ij}^{k}(x) y_j^{(k)}(x) + \sum_{r=0}^{m_2}\sum_{j=1}^{J} Q_{ij}^{r}(x)\, y_j^{(r)}\big(\alpha_{jr}x + \beta_{jr}\big)$$

and Volterra-Fredholm integral part is

$$I(x) = \int_a^b \sum_{j=1}^{J} K_{ij}^{f}(x,t) y_j(t)dt + \int_a^x \sum_{j=1}^{J} K_{ij}^{v}(x,t) y_j(t)dt.$$

2.1 MATRIX RELATION FOR THE DIFFERENTIAL PART D(x)

The solution $y_j(x)$ of (1) and its k-th derivative $y_j^{(k)}(x)$ of defined by the truncated Bell series (3). Then, we substitute the finite series (3) in the matrix form

$$y_j(x) = \mathbf{B}(x)\mathbf{A}_j\,,\; y_j^{(k)}(x) = \mathbf{B}^{(k)}(x)\,\mathbf{A}_j \tag{7}$$

where

$$\mathbf{A}_j = [a_{j0} \quad a_{j1} \quad \cdots \quad a_{jN}]^T, j=1,2,...,J.$$

The matrix form (5) is placed in the matrix relation (7) we obtain the matrix relation

$$y_j(x) = \mathbf{X}(x)\mathbf{S}\mathbf{A}_j \tag{8}$$

Besides, it is expressly seen that [19] the relation between the matrix $\mathbf{X}$(x) and its kth derivative $\mathbf{X}^{(k)}$(x) is

$$\mathbf{X}^{(k)}(x) = \mathbf{X}(x)\mathbf{M}^k \tag{9}$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \mathbf{M}^0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Thus, from the relations (8) and (9), we obtain the matrix relations

$$y_j^{(k)}(x) = \mathbf{X}(x)\mathbf{M}^k\mathbf{S}\mathbf{A}_j\,; j=1,2,...,J. \tag{10}$$

Similarly, if we put $x \to \alpha_{jk}x + \beta_{jk}$ into (8), we can write the matrix relation [20]

$$y_j^{(k)}(\alpha_{jk}x + \beta_{jk}) = \mathbf{X}\big(\alpha_{jk}x + \beta_{jk}\big)\mathbf{M}^k\mathbf{S}\mathbf{A}_j = \mathbf{X}(x)\mathcal{M}\big(\alpha_{jk}, \beta_{jk}\big)\mathbf{M}^k\mathbf{S}\mathbf{A}_j \tag{11}$$

if $\alpha_{jk} \neq 0 \; and \; \beta_{jk} \neq 0;$

$$\boldsymbol{\mathcal{M}}(\alpha_{jk},\beta_{jk})=\begin{bmatrix}\binom{0}{0}(\alpha_{jk})^0(\beta_{jk})^0 & \binom{1}{0}(\alpha_{jk})^0(\beta_{jk})^1 & \binom{2}{0}(\alpha_{jk})^0(\beta_{jk})^2 & \cdots & \binom{N}{0}(\alpha_{jk})^0(\beta_{jk})^N\\ 0 & \binom{1}{1}(\alpha_{jk})^1(\beta_{jk})^0 & \binom{2}{1}(\alpha_{jk})^1(\beta_{jk})^1 & \cdots & \binom{N}{1}(\alpha_{jk})^1(\beta_{jk})^{N-1}\\ 0 & 0 & \binom{2}{2}(\alpha_{jk})^2(\beta_{jk})^0 & \cdots & \binom{N}{2}(\alpha_{jk})^2(\beta_{jk})^{N-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \binom{N}{N}(\alpha_{jk})^N(\beta_{jk})^0\end{bmatrix}$$

and for $\alpha_{jk}\neq 0$, $\beta_{jk}=0$;

$$\boldsymbol{\mathcal{M}}(\alpha_{jk},0)=\begin{bmatrix}\binom{0}{0}(\alpha_{jk})^0 & \binom{1}{0}(\alpha_{jk})^0 & \binom{2}{0}(\alpha_{jk})^0 & \cdots & \binom{N}{0}(\alpha_{jk})^0\\ 0 & \binom{1}{1}(\alpha_{jk})^1 & \binom{2}{1}(\alpha_{jk})^1 & \cdots & \binom{N}{1}(\alpha_{jk})^1\\ 0 & 0 & \binom{2}{2}(\alpha_{jk})^2 & \cdots & \binom{N}{2}(\alpha_{jk})^2\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \binom{N}{N}(\alpha_{jk})^N\end{bmatrix}.$$

In this case [21], the matrix relations (10) and (11) can be expressed as

$$\mathbf{Y}^{(k)}(x)=\bar{\mathbf{X}}(x)\bar{\mathbf{M}}^k\bar{\mathbf{S}}\mathbf{A}_{\mathrm{j}}\,,\mathbf{Y}^{(r)}(\alpha_{jr}x+\beta_{jr})=\bar{\mathbf{X}}(x)\overline{\boldsymbol{\mathcal{M}}}(\alpha_{jr},\beta_{jr})\bar{\mathbf{M}}^r\bar{\mathbf{S}}\mathbf{A}_j \tag{12}$$

where

$$\mathbf{Y}^{(k)}(x)=\begin{bmatrix}y_{1,N}^{(k)}(x)\\ y_{2,N}^{(k)}(x)\\ \vdots\\ y_{k.N}^{(k)}(x)\end{bmatrix},\mathbf{Y}^{(r)}(\alpha_r x+\beta_r)=\begin{bmatrix}y_{1,N}^{(r)}(\alpha_{jr}x+\beta_{jr})\\ y_{2,N}^{(r)}(\alpha_{jr}x+\beta_{jr})\\ \vdots\\ y_{k.N}^{(r)}(\alpha_{jr}x+\beta_{jr})\end{bmatrix}$$

$$\bar{\mathbf{X}}(x)=\begin{bmatrix}\boldsymbol{X}(x) & 0 & \cdots & 0\\ 0 & \boldsymbol{X}(x) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \boldsymbol{X}(x)\end{bmatrix},\bar{\mathbf{M}}^k=\begin{bmatrix}\mathbf{M}^k & 0 & \cdots & 0\\ 0 & \mathbf{M}^k & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{M}^k\end{bmatrix},\bar{\mathbf{S}}=\begin{bmatrix}\mathbf{S} & 0 & \cdots & 0\\ 0 & \mathbf{S} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{S}\end{bmatrix}$$

$$\overline{\boldsymbol{\mathcal{M}}}(\alpha_{jr},\beta_{jr})=\begin{bmatrix}\boldsymbol{\mathcal{M}}(\alpha_{1r},\beta_{1r}) & 0 & \cdots & 0\\ 0 & \boldsymbol{\mathcal{M}}(\alpha_{2r},\beta_{2r}) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \boldsymbol{\mathcal{M}}(\alpha_{Jr},\beta_{jr})\end{bmatrix}.$$

Therefore, the system of differential part (6) of matrix relations can be written

$$\mathbf{D}(x)=\sum_{k=0}^{m_1}\mathbf{P}_k(x)\mathbf{Y}^{(k)}(x)+\sum_{r=0}^{m_2}\mathbf{Q}_r(x)\mathbf{Y}^{(r)}(\alpha_{jr}x+\beta_{jr}). \tag{13}$$

2.2 MATRIX RELATION FOR THE INTEGRAL PART I(x)

Fredholm-Volterra integral part is represented by Eqn. (1)

$$I(x)=\int_a^b\sum_{j=1}^{J}K_{ij}^{f}(x,t)y_j(t)dt+\int_a^x\sum_{j=1}^{J}K_{ij}^{v}(x,t)y_j(t)dt \quad (14)$$

where

$$I_f\ (x)=\int_a^b\sum_{j=1}^{J}K_{ij}^{f}(x,t)y_j(t)dt\ and\ V_i(x)=\int_a^x\sum_{j=1}^{J}K_{ij}^{v}(x,t)y_j(t)dt.$$

The matrix form of the kernel function in Fredholm integral part is $K_{ij}^{f}(x,t)$

$$\boldsymbol{K}_{ij}^{f}(x,t)\ =\mathbf{X}(x)\boldsymbol{K}_{ij}^{f}\mathbf{X}^{\mathrm{T}}(t) \quad (15)$$

where

$$K_{ij}^{f}(x,t)\ =\sum_{m=0}^{N}\sum_{n=0}^{N}k_{mn}^{f,ij}\mathrm{x}^m\mathrm{t}^n,\ k_{mn}^{f,ij}=\frac{1}{m!\,n!}\frac{\partial^{m+n}K_{ij}^{f}(0,0)}{\partial\mathrm{x}^m\partial\mathrm{t}^n},m,n=0,1,\dots,N$$

The kernel function matrix form (15) can be written Eq. (14) we have the matrix relation

$$[\boldsymbol{I}_f(x)]=\int_a^b\sum_{j=1}^{J}\mathbf{X}(x)\boldsymbol{K}_{ij}^{f}\mathbf{X}^{\mathrm{T}}(\mathrm{t})\mathbf{X}(t)\boldsymbol{S}\boldsymbol{A}_j dt=\sum_{\mathrm{j}=1}^{\mathrm{J}}\mathbf{X}(x)\boldsymbol{K}_{ij}^{f}\mathbf{Q}^f\mathbf{S}\mathbf{A}_{\mathrm{j}} \quad (16)$$

where

$$\mathbf{Q}^f=[q_{mn}]=\int_{\mathrm{a}}^{\mathrm{b}}\mathbf{X}^{\mathrm{T}}(t)\mathbf{X}(t)dt\quad and\quad q_{mn}=\frac{b^{m+n+1}-a^{m+n+1}}{m+n+1}\quad m,n=0,1,2,\dots,N.$$

By substuting matrix form of (10) into expression (16) we obtain the matrix relation for Fredholm integral part [22,23]

$$\boldsymbol{I}_f(x)=\begin{bmatrix}I_1(x)\\ I_2(x)\\ \vdots\\ I_J(x)\end{bmatrix}=\bar{\mathbf{X}}(x)\boldsymbol{K}_f\overline{\mathbf{Q}^f}\bar{\mathbf{S}}\boldsymbol{A} \quad (17)$$

where

$$\overline{\mathbf{Q}^f}=\begin{bmatrix}\mathbf{Q}^f & 0 & \cdots & 0\\ 0 & \mathbf{Q}^f & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{Q}^f\end{bmatrix},\boldsymbol{K}_f=\begin{bmatrix}K_{11}^f & K_{12}^f & \cdots & K_{1J}^f\\ K_{21}^f & K_{22}^f & \cdots & K_{2J}^f\\ \vdots & \vdots & \ddots & \vdots\\ K_{J1}^f & K_{J2}^f & \cdots & K_{JJ}^f\end{bmatrix}.$$

Similarly the matrix form of the kernel function in Volterra integral part is

$$\boldsymbol{K}_{ij}^{f}(x,t)\ =\mathbf{X}(x)\boldsymbol{K}_{ij}^{f}\mathbf{X}^{\mathrm{T}}(t) \quad (18)$$

and substituted the kernel function in the matrix relations (14) we have the matrix relation as

$$[\,\boldsymbol{V}_i(x)] = \int_a^x \sum_{j=1}^{J} \mathbf{X}(x)\boldsymbol{K}_{ij}^{v}\mathbf{X}^{\mathrm{T}}(t)\mathbf{X}(t)\mathbf{S}\mathbf{A}_{\mathrm{j}}dt = \sum_{\mathrm{j}=1}^{\mathrm{J}} \mathbf{X}(x)\boldsymbol{K}_{ij}^{v}\mathbf{\Theta}(\mathbf{x})\mathbf{S}\mathbf{A}_{\mathrm{j}} \tag{19}$$

where

$$\mathbf{\Theta}(\mathrm{x}) = [\varphi_{\mathrm{mn}}(x)] = \int_{\mathrm{a}}^{\mathrm{x}} \mathbf{X}^{\mathrm{T}}(\mathrm{t})\mathbf{X}(t)dt \;\;, \varphi_{mn}(\mathrm{x}) = \frac{\mathrm{x}^{\mathrm{m+n+1}} - \mathrm{a}^{\mathrm{m+n+1}}}{\mathrm{m+n+1}} \;\; m,n = 0,1,2,\dots,N.$$

So that using bu matrix relations (10) and (19) Volterra integral part is obtained as follows

$$\boldsymbol{V}(x) = \begin{bmatrix} V_1(x) \\ V_2(x) \\ \vdots \\ V_J(x) \end{bmatrix} = \overline{\mathbf{X}}(x)\boldsymbol{K}_v\overline{\mathbf{\Theta}}(\boldsymbol{x})\overline{\mathbf{S}}\mathbf{A} \tag{20}$$

where

$$\overline{\mathbf{\Theta}}(x) = \begin{bmatrix} \boldsymbol{\Theta}(x) & 0 & \cdots & 0 \\ 0 & \boldsymbol{\Theta}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\Theta}(x) \end{bmatrix}, \boldsymbol{K}_v = \begin{bmatrix} K_{11}^{v} & K_{12}^{v} & \cdots & K_{1J}^{v} \\ K_{21}^{v} & K_{22}^{v} & \cdots & K_{2J}^{v} \\ \vdots & \vdots & \ddots & \vdots \\ K_{J1}^{v} & K_{J2}^{v} & \cdots & K_{JJ}^{v} \end{bmatrix}.$$

Hence, the system of integral part (6) of matrix relations using (16) and (19) can be written

$$\mathbf{I}(x) = \overline{\mathbf{X}}(x)\boldsymbol{K}_f\overline{\mathbf{Q}^f}\overline{\mathbf{S}}\boldsymbol{A} + \overline{\mathbf{X}}(x)\boldsymbol{K}_v\overline{\mathbf{\Theta}}(\boldsymbol{x})\overline{\mathbf{S}}\mathbf{A}. \tag{21}$$

2.3 MATRIX RELATION FOR CONDITIONS

The matrix form of the conditions Eq. (2) are written using relations (8) for $j = 1,2,\dots,J$ and $k = 0,1,\dots,m-1$ as follows

$$\begin{bmatrix} y_1^{(k)}(a) \\ y_2^{(k)}(a) \\ \vdots \\ y_{jk}^{(k)}(a) \end{bmatrix} = \begin{bmatrix} \mathbf{X}(a)\mathbf{M}^{\mathrm{k}}\mathbf{S} & 0 & \cdots & 0 \\ 0 & \mathbf{X}(a)\mathbf{M}^{\mathrm{k}}\mathbf{S} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(a)\mathbf{M}^{\mathrm{k}}\mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_{\mathrm{J}} \end{bmatrix} = \begin{bmatrix} \lambda_{1k} \\ \lambda_{2k} \\ \vdots \\ \lambda_{Jk} \end{bmatrix}$$

or briefly

$$\mathbf{U}_{\mathrm{k}}\mathbf{A} = \lambda_k \text{ or } [\mathbf{U}_{\mathrm{k}};\lambda_k]\,,\; \mathbf{U}_{\mathrm{k}} = \sum_{j=0}^{m-1} a_j\overline{\mathbf{X}}(a)\overline{\mathbf{S}},\;\; j = 0,1,\dots,m-1. \tag{22}$$

3.COLLOCATION METHOD

The matrix relations (13) and (21) are placed in Eq. (1) we obtain that the fundamental matrix equation

$$\sum_{k=0}^{m} \boldsymbol{P}_k(x)\bar{\mathbf{X}}(x)\overline{\mathcal{M}}(\alpha_k,\beta_k)(\bar{\mathbf{M}})^k\bar{\mathbf{S}}\mathbf{A} = \boldsymbol{G}(x) + \bar{\mathbf{X}}(x)\boldsymbol{K}_f\overline{\mathbf{Q}^f}\bar{\mathbf{S}}\boldsymbol{A} + \bar{\mathbf{X}}(x)\boldsymbol{K}_v\bar{\boldsymbol{\Theta}}(x)\bar{\mathbf{S}}\mathbf{A} \tag{23}$$

The collocation points x_s are defined by

$$x_s = a + \frac{b-a}{N}s \ , s = 0,1,\ldots,N. \tag{24}$$

By using the collocation points (24), into Eq. (23) we obtain the system of the matrix equations

$$\begin{aligned}\sum_{k=0}^{m} \mathbf{P}_k\bar{\mathbf{X}}(x_s)\bar{\mathbf{M}}^k\bar{\mathbf{S}} + \mathbf{Q}_k(x_s)\bar{\mathbf{X}}(x_s)\overline{\mathcal{M}}(\alpha_k,\beta_k)(\bar{\mathbf{M}})^k\bar{\mathbf{S}}\mathbf{A} \\ = \boldsymbol{G}(x_s) + \bar{\mathbf{X}}(x_s)\boldsymbol{K}_{ij}^f\overline{\mathbf{Q}^f}\bar{\mathbf{S}}\boldsymbol{A} + \bar{\mathbf{X}}(x_s)\boldsymbol{K}_v\bar{\boldsymbol{\Theta}}(x_s)\bar{\mathbf{S}}\boldsymbol{A}\end{aligned} \tag{25}$$

or briefly, the fundamental matrix equation is expressed as

$$\left\{\sum_{k=0}^{m} \mathbf{P}_k\bar{\mathbf{X}}\bar{\mathbf{M}}^k\bar{\mathbf{S}} + \overline{\mathbf{Q}_k}\,\bar{\mathbf{X}}^*\overline{\mathcal{M}}(\alpha_k,\beta_k)(\bar{\mathbf{M}})^k\bar{\mathbf{S}} - \bar{\mathbf{X}}^*\boldsymbol{K}_f\overline{\mathbf{Q}^f}\bar{\mathbf{S}} - \bar{\mathbf{X}}\overline{\boldsymbol{K}_v}\bar{\boldsymbol{\Theta}}^*\,\bar{\mathbf{S}}\right\}\mathbf{A} = \mathbf{G} \tag{26}$$

where

$$\mathbf{P}_k(x) = \begin{bmatrix} P_{11}^k(x) & P_{12}^k(x) & \cdots & P_{1J}^k(x) \\ P_{21}^k(x) & P_{22}^k(x) & \cdots & P_{2J}^k(x) \\ \vdots & \vdots & \ddots & \vdots \\ P_{J1}^k(x) & P_{J2}^k(x) & \cdots & P_{JJ}^k(x) \end{bmatrix}, \bar{\mathbf{P}}_k = \begin{bmatrix} \mathbf{P}_k(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{P}_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_k(x_N) \end{bmatrix},$$

$$\mathbf{Q}_r(x) = \begin{bmatrix} Q_{11}^r(x) & Q_{12}^r(x) & \cdots & Q_{1J}^r(x) \\ Q_{21}^r(x) & Q_{22}^r(x) & \cdots & Q_{2J}^r(x) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{J1}^r(x) & Q_{J2}^r(x) & \cdots & Q_{JJ}^r(x) \end{bmatrix}, \bar{\mathbf{Q}}_{\boldsymbol{r}} = \begin{bmatrix} \mathbf{Q}_r(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{Q}_r(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{Q}_r(x_N) \end{bmatrix},$$

$$\bar{\mathbf{X}}^* = \begin{bmatrix} \bar{\mathbf{X}}(x_0) \\ \bar{\mathbf{X}}(x_1) \\ \vdots \\ \bar{\mathbf{X}}(x_N) \end{bmatrix}, \overline{\boldsymbol{K}_v} = \begin{bmatrix} \boldsymbol{K}_v & 0 & \cdots & 0 \\ 0 & \boldsymbol{K}_v & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{K}_v \end{bmatrix}, \bar{\boldsymbol{\Theta}}^* = \begin{bmatrix} \bar{\boldsymbol{\Theta}}(x_0) \\ \bar{\boldsymbol{\Theta}}(x_1) \\ \vdots \\ \bar{\boldsymbol{\Theta}}(x_N) \end{bmatrix}, \mathbf{G} = \begin{bmatrix} \mathbf{G}(x_0) \\ \mathbf{G}(x_1) \\ \vdots \\ \mathbf{G}(x_N) \end{bmatrix}.$$

The fundametnal matrix relation (26) is written as

$$\mathbf{WA} = \mathbf{G} \quad or \ \ [\mathbf{W};\mathbf{G}] \tag{27}$$

where

$$\mathbf{W} = \sum_{k=0}^{m} \mathbf{P}_k\bar{\mathbf{X}}\bar{\mathbf{M}}^k\bar{\mathbf{S}} + \overline{\mathbf{Q}_k}\,\bar{\mathbf{X}}^*\overline{\mathcal{M}}(\alpha_k,\beta_k)(\bar{\mathbf{M}})^k\bar{\mathbf{S}} - \bar{\mathbf{X}}^*\boldsymbol{K}_f\overline{\mathbf{Q}^f}\bar{\mathbf{S}} - \bar{\mathbf{X}}\overline{\boldsymbol{K}_v}\bar{\boldsymbol{\Theta}}^*\,\bar{\mathbf{S}} \tag{28}$$

$$\mathbf{W}=\sum_{k=0}^{m}\mathbf{P}_k\overline{\mathbf{X}}\overline{\mathbf{M}}^k\overline{\mathbf{S}}+\overline{\mathbf{Q}_k}\,\overline{\mathbf{X}}^*\overline{\mathcal{M}}(\alpha_k,\boldsymbol{\beta}_k)(\overline{\mathbf{M}})^k\overline{\mathbf{S}}-\overline{\mathbf{X}}^*\boldsymbol{K}_f\overline{\mathbf{Q}^f}\overline{\mathbf{S}}-\overline{\mathbf{X}\boldsymbol{K}_v}\overline{\mathbf{\Theta}}^*\,\overline{\mathbf{S}}$$

which is a linear system of *k (N + 1)* algebraic equations in *k (N + 1)* unknown Bell coefficients an $a_{j0}, a_{j1}, \ldots, a_{jN}, n = 0,1,\cdots,N, i = 1,2,\cdots,k.$

Consequently, by replacing the rows of the matrix $\mathbf{U}_k$and λ_k with the rows of the matrix **W** and **G**, respectively, we obtain

$$\widetilde{\mathbf{W}}\mathbf{A}=\widetilde{\mathbf{G}} \quad or\ [\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}] \tag{29}$$

If $rank(\widetilde{\mathbf{W}})=rank[\widetilde{\mathbf{W}}\,;\,\widetilde{\mathbf{G}}]=k(N+1)$,then we can write

$$\mathbf{A}_j=(\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{G}}\,.$$

Thus the matrix **A** (representing unknown Bell coefficients) is uniquely determined by Eqn.(1) under the coefficient equation (2) has unique solution. This solution is given by truncated Bell series

$$y_j(x)\ \cong y_{j,N}(x)=\sum_{n=0}^{N}a_{jn}B_n(x).$$

4. ERROR ESTIMATION BASED ON RESIDUAL FUNCTION

We can easily check the accuracy of the obtained solutions as follows. Since the truncated Bell series (3) approximate the solution of Volterra-Fredholm integral the system (1), using the residual correction method. The residual function of the method can be defined as

$$R_{iN}(x)=L[y_{iN}(x)]-g_i(x), i=1,2,\ldots,k \tag{30}$$

where $[L(y_{iN}(x)]\cong g_i(x)$ and $y_{iN}(x)$ are the Bell polynomial solutions (3) of the problems $(1)-(2)$. Then $y_{iN}(x)$ correspond the problem

$$L[y_{iN}(x)]=\sum_{k=0}^{m_1}\sum_{j=1}^{J}P_{ij}^k(x)y_{jN}^{(k)}(x)+\sum_{r=0}^{m_2}\sum_{j=1}^{J}Q_{ij}^r(x)\,y_{jN}^{(r)}\left(\alpha_{jr}x+\beta_{jr}\right)=$$
$$-\int_a^b\sum_{j=1}^{J}K_{ij}^f(x,t)y_{jN}(t)dt-\int_a^x\sum_{j=1}^{J}K_{ij}^v(x,t)y_{jN}(t)dt=g_i(x)+R_{iN}(x).$$

Furthermore, the exact solution $y_j(x)$ and the approximate solution $y_{jN}(x)$ are called, the error function $e_{jN}(x)$ is calculated by the following form

$$e_{jN}(x)=y_j(x)-y_{jN}(x). \tag{31}$$

Substituting (30) and (2) into (31), and by simplifying the result, the error problem is found

$$\sum_{k=0}^{m_1}\sum_{j=1}^{J} P_{ij}^{k}(x)e_{jN}^{(k)}(x)+\sum_{r=0}^{m_2}\sum_{j=1}^{J} Q_{ij}^{r}(x)\, e_{jN}^{(r)}\left(\alpha_{jr}x+\beta_{jr}\right)$$
$$-\int_a^b\sum_{j=1}^{J} K_{ij}^{f}(x,t)e_{jN}(t)dt\ -\int_a^x\sum_{j=1}^{J} K_{ij}^{v}(x,t)e_{jN}(t)dt=-R_{iN}(x). \qquad (32)$$
$$\sum_{k=0}^{m-1} a_{jk}e_N^{(k)}(a)=0, j=0,1,2,\ldots,m-1.$$

Bell collacation method is applied to Eq. (32), the approximation $e_{N,M}(x)$ to $e_N(x)$ is obtained, $(M\geq N)$ which is error function based on residual function$R_N(x)$. If $e_{jN}(x)\to 0$ when N is sufficiently large enough, then the error decreases [24-28].

5.NUMERICAL EXAMPLES

Example 5.1. Consider first the system of the linear, delay Fredholm-Volterra integro-differential equation

$$\begin{cases} y_1'-y_2(x-1)=x^2-2x-\dfrac{5}{2}+\displaystyle\int_0^1\left(y_1(t)+y_2(t)\right)dt-\int_0^x y_2(t)dt \\ y_2'+2y_1(x+1)=-\dfrac{x^2}{2}+2x-1+\displaystyle\int_0^1 y_2(t)dt+\int_0^x y_1(t)dt \end{cases}$$

with the initial conditions $y_1(0)=0, y_2(0)=0$ and the approximate solution $y_i(x)$ by the truncated Bell series

$$y_{j,2}(x)=\sum_{n=0}^{2} a_{jn}B_n(x)\,, j=1,2$$

where $N=2, m=1, J=2, g_1(x)=x^2-2x-\frac{5}{2}, g_2(x)=-\frac{x^2}{2}+2x-1, Q_{11}^0=0, Q_{12}^0=-1, Q_{21}^0=2, Q_{22}^0=0, P_{11}^1=1, P_{12}^1=0, P_{21}^1=0, P_{22}^1=1, \alpha_{10}=1, \beta_{10}=-1\,, \alpha_{20}=1, \beta_{20}=1\,, K_{11}^f(x,t)=1, K_{12}^f(x,t)=1\,, K_{21}^f(x,t)=0, K_{22}^f(x,t)=1, K_{11}^v(x,t)=0, K_{12}^v(x,t)=-1, K_{21}^v(x,t)=1, K_{22}^v(x,t)=0, y_1(x)=x$ and $y_2(x)=2x$ are exact solution.

The set of collocation points (24) for N = 2 is computed as

$$\left\{x_0=0, x_1=\frac{1}{2}, x_1=1\right\}$$

and from the Eq. (25), the Fundamental matrix equation of the problem is

$$\left\{\mathbf{P}_1\bar{\mathbf{X}}\bar{\mathbf{M}}\bar{\mathbf{S}}+\mathbf{Q}_0\bar{\mathbf{X}}\bar{\boldsymbol{\mathcal{M}}}(\alpha_0,\beta_0)(\bar{\mathbf{M}})^{\mathbf{0}}\bar{\mathbf{S}}-\bar{\mathbf{X}}\,\boldsymbol{K}_f\overline{\mathbf{Q}^f}\bar{\mathbf{S}}-\bar{\mathbf{X}}^*\overline{\boldsymbol{K}_v}\bar{\mathbf{Q}}^*\,\bar{\mathbf{S}}\right\}\mathbf{A}=\mathbf{G}$$

where

$$\mathbf{P}_1(x)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \mathbf{P}_1=\begin{bmatrix}\mathbf{P}_1(0) & 0 & 0\\ 0 & \mathbf{P}_1(1/2) & 0\\ 0 & 0 & \mathbf{P}_1(1)\end{bmatrix}$$

$$\mathbf{Q}_0(x)=\begin{bmatrix}0 & -1\\ 2 & 0\end{bmatrix},\ \mathbf{Q}_0=\begin{bmatrix}\mathbf{Q}_0(0) & 0 & 0\\ 0 & \mathbf{Q}_0(1/2) & 0\\ 0 & 0 & \mathbf{Q}_0(1)\end{bmatrix},$$

$$\overline{\mathbf{X}}(x)=\begin{bmatrix}\mathbf{X}(x) & 0\\ 0 & \mathbf{X}(x)\end{bmatrix},\ \overline{\mathbf{X}}=\begin{bmatrix}\overline{\mathbf{X}}(0) & 0 & 0\\ 0 & \overline{\mathbf{X}}(1/2) & 0\\ 0 & 0 & \overline{\mathbf{X}}(1)\end{bmatrix},\ \overline{\mathbf{X}}^*=\begin{bmatrix}\overline{\mathbf{X}}(0)\\ \overline{\mathbf{X}}(1/2)\\ \overline{\mathbf{X}}(1)\end{bmatrix},$$

$$\overline{\mathbf{M}}=\begin{bmatrix}\mathbf{M} & 0\\ 0 & \mathbf{M}\end{bmatrix},\ \mathbf{M}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{bmatrix},\ \overline{\mathbf{S}}=\begin{bmatrix}\mathbf{S} & 0\\ 0 & \mathbf{S}\end{bmatrix},\ \mathbf{S}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix},$$

$$\overline{\boldsymbol{\mathcal{M}}}(\alpha_0,\beta_0)=\begin{bmatrix}\boldsymbol{\mathcal{M}}(1,-1) & 0\\ 0 & \boldsymbol{\mathcal{M}}(1,1)\end{bmatrix},\ \overline{\boldsymbol{\mathcal{M}}}(\alpha_1,\beta_1)=\begin{bmatrix}\boldsymbol{\mathcal{M}}(1,0) & 0\\ 0 & \boldsymbol{\mathcal{M}}(1,0)\end{bmatrix},$$

$$\boldsymbol{\mathcal{M}}(1,-1)=\begin{bmatrix}1 & -1 & 1\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix},\ \boldsymbol{\mathcal{M}}(1,1)=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix},\ \boldsymbol{\mathcal{M}}(1,0)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

$$\boldsymbol{K}_f=\begin{bmatrix}1&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix},\ \boldsymbol{K}_v=\begin{bmatrix}0&0&0&-1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix},\ \overline{\boldsymbol{K}_v}=\begin{bmatrix}\boldsymbol{K}_v & 0\\ 0 & \boldsymbol{K}_v\end{bmatrix},$$

$$\mathbf{Q}^{\mathrm{f}}=\begin{bmatrix}1 & 1/2 & 1/3\\ 1/2 & 1/3 & 1/4\\ 1/3 & 1/4 & 1/5\end{bmatrix},\ \overline{\mathbf{Q}}=\begin{bmatrix}\mathbf{Q}^{\mathrm{f}} & 0\\ 0 & \mathbf{Q}^{\mathrm{f}}\end{bmatrix},\ \overline{\boldsymbol{\Theta}}^*=\begin{bmatrix}\overline{\boldsymbol{\Theta}}(0)\\ \overline{\boldsymbol{\Theta}}(\frac{1}{2})\\ \overline{\boldsymbol{\Theta}}(1)\end{bmatrix},\ \mathbf{G}=\begin{bmatrix}g(0)\\ g(1/2)\\ g(1)\end{bmatrix}.$$

The agumented matrix for this fundamental matrix is

$$[\mathbf{W};\mathbf{G}]=\begin{bmatrix}-1 & 1/2 & -1/6 & -2 & -3/2 & -17/6 & ; & -5/2\\ 2 & -2 & 0 & -1 & 1/2 & 1/6 & ; & -1\\ -1 & 1/2 & 7/6 & -3/2 & -15/8 & -53/12 & ; & -13/4\\ -3/2 & -9/8 & -2/3 & -1 & 1/2 & 7/6 & ; & -1/8\\ -1 & 1/2 & 13/6 & -1 & -2 & -6 & ; & 0\\ 1 & -1/2 & -5/6 & -1 & -1/2 & 13/6 & ; & 0\end{bmatrix}$$

From Eqn. (22), the matrix form for initial conditions is

$$[\mathbf{U};\lambda]=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & ; & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & ; & 0\end{bmatrix}.$$

Hence, the new augmented matrix based on conditions from system (29) can be obtained as follows

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]=\begin{bmatrix} -1 & 1/2 & -1/6 & -2 & -3/2 & -17/6 & ; & -5/2 \\ 2 & -2 & 0 & -1 & 1/2 & 1/6 & ; & -1 \\ -1 & 1/2 & 7/6 & -3/2 & -15/8 & -53/12 & ; & -13/4 \\ -3/2 & -9/8 & -2/3 & -1 & 1/2 & 7/6 & ; & -1/8 \\ 1 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & ; & 0 \end{bmatrix}.$$

By solving this system, the unknown Bell coefficients matrix is obtained as

$$\mathbf{A}=[0\quad 1\quad 0\quad 0\quad 2\quad 0]^T.$$

Substituting the elements of the column matrix into Eq. (3), we have $y_1(x)=x$, $y_2(x)=2x$ which are the exact solutions of the problem.

Example 5.2. Consider the system of the linear, delay Fredholm-Volterra integro-differential equation given by

$$\begin{cases} y_1' - y_2(x+1) = e^{-1} - e^{-(x+1)} + \int_0^1 y_1(t)dt + \int_0^x y_1(t)dt \\ y_2' + y_1(x-1) = e^{x-1} - e + \int_0^1 y_1(t)dt + \int_0^x y_2(t)dt \end{cases}$$

with the initial conditions $y_1(0)=0, y_2(0)=0$ and the exact solution $y_1(x)=e^x$ and $y_2(x)=e^{-x}$.Here,$m=1$,$J=2, g_1(x)=e^{-1}-e^{-(x+1)}, g_2(x)=e^{x-1}-e, Q_{11}^0=0, Q_{12}^0=-1, Q_{21}^0=1, Q_{22}^0=0, P_{11}^1=1, P_{12}^1=0, P_{21}^1=0, P_{22}^1=1, \alpha_{10}=1, \beta_{10}=1$ and $\alpha_{20}=1, \beta_{20}=-1$, $K_{11}^f(x,t)=1, K_{12}^f(x,t)=0$, $K_{21}^f(x,t)=1, K_{22}^f(x,t)=0, K_{11}^v(x,t)=1, K_{12}^v(x,t)=0, K_{21}^v(x,t)=0, K_{22}^v(x,t)=1$ and from the Eq. (25), the fundamental matrix equation of the problem is

$$\left\{\mathbf{P}_1\overline{\mathbf{X}}\overline{\mathbf{M}}\overline{\mathbf{S}}+\mathbf{Q}_0\overline{\boldsymbol{X}}\overline{\boldsymbol{\mathcal{M}}}(\alpha_0,\beta_0)(\overline{\mathbf{M}})^{\mathbf{0}}\overline{\mathbf{S}}-\overline{\mathbf{X}}\,\boldsymbol{K}_f\overline{\mathbf{Q}^f}\overline{\mathbf{S}}-\overline{\mathbf{X}}^*\overline{\boldsymbol{K}_v}\overline{\mathbf{\Theta}}^*\,\overline{\mathbf{S}}\right\}\mathbf{A}=\mathbf{G}$$

Therefore, necessary operations are calculated, we obtain the approximate solution by the Bell polynomials of the problem for $i=1{,}2$ and $N=6{,}8$ and 9 respectively

$$\begin{aligned} y_{1,6}(x) = 1 &+ 1.002695193763711x + 0.501508293061786x^2 \\ &+ 0.166630522608720x^3 + 0.043697945354716x^4 \\ &+ 0.006229284335894x^5 + 0.004158173664900x^6 \end{aligned}$$

$$\begin{aligned} y_{2,6}(x) = 1 &- 1.003663743331868x + 0.518401221221304x^2 \\ &- 0.193481788363343x^3 + 0.064154235060393x^4 \\ &- 0.018659015113252x^5 + 0.003168744341528x^6 \end{aligned}$$

$$\begin{aligned} y_{1,8}(x) = 1 &+ 1.000088966681897x + 0.500014529945630x^2 \\ &+ 0.166677709377466x^3 + 0.041685866848629x^4 \\ &+ 0.008292410152932x^5 + 0.001473446627363x^6 \\ &+ 1.315812272007100e-04x^7 + 7.354743809757294e-05x^8 \end{aligned}$$

$$y_{2,8}(x) = 1 - 1.000143525881152x + 0.500821438113966x^2 - 0.168244599318814x^3 + 0.043576441070650x^4 - 0.009870464523554x^5 + 0.002192834129431x^6 - 4.444385024704929e-04x^7 + 5.394143308902359e-05x^8$$

$$y_{1,9}(x) = 1 + 1.000365219821865x + 0.500010232764820x^2 + 0.166614728602149x^3 + 0.400341292738178x^4 + 0.0083x^5 + 0.100692130094458x^6 + 0.010659392770167x^7 + 3.926306115665755e-04x^8 + 5.721142912719674e-06x^9$$

$$y_{2,9}(x) = 1 - 0.999998678360542x + 0.499967405863498x^2 - 0.166487110941700x^3 + 0.041417181896475x^4 + 0.008094228602448x^5 + 0.001438737471000x^6 + 1.246312139079178e-04x^7 + 2.825678766922047e-06x^8 - 9.246006649727983e-07x^9.$$

Table 1. Comparison of the absolute errors of $y_1(x)$ for N= 6, 8, 9.

x	$y(x) = e^x$	$\|e_6(x)\|$	$\|e_8(x)\|$	$\|e_9(x)\|$
0	1	0	0	0
0.2	0.8187	1.6281e-04	1.9765e-05	3.1003e-07
0.4	0.6703	4.8181e-05	4.4079e-05	2.6883e-06
0.6	0.5488	1.0732e-03	7.4686e-05	9.3911e-06
0.8	0.4493	2.2546e-03	1.1699e-04	2.2193e-06
1	0.3679	4.7685e-03	1.8823e-05	4.1426e-05

Table 2. Comparison of the absolute errors of $y_2(x)$ for N= 6, 8, 9.

x	$y(x) = e^{-x}$	$\|e_6(x)\|$	$\|e_8(x)\|$	$\|e_9(x)\|$
0	1	0	0	0
0.2	0.8187	1.7843e-04	5.8590e-05	5.6909e-07
0.4	0.6703	2.4008e-04	9.0917e-06	6.7357e-07
0.6	0.5488	8.3380e-04	2.7886e-05	2.1252e-06
0.8	0.4493	1.4482e-03	4.5940e-04	3.4233e-06
1	0.3679	2.0402e-03	6.2185e-05	4.4943e-06

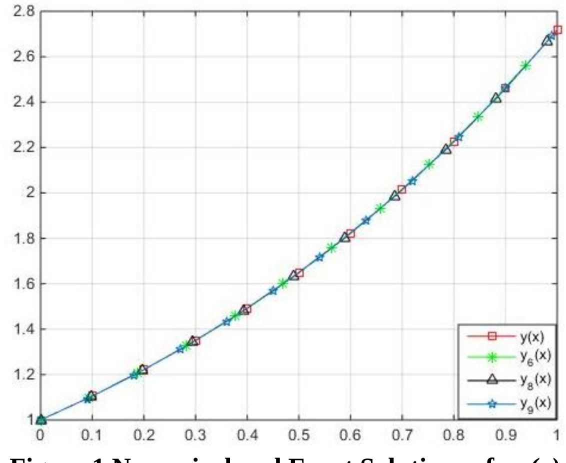

Figure 1.Numerical and Exact Solutions of $y_1(x)$ for N = 6,8,9

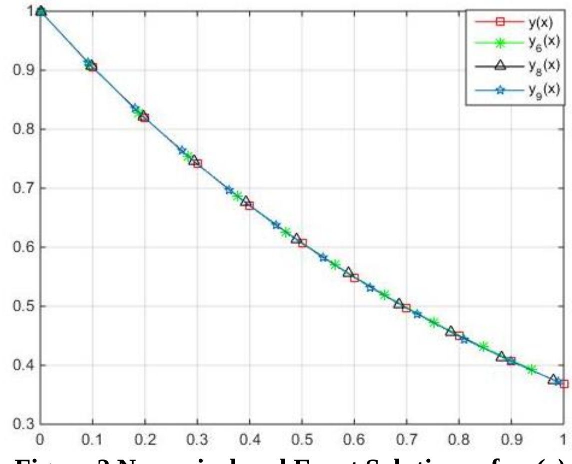

Figure 2.Numerical and Exact Solutions of $y_2(x)$ for N = 6,8,9

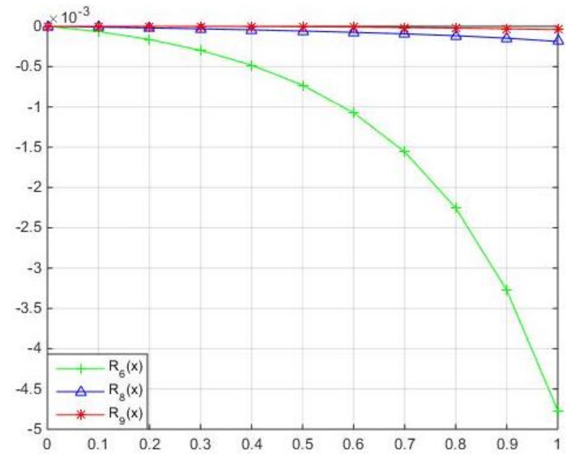

Figure 3. Residual Error Functions of $y_1(x)$ for N =6,8,9

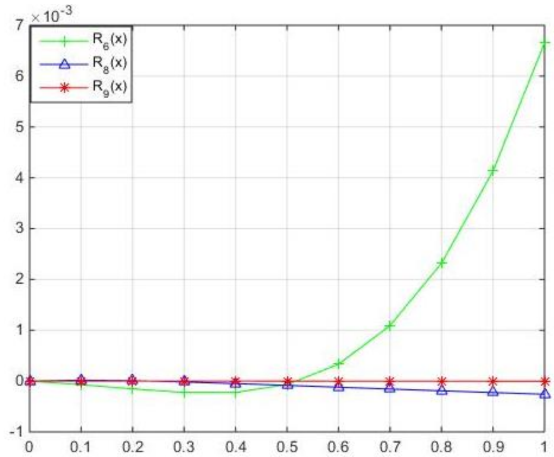

Figure 4. Residual Error Functions of $y_2(x)$ for N =6,8,9.

6.CONCLUSIONS

The system of delay Fredholm Volterra integro differential equations which plays a very significant role in physics and engineering, are generally difficult to solve analytically. For this reason, it is necessary to obtain approximate solutions. A method has been proposed for obtaining exact and approximate solutions. To illustrate the availability and applicability of this method, illustrative examples are solved, and an error analysis based on the residual function is implemented to show the accuracy of the results. Calculations related to the examples are performed on a computer with the help of a computer code written in Matlab. The method can be developed for application to systems of nonlinear differential equations, nonlinear integral systems; but some additional changes need to be revised.

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