ORIGINAL PAPER

ON DUAL BICOMPLEX BALANCING AND LUCAS-BALANCING NUMBERS

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Abstract. *In this paper, dual bicomplex Balancing and Lucas-Balancing numbers are defined, and some identities analogous to the classic properties of the Fibonacci and Lucas sequences are produced. We give the relationship between these numbers and Pell and Pell-Lucas numbers. From these, the basic bicomplex properties for the norm and its conjugate of these numbers are also developed. These in turn lead to the Binet formula, the generating functions and exponential generating functions, which are important concepts for number sequences. The Cassini identity, which is important for number sequences, actuallyemerged to solve the famous Curry paradox. We calculated the Cassini, Catalan, Vajda and d'Ocagne identities for these numbers.*

Keywords: *Balancing numbers; dual bicomplex Balancing numbers; Binet formula; Cassini identity.*

1. INTRODUCTION

A reason for continuing to develop basic properties of numbers like the Balancing numbers is that it provides the tools for extending applications of difference equations. An advantage of difference equations over differential equations can be for instance in medical applications in that medical data is generally extracted from the patient at discrete intervals. Assuming smooth transitions from one time-point to the next can result in ill-conditioning problems which are often ignored. For example, the Fibonacci rabbit population model is often regarded as one of the first studies of population growth using discrete mathematics [1]. Later, an analytic model of population dynamics was introduced by Volterra [2]. Dubeau [3], in revisiting the Fibonacci rabbit growth model, developed an approach that can be applied to population dynamics and epidemiology where censoring occurs either by inability to procreate or by death. Moreover, properties of generalized Fibonacci numbers can frequently throw light on the connections within the original properties; for example, where these connections are accidental, rather than essential or integral to the nature of the sequence. Sometimes these generalizations are elegant analogues of the original identities, but sometimes they are quite different as for instance with generating functions of powers of generalized Fibonacci numbers which depend in turn on generalizations of Simson's identity. Generalizations of many number sequences, especially Fibonacci numbers, have been studied and many generalizations have been made [4-13] and these provide templates.

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In particular, Balancing numbers $\{B_n\}_{n=0}^{\infty}$. were introduced by Behera and Panda [14]. The sequence of Balancing numbers has been studied extensively and many generalizations of these numbers have been made [15-25]. This paper extends them to the bicomplex numbers initiated by Corrado Segre in 1892 [26] and to quaternions. Quaternions have generated a growing interest in algebra by combining quaternions with algebraic objects [27]. Bicomplex numbers form an algebra over $\mathbb{C}$ of dimension two, and since $\mathbb{C}$ is of dimension two over $\mathbb{R}$, the bicomplex numbers are an algebra over $\mathbb{R}$ of dimension four. Bicomplex numbers are a generalization of complex numbers, similar to quaternions. Bicomplex numbers are commutative while quaternions are not commutative. Another difference is quaternions form a division algebra, but bicomplex quaternions do not form a division algebra [28].

The Balancing numbers sequence is denoted by $\{B_n\}_{n=0}^{\infty}$. The recurrence relation for Balancing numbers is

$$B_{n+1} = 6B_n - B_{n-1}, \qquad n \geq 2. \tag{1.1}$$

with initial terms, $B_1 = 1,\ \ B_2 = 6$ [23].

Note that, we can set $B_2 = 6B_1 - B_0,\ B_0 = 0$ by using (1.1)

The corresponding Binet formula for Balancing numbers is then

$$B_n = \frac{a^n - b^n}{a - b}$$

in which

$$a = 3 + \sqrt{8}, b = 3 - \sqrt{8}, ab = 1, a + b = 6 \ \ [23].$$

The Lucas-Balancing numbers sequence is denoted by $\{C_n\}_{n=0}^{\infty}$ with $C_n = \sqrt{8{B_n}^2 + 1}$ [24] and they satisfy the recurrence relation

$$C_{n+1} = 6C_n - C_{n-1}, \qquad n \geq 2. \tag{1.2}$$

with the initial terms, $C_1 = 3,\ \ C_2 = 17$.

Note that, we can set $C_2 = 6C_1 - C_0,\ C_0 = 1$ by using (1.2).

The Binet formula for Lucas-Balancing numbers is

$$C_n = \frac{a^n - b^n}{2} \ \ [24].$$

Alternatively, the following equations are also provided [17].

$$B_{n+1} = 3B_n + \sqrt{8{B_n}^2 + 1} \text{ and } B_{n-1} = 3B_n - \sqrt{8{B_n}^2 + 1}.$$

As a result of these equations, the following relationship is obtained.

$$B_{n+1} - B_{n-1} = 2C_n.$$

These sequences can be extended to negative indices n [24].

$$B_{-n} = -B_n \text{ and } C_{-n} = C_n.$$

There is a relationship between Balancing numbers and Pell numbers P_n as follows [25].

$$P_n = 2B_n. \tag{1.3}$$

Similarly, there is a relationship between Balancing numbers and Pell-Lucas numbers Q_n as follows [25].

$$B_n = P_n Q_n. \tag{1.4}$$

The Bicomplex number sequence is defined by

$$\mathbb{BC} = \{z_1 + z_2 j \mid z_1, z_2 \in \mathbb{C}, j^2 = -1\}$$

where z_1, z_2 are complex numbers. According to this definition, bicomplex numbers have the representation

$$z = z_1 + z_2 j = x + yi + (t + ki)j = x + yi + tj + kij$$

where $x, y, t, k \in \mathbb{R}$ [28].

This leads to the multiplication of bicomplex units as shown in Table 1.

Table 1. Multiplication rule of bicomplex units.

×	**1**	***i***	***j***	***ij***
1	1	i	j	ij
i	i	-1	ij	$-j$
j	j	ij	-1	$-i$
ij	ij	$-j$	$-i$	1

The three conjugates and the three moduli of bicomplex numbers are displayed in Table 2.

Table 2. Conjugates and moduli of bicomplex numbers [29].

Conjugates	**Moduli**
$z_i = x - yi + (t - ki)j$	$\|z\|_i = \sqrt{z \times z_i}$
$z_j = x + yi - (t + ki)j$	$\|z\|_j = \sqrt{z \times z_j}$
$z_{ij} = x - yi - (t - ki)j$	$\|z\|_{ij} = \sqrt{z \times z_{ij}}$

Dual numbers were introduced in 1873 by William Clifford and were used later by Eduard Study to represent the dual angle which measures the relative position of two skew lines in space. Study defined a dual angle as $\theta + d\varepsilon$, where θ is the angle between the directions of two lines in three-dimensional space and d is the distance between them. The $n-$ dimensional generalization, the Grassmann number, was introduced by Hermann Grassmann. They are expressions of the form $a + b\varepsilon$, where a and b are real numbers, and ε is a symbol taken to satisfy $\varepsilon^2 = 0$ [30]. Dual numbers are defined by

$$\mathbb{D} = \{a + b\varepsilon \mid \varepsilon^2 = 0, \qquad a, b \in \mathbb{R}\}.$$

Dual numbers are associated with other numbers and studied [31-35].

The Cassini identity emerged in the solution of the famous Curry paradox. Now let's explain the Curry paradox. From a right triangle with side lengths of 8 and 13 units, let's cut right triangles with side lengths of 8, 3 and 5, 2, respectively. These small triangles cut can be placed in two different ways according to the angles of the given triangle. In this case, a rectangle with side lengths of 5 and 3 units and an area of 15 square units is left. In the other case, the remaining rectangle has dimensions 8 × 2 and area 16. This is quite surprising. he applet below displays a single parameter (call it n) such that the dissection applies to a right triangle with legs $F_{n+1}$and F_{n-1}, where F_k is the kth Fibonacci number. The two rectangles then have dimensions $F_{n-1} \times F_{n-2}$ and $F_n \times F_{n-3}$ with areas that always differ by 1 which, like Cassini identity [36].

In this study, we define dual bicomplex Balancing and Lucas-Balancing numbers. This work is similar to the methods and techniques used in articles [31-35, 37, 38]. We give some identities analogous to the classic properties of the Fibonacci and Lucas sequences. We give the relationship between these numbers and Pell, Pell-Lucas numbers by Theorem 3.2. Thus, all existing connections between Pell, Pell-Lucas sequences and other sequences were established between dual bicomplex Balancing, Lucas-Balancing numbers sequences and other sequences by means of Theorem 3.2. For these numbers, we calculated the basic bicomplex properties, the norm and it's conjugate. We have given the generating function and the Binet formula, which are important for the number sequences of these numbers. We calculate the identities Cassini, Catalan, Vajda and d'Ocagne which are important for these numbers.

2. DUAL BICOMPLEX BALANCING AND LUCAS-BALANCING NUMBERS

Definition 2.1. Let $n \geq 0$ be integer, dual bicomplex Balancing and Lucas-Balancing numbers $\widetilde{BB}_n$ and $\widetilde{BC}_n$ are defined respectively by,

$$\begin{aligned}\widetilde{BB}_n &= BB_n + \varepsilon BB_{n+1} \\ &= B_n + B_{n+1}i_1 + B_{n+2}i_2 + B_{n+3}i_3 + \varepsilon(B_{n+1} + B_{n+2}i_1 + B_{n+3}i_2 + B_{n+4}i_3)\end{aligned}$$

and

$$\begin{aligned}\widetilde{BC}_n &= BC_n + \varepsilon BC_{n+1} \\ &= C_n + C_{n+1}i_1 + C_{n+2}i_2 + C_{n+3}i_3 + \varepsilon(C_{n+1} + C_{n+2}i_1 + C_{n+3}i_2 + C_{n+4}i_3)\end{aligned}$$

in which BB_n is the nth bicomplex Balancing number and BC_n is the nth bicomplex Lucas-Balancing number, and B_n is the nth Balancing number and C_n is the nth Lucas- Balancing numbers. $i_1, i_2, i_3,$ are imaginary units and ε is dual unit. That is,

$$i_1{}^2 = i_2{}^2 = i_3{}^2 = -1, \ \ \varepsilon^2 = 0, \varepsilon \neq 0$$
$$i_1 i_2 = i_2 i_1 = -i_3, \quad i_1 i_3 = i_3 i_1 = -i_2, \quad i_2 i_3 = i_3 i_2 = -i_1.$$

We also then have

$$\widetilde{BB_n} = (B_n + \varepsilon B_{n+1}) + (B_{n+1} + \varepsilon B_{n+2})i_1 + (B_{n+2} + \varepsilon B_{n+3})i_2 + (B_{n+1} + \varepsilon B_{n+2})i_3,$$

$$\widetilde{BB_n} = \widetilde{B_n} + \widetilde{B_{n+1}}i_1 + \widetilde{B_{n+2}}i_2 + \widetilde{B_{n+3}}i_3,$$

and

$$\widetilde{BC_n} = (C_n + \varepsilon C_{n+1}) + (C_{n+1} + \varepsilon C_{n+2})i_1 + (C_{n+2} + \varepsilon C_{n+3})i_2 + (C_{n+1} + \varepsilon C_{n+2})i_3,$$

$$\widetilde{BC_n} = \widetilde{C_n} + \widetilde{C_{n+1}}i_1 + \widetilde{C_{n+2}}i_2 + \widetilde{C_{n+3}}i_3$$

where, $\widetilde{B_n}$ is the nth dual Balancing number and $\widetilde{C_n}$ is the nth dual Lucas-Balancing number.

Theorem 2.2. We have

a) $\widetilde{BB_n} = \frac{1}{2}[(P_{2n} + \varepsilon P_{2n+2}) + (P_{2n+2} + \varepsilon P_{2n+4})i_1 + (P_{2n+4} + \varepsilon P_{2n+6})i_2$
$+(P_{2n+6} + \varepsilon P_{2n+8})i_3,$

b) $\widetilde{BB_n} = (P_n Q_n + \varepsilon P_{n+1} Q_{n+1}) + (P_{n+1} Q_{n+1} + \varepsilon P_{n+2} Q_{n+2})i_1$
$+(P_{n+2} Q_{n+2} + \varepsilon P_{n+3} Q_{n+3})i_2 + (P_{n+3} Q_{n+3} + \varepsilon P_{n+4} Q_{n+4})i_3.$

Proof: The proof can be easily done using the relations (1.3) and (1.4).

Definition 2.3. These sequences can be extended to negative indices n. $\widetilde{BB_{-n}}$ and $\widetilde{BC_{-n}}$ are defined respectively by

$$\widetilde{BB_{-n}} = -B_n - B_{n-1}i_1 - B_{n-2}i_2 - B_{n-3}i_3 - \varepsilon(B_{n-1} + B_{n-2}i_1 + B_{n-3}i_2 + B_{n-4}i_3)$$

and

$$\widetilde{BC_{-n}} = C_n + C_{n+1}i_1 + C_{n+2}i_2 + C_{n+3}i_3 + \varepsilon(C_{n+1} + C_{n+2}i_1 + C_{n+3}i_2 + C_{n+4}i_3).$$

Theorem 2.4. The recurrence relations for dual bicomplex Balancing and Lucas-Balancing numbers are as follows.

a) $\widetilde{BB_{n+1}} = 6\widetilde{BB_n} - \widetilde{BB_{n-1}},$
b) $\widetilde{BC_{n+1}} = 6\widetilde{BC_n} - \widetilde{BC_{n-1}}.$

Proof:
a) $\widetilde{BB_{n+1}} = \widetilde{B_{n+1}} + \widetilde{B_{n+2}}i_1 + \widetilde{B_{n+3}}i_2 + \widetilde{B_{n+4}}i_3$
$= (B_{n+1} + \varepsilon B_{n+2}) + (B_{n+2} + \varepsilon B_{n+3})i_1 + (B_{n+3} + \varepsilon B_{n+4}) + (B_{n+4} + \varepsilon B_{n+5})i_3$
$= \left((6B_n - B_{n-1}) + \varepsilon\big((6B_{n+1} - B_n)\big) + \big((6B_{n+1} - B_n) + \varepsilon(6B_{n+2} - B_{n+1})\big)i_1\right.$

$$+\big((6B_{n+2}-B_{n+1})+\varepsilon(6B_{n+3}-B_{n+2})\big)i_2$$
$$+\big((6B_{n+3}-B_{n+2})+\varepsilon(6B_{n+4}-B_{n+3})\big)i_3$$

$$=6\big((B_n+\varepsilon B_{n+1})+(B_{n+1}+\varepsilon B_{n+2})i_1+(B_{n+2}+\varepsilon B_{n+3})i_2+(B_{n+3}+\varepsilon B_{n+4})i_3\big)$$
$$-\big((B_{n-1}+\varepsilon B_n)+(B_n+B_{n+1})i_1+(B_{n+1}+B_{n+2})i_2+(B_{n+2}+B_{n+3})i_3\big)$$
$$=6\widetilde{BB}_n-\widetilde{BB_{n-1}}.$$

The proof of $b)$ is done similarly to $a)$.

Theorem 2.5. The Binet’s formula of $\widetilde{BB}_n$ and $\widetilde{BC}_n$ as follows.

a) $\widetilde{BB_n}=\frac{a^n\tilde{a}-b^n\tilde{b}}{a-b}$, (2.1)

b) $\widetilde{BC_n}=\frac{a^n\tilde{a}-b^n\tilde{b}}{2}$ (2.2)

where

$$\tilde{a}=\big(1+ai_1+a^2i_2+a^3i_3+\varepsilon(a+a^2i_1+a^3i_2+a^4i_3)\big)$$
$$\tilde{b}=\big(1+bi_1+b^2i_2+b^3i_3+\varepsilon(b+b^2i_1+b^3i_2+b^4i_3)\big).$$

Proof:

a) $\widetilde{BB_n}=(B_n+B_{n+1}i_1+B_{n+2}i_2+B_{n+3}i_3)+\varepsilon(B_{n+1}+B_{n+2}i_1+B_{n+3}i_2+B_{n+4}i_3)$ from Binet formula for the Balancing numbers, we have

$$=\left(\frac{a^n-b^n}{a-b}\right)+\left(\frac{a^{n+1}-b^{n+1}}{a-b}\right)i_1+\left(\frac{a^{n+2}-b^{n+2}}{a-b}\right)i_2+\left(\frac{a^{n+3}-b^{n+3}}{a-b}\right)i_3$$
$$+\varepsilon\left(\left(\frac{a^{n+1}-b^{n+1}}{a-b}\right)+\left(\frac{a^{n+2}-b^{n+2}}{a-b}\right)i_1+\left(\frac{a^{n+3}-b^{n+3}}{a-b}\right)i_2+\left(\frac{a^{n+4}-b^{n+4}}{a-b}\right)i_3\right)$$
$$=\left(\frac{a^n}{a-b}\right)\big(1+ai_1+a^2i_2+a^3i_3+\varepsilon(a+a^2i_1+a^3i_2+a^4i_3)\big)$$
$$-\left(\frac{b^n}{a-b}\right)\big(1+bi_1+b^2i_2+b^3i_3+\varepsilon(b+b^2i_1+b^3i_2+b^4i_3)\big)$$
$$=\frac{a^n\tilde{a}-b^n\tilde{b}}{a-b}.$$

The proof of $b)$ is done similarly to $a)$.

Corollary 2.6. We have

$$\boldsymbol{a})\ \widetilde{BB_{-n}}=\frac{b^n\tilde{a}-a^n\tilde{b}}{a-b},$$
$$\boldsymbol{b})\ \widetilde{BC_{-n}}=\frac{b^n\tilde{a}+a^n\tilde{b}}{2}.$$

Proof: From equation (2.1), we have

$$\boldsymbol{a})\ \widetilde{BB_{-n}} = \frac{a^{-n}\tilde{a} - b^{-n}\tilde{b}}{a-b}$$

$$= \frac{\frac{\tilde{a}}{a^n} - \frac{\tilde{b}}{b^n}}{a-b}$$

$$= \frac{b^n\tilde{a} - a^n\tilde{b}}{a^n b^n}\frac{1}{a-b}$$

$$= \frac{b^n\tilde{a} - a^n\tilde{b}}{a-b}.$$

The proof of $b)$ is done similarly to $a)$.

Theorem 2.7. The generating functions for $\widetilde{BB_n}$ and $\widetilde{BC_n}$ are given, respectively, by

$$\boldsymbol{a})\ g(x) = \sum_{n=0}^{\infty} \widetilde{BB_n}\, x^n = \frac{\tilde{a} - \tilde{b} + (\tilde{b}a - \tilde{a}b)x}{4\sqrt{2}(1-6x+x^2)},$$

$$\boldsymbol{b})\ h(x) = \sum_{n=0}^{\infty} \widetilde{BC_n}\, x^n = \frac{\tilde{a} - \tilde{b} + (\tilde{b}a - \tilde{a}b)x}{2(1-6x+x^2)},$$

$$\boldsymbol{c})\ k(x) = \sum_{n=0}^{\infty} \widetilde{BB_{-n}}\, x^n = \frac{\tilde{a} - \tilde{b} + (\tilde{b}b - \tilde{a}a)x}{4\sqrt{2}(1-6x+x^2)},$$

$$\boldsymbol{d})\ l(x) = \sum_{n=0}^{\infty} \widetilde{BC_{-n}}\, x^n = \frac{\tilde{a} - \tilde{b} + (\tilde{b}b - \tilde{a}a)x}{2(1-6x+x^2)}$$

Proof: By using (2.1), we have

$$\boldsymbol{a})\ g(x) = \sum_{n=0}^{\infty} \widetilde{BB_n}\, x^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{a^n\tilde{a} - b^n\tilde{b}}{a-b}\right)x^n$$

$$= \frac{\tilde{a}}{a-b}\sum_{n=0}^{\infty}(ax)^n - \frac{\tilde{b}}{a-b}\sum_{n=0}^{\infty}(bx)^n$$

$$= \left(\frac{\tilde{a}}{a-b}\right)\left(\frac{1}{1-ax}\right) - \left(\frac{\tilde{b}}{a-b}\right)\left(\frac{1}{1-bx}\right)$$

$$= \frac{\tilde{a} - \tilde{a}bx - \tilde{b} + \tilde{b}ax}{(a-b)(1-(a+b)x+abx^2)}$$

$$= \frac{\tilde{a} - \tilde{b} + (\tilde{b}a - \tilde{a}b)x}{4\sqrt{2}(1-6x+x^2)}.$$

The proofs of $b)$, $c)$ and $d)$ are done similarly to $a)$.

Theorem 2.8. The exponential generating functions for $\widetilde{BB}_n$ and $\widetilde{BC}_n$ are given by respectively, by

a) $g'(x) = \sum_{n=0}^{\infty} \widetilde{BB}_n \frac{x^n}{n!} = \frac{\tilde{a}e^{ax} - \tilde{b}e^{bx}}{a-b}$,

b) $h'(x) = \sum_{n=0}^{\infty} \widetilde{BC}_n \frac{x^n}{n!} = \frac{\tilde{a}e^{ax} + \tilde{b}e^{bx}}{2}$,

c) $k'(x) = \sum_{n=0}^{\infty} \widetilde{BB}_{-n} \frac{x^n}{n!} = \frac{\tilde{a}e^{bx} - \tilde{b}e^{ax}}{a-b}$,

d) $l'(x) = \sum_{n=0}^{\infty} \widetilde{BC}_{-n} \frac{x^n}{n!} = \frac{\tilde{a}e^{bx} + \tilde{b}e^{ax}}{2}$.

Proof:

a)
$$\begin{aligned} g'(x) &= \sum_{n=0}^{\infty} \widetilde{BB}_n \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{a^n\tilde{a} - b^n\tilde{b}}{a-b}\right)\frac{x^n}{n!} \\ &= \frac{\tilde{a}}{a-b}\sum_{n=0}^{\infty}\frac{(ax)^n}{n!} - \frac{\tilde{b}}{a-b}\sum_{n=0}^{\infty}\frac{(bx)^n}{n!} \\ &= \frac{\tilde{a}e^{ax} - \tilde{b}e^{bx}}{a-b}. \end{aligned}$$

The proofs of b), c) and d) are done similarly to a).

Definition 2.9. The conjugates of $\widetilde{BB}_n$ and $\widetilde{BC}_n$ are as follows.

a) $\overline{(\widetilde{BB}_n)}^{\varepsilon} = \overline{\widetilde{B_n}} + \overline{\widetilde{B_{n+1}}}i_1 + \overline{\widetilde{B_{n+2}}}i_2 + \overline{\widetilde{B_{n+3}}}i_3$,

b) $\overline{(\widetilde{BB}_n)}^{i_1} = \widetilde{B_n} - \widetilde{B_{n+1}}i_1 + \widetilde{B_{n+2}}i_2 - \widetilde{B_{n+3}}i_3$,

c) $\overline{(\widetilde{BB}_n)}^{i_2} = \widetilde{B_n} + \widetilde{B_{n+1}}i_1 - \widetilde{B_{n+2}}i_2 - \widetilde{B_{n+3}}i_3$,

d) $\overline{(\widetilde{BB}_n)}^{i_3} = \widetilde{B_n} - \widetilde{B_{n+1}}i_1 - \widetilde{B_{n+2}}i_2 + \widetilde{B_{n+3}}i_3$,

e) $\overline{(\widetilde{BC}_n)}^{\epsilon} = \overline{\widetilde{C_n}} + \overline{\widetilde{C_{n+1}}}i_1 + \overline{\widetilde{C_{n+2}}}i_2 + \overline{\widetilde{C_{n+3}}}i_3$,

f) $\overline{(\widetilde{BC}_n)}^{i_1} = \widetilde{C_n} - \widetilde{C_{n+1}}i_1 + \widetilde{C_{n+2}}i_2 - \widetilde{C_{n+3}}i_3$,

g) $\overline{(\widetilde{BC}_n)}^{i_2} = \widetilde{C_n} + \widetilde{C_{n+1}}i_1 - \widetilde{C_{n+2}}i_2 - \widetilde{C_{n+3}}i_3$,

h) $\overline{(\widetilde{BC}_n)}^{i_3} = \widetilde{C_n} - \widetilde{C_{n+1}}i_1 + \widetilde{C_{n+2}}i_2 + \widetilde{C_{n+3}}i_3$.

Theorem 2.10. There are the norms for numbers $\widetilde{BB}_n$ and $\widetilde{BC}_n$ as follows.

a) $\left(N(\widetilde{BB}_n)^{i_1}\right)^{\mathbf{2}} = (\widetilde{B_n})^2 + (\widetilde{B_{n+1}})^2 - (\widetilde{B_{n+2}})^2 + (\widetilde{B_{n+3}})^2 + 2i_2(\widetilde{B_{n+3}}\widetilde{B_{n+1}} + \widetilde{B_n}\widetilde{B_{n+2}})$,

b) $\left(N(\widetilde{BB}_n)^{i_2}\right)^{\mathbf{2}} = (\widetilde{B_n})^2 + (\widetilde{B_{n+1}})^2 + (\widetilde{B_{n+2}})^2 + (\widetilde{B_{n+3}})^2 + 2i_1(\widetilde{B_n}\widetilde{B_{n+1}} + \widetilde{B_{n+2}}\widetilde{B_{n+3}})$,

c) $\left(N\left(\widetilde{BB_n}\right)^{i_3}\right)^{\mathbf{2}} = \left(\widetilde{B_n}\right)^2 + \left(\widetilde{B_{n+1}}\right)^2 + \left(\widetilde{B_{n+2}}\right)^2 + \left(\widetilde{B_{n+3}}\right)^2 + 2i_3\left(\widetilde{B_n}\widetilde{B_{n+3}} - \widetilde{B_{n+1}}\widetilde{B_{n+2}}\right),$
d) $\left(N\left(\widetilde{BC_n}\right)^{i_1}\right)^{\mathbf{2}} = \left(\widetilde{C_n}\right)^2 + \left(\widetilde{C_{n+1}}\right)^2 - \left(\widetilde{C_{n+2}}\right)^2 + \left(\widetilde{C_{n+3}}\right)^2 + 2i_2\left(\widetilde{C_{n+3}}\widetilde{C_{n+1}} + \widetilde{C_n}\widetilde{C_{n+2}}\right),$
e) $\left(N\left(\widetilde{BC_n}\right)^{i_2}\right)^{\mathbf{2}} = \left(\widetilde{C_n}\right)^2 + \left(\widetilde{C_{n+1}}\right)^2 + \left(\widetilde{C_{n+2}}\right)^2 + \left(\widetilde{C_{n+3}}\right)^2 + 2i_1\left(\widetilde{C_n}\widetilde{C_{n+1}} + \widetilde{C_{n+2}}\widetilde{C_{n+3}}\right),$
f) $\left(N\left(\widetilde{BC_n}\right)^{i_2}\right)^{\mathbf{2}} = \left(\widetilde{C_n}\right)^2 + \left(\widetilde{C_{n+1}}\right)^2 + \left(\widetilde{C_{n+2}}\right)^2 + \left(\widetilde{C_{n+3}}\right)^2 + 2i_2\left(\widetilde{C_n}\widetilde{C_{n+3}} - \widetilde{C_{n+1}}\widetilde{C_{n+2}}\right).$

Proof:

a)
$$\begin{aligned}\left(N\left(\widetilde{BB_n}\right)^{i_1}\right)^{\mathbf{2}} &= \left(\widetilde{BB_n}\right)\overline{\left(\widetilde{BB_n}\right)}^{i_1} \\ &= \left(\widetilde{B_n}\right)^2 + \widetilde{B_n}\widetilde{B_{n+1}}i_1 + \widetilde{B_n}\widetilde{B_{n+2}}i_2 + \widetilde{B_n}\widetilde{B_{n+3}}i_3 - \widetilde{B_n}\widetilde{B_{n+1}}i_1 + \left(\widetilde{B_{n+1}}\right)^2 \\ &\quad -\widetilde{B_{n+1}}\widetilde{B_{n+2}}i_3 + \widetilde{B_{n+1}}\widetilde{B_{n+3}}i_2 + \widetilde{B_n}\widetilde{B_{n+2}}i_2 + \widetilde{B_{n+1}}\widetilde{B_{n+2}}i_3 - \left(\widetilde{B_{n+2}}\right)^2 \\ &\quad - \widetilde{B_{n+2}}\widetilde{B_{n+3}}i_1 + \widetilde{B_n}\widetilde{B_{n+3}}i_3 - \widetilde{B_n}\widetilde{B_{n+3}}i_3 + \widetilde{B_{n+1}}\widetilde{B_{n+3}}i_2 + \widetilde{B_{n+1}}\widetilde{B_{n+2}}i_3 + \left(\widetilde{B_{n+3}}\right)^2 \\ &= \left(\widetilde{B_n}\right)^2 + \left(\widetilde{B_{n+1}}\right)^2 - \left(\widetilde{B_{n+2}}\right)^2 + \left(\widetilde{B_{n+3}}\right)^2 + 2i_2\left(\widetilde{B_{n+3}}\widetilde{B_{n+1}} + \widetilde{B_n}\widetilde{B_{n+2}}\right).\end{aligned}$$

The proofs of $b)$, $c)$, $d)$, $e)$ and $f)$ are done similarly to $a)$.

Theorem 2.11. We have

a) $\overline{\left(\widetilde{BB_n}\right)}^{\varepsilon} + \widetilde{BB_n} = 2BB_n,$
b) $\overline{\left(\widetilde{BB_n}\right)}^{i_1} + \overline{\left(\widetilde{BB_n}\right)}^{i_2} = 2\left(\widetilde{B_n} - \widetilde{B_{n+3}}i_3\right),$
c) $\overline{\left(\widetilde{BB_n}\right)}^{i_1} + \overline{\left(\widetilde{BB_n}\right)}^{i_3} = 2\left(\widetilde{B_n} - \widetilde{B_{n+1}}i_1\right),$
d) $\overline{\left(\widetilde{BB_n}\right)}^{i_2} + \overline{\left(\widetilde{BB_n}\right)}^{i_3} = 2\left(\widetilde{B_n} - \widetilde{B_{n+2}}i_2\right),$
e) $\overline{\left(\widetilde{BB_n}\right)}^{i_1} - \overline{\left(\widetilde{BB_n}\right)}^{i_2} = 2\left(\widetilde{B_{n+2}}i_2 - \widetilde{B_{n+1}}i_1\right),$
f) $\overline{\left(\widetilde{BB_n}\right)}^{i_1} - \overline{\left(\widetilde{BB_n}\right)}^{i_3} = 2\left(\widetilde{B_{n+1}}i_1 - \widetilde{B_{n+2}}i_2\right),$
g) $\overline{\left(\widetilde{BB_n}\right)}^{i_2} - \overline{\left(\widetilde{BB_n}\right)}^{i_3} = 2\left(\widetilde{B_{n+1}}i_1 - \widetilde{B_{n+3}}i_3\right),$
h) $\overline{\left(\widetilde{BB_n}\right)}^{i_1} + \overline{\left(\widetilde{BB_n}\right)}^{i_2} + \overline{\left(\widetilde{BB_n}\right)}^{i_3} = 3\widetilde{B_n} - \widetilde{B_{n+1}}i_1 - \widetilde{B_{n+2}}i_2 - \widetilde{B_{n+3}}i_3,$
i) $\overline{\left(\widetilde{BC_n}\right)}^{\varepsilon} + \widetilde{BC_n} = 2BC_n,$
j) $\overline{\left(\widetilde{BC_n}\right)}^{i_1} + \overline{\left(\widetilde{BC_n}\right)}^{i_2} = 2\left(\widetilde{C_n} - \widetilde{C_{n+3}}i_3\right),$
k) $\overline{\left(\widetilde{BC_n}\right)}^{i_1} + \overline{\left(\widetilde{BC_n}\right)}^{i_3} = 2\left(\widetilde{C_n} - \widetilde{C_{n+1}}i_1\right),$
l) $\overline{\left(\widetilde{BC_n}\right)}^{i_2} + \overline{\left(\widetilde{BC_n}\right)}^{i_3} = 2\left(\widetilde{C_n} - \widetilde{C_{n+2}}i_2\right),$
m) $\overline{\left(\widetilde{BC_n}\right)}^{i_1} - \overline{\left(\widetilde{BC_n}\right)}^{i_2} = 2\left(\widetilde{C_{n+2}} - \widetilde{C_{n+1}}i_1\right),$
n) $\overline{\left(\widetilde{BC_n}\right)}^{i_1} - \overline{\left(\widetilde{BC_n}\right)}^{i_3} = 2\left(\widetilde{C_{n+1}} - \widetilde{C_{n+2}}i_2\right),$
o) $\overline{\left(\widetilde{BC_n}\right)}^{i_2} - \overline{\left(\widetilde{BC_n}\right)}^{i_3} = 2\left(\widetilde{C_{n+1}} - \widetilde{C_{n+3}}i_3\right),$
p) $\overline{\left(\widetilde{BC_n}\right)}^{i_1} + \overline{\left(\widetilde{BC_n}\right)}^{i_2} + \overline{\left(\widetilde{BC_n}\right)}^{i_3} = 3\widetilde{C_n} - \widetilde{C_{n+1}}i_1 - \widetilde{C_{n+2}}i_2 - \widetilde{C_{n+3}}i_3.$

Proof: The proofs follow immediately from the definitions, as before.

Theorem 2.12. There are the following equations.

a) $\widetilde{BB_{n+1}} - \widetilde{BB_{n-1}} = 2\widetilde{BC}_n$,
b) $\widetilde{BC_{n+1}} - 3\widetilde{BC}_n = 8\widetilde{BB}_n$,
c) $\widetilde{BC_{n-1}} - 3\widetilde{BC}_n = -8\widetilde{BB}_n$,
d) $\widetilde{BC_{n+1}} - \widetilde{BC_{n-1}} = 16\widetilde{BB}_n$,
e) $\widetilde{BB_{n-1}} = 3\widetilde{BB}_n - \widetilde{BC_n}$.

Proof:
a) $\widetilde{BB_{n+1}} - \widetilde{BB_{n-1}}$
$= (B_{n+1} + \varepsilon B_{n+2}) + (B_{n+2} + \varepsilon B_{n+3})i_1 + (B_{n+3} + \varepsilon B_{n+4})i_2 + (B_{n+4} + \varepsilon B_{n+5})i_3$
$- (B_{n-1} + \varepsilon B_n) + (B_n + \varepsilon B_{n+1})i_1 + (B_{n+1} + \varepsilon B_{n+2})i_2 + (B_{n+2} + \varepsilon B_{n+3})i_3$
$= \big((B_{n+1} - B_{n-1}) + \varepsilon(B_{n+2} - B_n)\big) + \big((B_{n+2} - B_n) + \varepsilon(B_{n+3} - B_{n+1})\big)i_1$
$+\big((B_{n+3} - B_{n+1}) + \varepsilon(B_{n+4} - B_{n+2})\big)i_2 + \big((B_{n+4} - B_{n+2}) + \varepsilon(B_{n+5} - B_{n+3})\big)i_3,$
from $B_{n+1} - B_{n-1} = 2C_n$, we have
$= (2C_n + 2\varepsilon C_{n+1}) + (2C_{n+1} + 2\varepsilon C_{n+2})i_1 + (2C_{n+2} + 2\varepsilon C_{n+3})i_2 + (2C_{n+3} + 2\varepsilon C_{n+4})i_3$
$= 2\widetilde{BC}_n$.

The proofs of $b), c), d)$ and $e)$ are done similarly to $a)$.

Theorem 2.13. (Cassini identity) We have

$$\textbf{a})\ \widetilde{BB_{n-1}}\widetilde{BB_{n+1}} - \widetilde{BB}_n^{\ 2} = \frac{b\tilde{a}\tilde{b} - a\tilde{b}\tilde{a}}{a-b},$$

$$\textbf{b})\ \widetilde{BC_{n-1}}\widetilde{BC_{n+1}} - \widetilde{BC}_n^{\ 2} = \frac{a-b}{4}\big(b\tilde{a}\tilde{b} - a\tilde{b}\tilde{a}\big).$$

Proof. ***a***) $\widetilde{BB_{n-1}}\widetilde{BB_{n+1}} - \widetilde{BB}_n^{\ 2}$

$$= \left(\frac{a^{n-1}\tilde{a} - b^{n-1}\tilde{b}}{a-b}\right)\left(\frac{a^{n+1}\tilde{a} - b^{n+1}\tilde{b}}{a-b}\right) - \left(\frac{a^n\tilde{a} - b^n\tilde{b}}{a-b}\right)^2$$
$$= \frac{-a^{n-1}b^{n+1}\tilde{a}\tilde{b} - b^{n-1}a^{n+1}\tilde{b}\tilde{a} + a^n b^n\tilde{a}\tilde{b} + b^n a^n\tilde{b}\tilde{a}}{(a-b)^2}$$
$$= \frac{a^n b^n\tilde{a}\tilde{b}\left(1 - \frac{b}{a}\right) - b^n a^n\tilde{b}\tilde{a}\left(\frac{a}{b} - 1\right)}{(a-b)^2}$$
$$= \frac{(a-b)\left(\frac{\tilde{a}\tilde{b}}{a} - \frac{\tilde{b}\tilde{a}}{b}\right)}{(a-b)^2} = \frac{b\tilde{a}\tilde{b} - a\tilde{b}\tilde{a}}{a-b}.$$

The proof of $b)$ is done similarly to $a)$.

Theorem 2.14. (Catalan identity) We have

$$\textbf{a})\ \widetilde{BB_{n-t}}\widetilde{BB_{n+t}} - \widetilde{BB}_n^{\ 2} = B_t\left(\frac{b^t\tilde{a}\tilde{b} - a^t\tilde{b}\tilde{a}}{a-b}\right),$$

$$\textbf{b})\ \widetilde{BC_{n-t}}\widetilde{BC_{n+t}} - \widetilde{BC}_n^{\ 2} = \frac{C_t}{2}\big(b^t\tilde{a}\tilde{b} - a^t\tilde{b}\tilde{a}\big).$$

Proof:

a) $\widetilde{BB_{n-1}}\widetilde{BB_{n+1}}-\widetilde{BB_n}^2$

$$=\left(\frac{a^{n-t}\tilde{a}-b^{n-t}\tilde{b}}{a-b}\right)\left(\frac{a^{n+t}\tilde{a}-b^{n+t}\tilde{b}}{a-b}\right)-\left(\frac{a^{n}\tilde{a}-b^{n}\tilde{b}}{a-b}\right)^2$$

$$=\frac{-a^{n-t}b^{n+t}\tilde{a}\tilde{b}-b^{n-t}a^{n+t}\tilde{b}\tilde{a}+a^nb^n\tilde{a}\tilde{b}+b^na^n\tilde{b}\tilde{a}}{(a-b)^2}$$

$$=\frac{a^nb^n\tilde{a}\tilde{b}\left(1-\left(\frac{b}{a}\right)^t\right)-b^na^n\tilde{b}\tilde{a}\left(\left(\frac{a}{b}\right)^t-1\right)}{(a-b)^2}$$

$$=\frac{(a^t-b^t)\left(\frac{\tilde{a}\tilde{b}}{a^t}-\frac{\tilde{b}\tilde{a}}{b^t}\right)}{(a-b)^2}=B_t\left(\frac{b^t\tilde{a}\tilde{b}-a^t\tilde{b}\tilde{a}}{a-b}\right).$$

Note that, if we take $t=1$, we return to the Cassini identity.
The proof of b) is done similarly to a).

Theorem 2.15. (Vajda identity) We have

$$\textbf{\textit{a}})\ \widetilde{BB_{n+r}}\widetilde{BB_{n-r}}-\widetilde{BB_n}\ \widetilde{BB_{n+r+k}}=B_r\left(\frac{a^k\tilde{b}\tilde{a}-b^k\tilde{a}\tilde{b}}{a-b}\right),$$

$$\textbf{\textit{b}})\ \widetilde{BC_{n+r}}\widetilde{BC_{n-r}}-\widetilde{BC_n}\widetilde{BC_{n+r+k}}=\frac{C_r}{2}\left(\frac{a^k\tilde{b}\tilde{a}-b^k\tilde{a}\tilde{b}}{a-b}\right).$$

Proof:

a) $\widetilde{BB_{n+r}}\widetilde{BB_{n-r}}-\widetilde{BB_n}\ \widetilde{BB_{n+r+k}}$

$$=\left(\frac{a^{n+r}\tilde{a}-b^{n+r}\tilde{b}}{a-b}\right)\left(\frac{a^{n-r}\tilde{a}-b^{n-r}\tilde{b}}{a-b}\right)-\left(\frac{a^{n}\tilde{a}-b^{n}\tilde{b}}{a-b}\right)\left(\frac{a^{n+r+k}\tilde{a}-b^{n+r+k}\tilde{b}}{a-b}\right)$$

$$=\frac{-a^{n+r}b^{n-r}\tilde{a}\tilde{b}-b^{n+r}a^{n-r}\tilde{b}\tilde{a}+a^nb^{n+r+k}\tilde{a}\tilde{b}+b^na^{n+r+k}\tilde{b}\tilde{a}}{(a-b)^2}$$

$$=\frac{a^nb^{n+k}\tilde{a}\tilde{b}(b^r-a^r)+b^na^{n+k}\tilde{b}\tilde{a}(a^r-b^r)}{(a-b)^2}$$

$$=\frac{(a^r-b^r)\left(a^k\tilde{b}\tilde{a}-b^k\tilde{a}\tilde{b}\right)}{(a-b)^2}=B_r\left(\frac{a^k\tilde{b}\tilde{a}-b^k\tilde{a}\tilde{b}}{a-b}\right).$$

The proof of b) is done similarly to a).

Theorem 3.12. (d'Ocagne identity) For $n\leq m$, we have

$$\textbf{\textit{a}})\ \widetilde{BB_m}\widetilde{BB_{n+1}}-\widetilde{BB_n}\ \widetilde{BB_{m+1}}=B_{m-n}\left(\frac{a\tilde{b}\tilde{a}-b\tilde{a}\tilde{b}}{a-b}\right),$$

$$\boldsymbol{b})\ \widetilde{BC_m}\widetilde{BC_{n+1}} - \widetilde{BC_n}\widetilde{BC_{m+1}} = \frac{C_{m-n}}{2}\left(a\tilde{b}\tilde{a} - b\tilde{a}\tilde{b}\right).$$

Proof:

$\boldsymbol{a})\ \widetilde{BB_{n+r}}\widetilde{BB_{n-r}} - \widetilde{BB_n}\ \ \widetilde{BB_{n+r+k}}$

$$= \left(\frac{a^m\tilde{a} - b^m\tilde{b}}{a-b}\right)\left(\frac{a^{n+1}\tilde{a} - b^{n+1}\tilde{b}}{a-b}\right) - \left(\frac{a^n\tilde{a} - b^n\tilde{b}}{a-b}\right)\left(\frac{a^{m+1}\tilde{a} - b^{m+1}\tilde{b}}{a-b}\right)$$

$$= \frac{-a^m b^{n+1}\tilde{a}\tilde{b} - b^m a^{n+1}\tilde{b}\tilde{a} + a^n b^{m+1}\tilde{a}\tilde{b} + b^n a^{m+1}\tilde{b}\tilde{a}}{(a-b)^2}$$

$$= \frac{a^n b^{n+1}\tilde{a}\tilde{b}(b^{m-n} - a^{m-n}) + b^n a^{n+1}\tilde{b}\tilde{a}(a^{m-n} - b^{m-n})}{(a-b)^2}$$

$$= \frac{(a^{m-n} - b^{m-n})\left(a\tilde{b}\tilde{a} - b\tilde{a}\tilde{b}\right)}{(a-b)^2}$$

$$= B_{m-n}\left(\frac{a\tilde{b}\tilde{a} - b\tilde{a}\tilde{b}}{a-b}\right).$$

The proof of $b)$ is done similarly to $a)$.

3. CONCLUSIONS

Dual bicomplex Balancing and Lucas-Balancing numbers have been defined with properties analogous to the classic properties of the Fibonacci and Lucas sequences such as the Binet type general formula. Given that their defining second order, homogenous, linear recurrence relation is a particular case of Horadam's well-known

$$H_n = pH_{n-1} - qH_{n-2} \quad , n \geq 2$$

with p = 6 and q =1, the related Cassini, Catalan, Vajda and d'Ocagne identity types developed in this paper could be further developed from varying the coefficients in the recurrence relations as in [11]. Also, we found the relationship between these numbers and Pell, Pell-Lucas numbers by Theorem 2.2. Thus, all existing connections among Pell, Pell Lucas sequences and other sequences have been established among dual bicomplex balancing and other second order sequences by this theorem. By way of conclusion, here are three examples where light has yet to be shone succinctly and which may interest conference participants. The references can assist with these extensions which can build on Balancing number generalizations of the Fibonacci sequence.

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