ORIGINAL PAPER

WAVELET BASED PARAMETER ESTIMATION OF MIXTURE OF GAUSSIANS

SHRAVANI MOTHE[1], SRINIVAS CHARY VARANASI[1], MURALI GUNDAGANI[2]

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Abstract. *This paper presents a family of new continuous wavelet function and describes few properties of it. A method of estimating parameters for a Gaussians using this new continuous wavelet transform is described. The robustness of this method for estimating the parameters in the presence of noise is also studied by simulating different possible cases.*
Keywords: *Wavelet Transform, properties; time-frequency bandwidth; Gaussian distribution.*

1. INTRODUCTION

Continuous wavelet transform (CWT) is used to analyze how the frequency content of a signal changes over time [1]. Most of the continuous wavelets are used in applications involving the detection of minute change of signals over a continuous set of scales [2]. The applications of CWT range from finding a crack or damage in materials via non destructive testing [3, 4], seismic studies, analysis of financial time series [5-7], speech processing, pulmonary microvascular pressure [8], geophysical, metrological studies [9, 10] etc.

In this paper we present an application of CWT for estimating parameters of Gaussians using a new mother wavelet function. We also study few interesting properties of the proposed new wavelet function.In this paper we intend to develop a new family of continuous wavelets, study some important properties of it and establish a relation between the transform coefficients with fifth member of the proposed wavelet and parameters of Gaussian distribution at different scales. We also propose a schema of parameter estimation of Gaussian distribution using this new wavelet function.

2. MATHEMATICAL FORMULATION

2.1. CONTINUOUS WAVELET TRANSFORM

A wavelet is an oscillatory, real or complex-valued function $\psi(x) \in L^2(R)$infinite interval with an average value of zero. The square integrable measurable function $\psi(x)$ is called a mother wavelet. The mother wavelet$\psi(x)$is localized in both time and frequency

[1]Malla Reddy University, Mathematics, 500043 Hyderabad, India. E-mail: sravithareddy11@gmail.com; srinivaschary.varanasi@gmail.com.
[2]Sreenidhi University, Mathematics, 501301 Hyderabad, India. E-mail: murali.maths81@gmail.com.

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domains and generates afamily of wavelets $\psi_{a,b}(x)$formulated as$\psi_{a,b}(x) = \frac{1}{\sqrt{b}}\psi\left(\frac{x-a}{b}\right)$, where the numbers $a \in R$ and $b \in R^{+}$ denote the scale and translation parameters, respectively.

For a given real signal $f(x) \in L^2(R)$, the continuous wavelet transform(CWT) is defined as the inner product of the signal function with the wavelet functions [Mallat, Daubechies, Antoine]

$$Wf(a,b) = \langle f, \psi_{a,b}\rangle = \frac{1}{\sqrt{b}}\int_{-\infty}^{\infty} f(x)\,\psi\left(\frac{x-a}{b}\right)dx,$$

where $Wf(a,b)$ is called a wavelet coefficient for the wavelet $\psi_{a,b}(x)$ and it measures the variation of the signalin the vicinity of a whose size is proportional to b.

The choice of the mother wavelet depends on the nature of the application at hand which gives a scope for closer understanding of the application and provides motivation for development of new wavelet functions. Here we study few existing continuous wavelet functions and their properties.

The non-redundant wavelet transform using discrete wavelets de-correlates the wavelet coefficients at octave scale resolutions and is often inadequate for applications demanding identification of features on a finer scale as might be required in crack or damage detection in metal plates. Hence, for practical implementation we choose adequately discretized time (a) and frequency (b) scales and approximating the CWT integral as a summation over this grid.Mexican hat wavelet and Morlet wavelet are frequently used continuous wavelet functions for various applications.

The explicit expression for the Mexican wavelet is

$$\psi(t) = (1-t^2)e^{-\frac{t^2}{2}}.$$

It is the negative normalized second derivative of a Gaussian function.The Morlet wavelet (or Gabor wavelet) is a wavelet composed of cos(5t) multiplied by a Gaussian window(envelope) i.e,$\psi(t) = \cos(5t)\,e^{-t^2}$

2.2 NEW CONTINUOUS WAVELET

The family is built with different orders starting from the continuous function $\theta(t) = \frac{t}{1+t^2}$and taking the k^{th} derivative of $\theta(t)$.The integer k is the parameter of this family and represents the order of this family i.e. $\psi_k(t)$is a continuous wavelet for each k and$\psi_k(t) = N_k\frac{d^k}{dt^k}\theta(t)$.

The constant N_k has to be chosen$\|\psi_k(t)\|^2 = 1$, imposing this condition we get

$$N_k = \left(\int_{-\infty}^{\infty}\left|\frac{d^k}{dt^k}\theta(t)\right|^2 dt\right)^{-1/2}.$$

Table1 provides the values of $N_k$for different values of $k = 1,2,3,4,5,6,7,8$.

Table1.Values of C_{ψ_k} *and* N_k for different orders of *the* Wavelet $(k) = 1,2,3,4,5,6,7,8$.

Order of the Wavelet (k)	C_{ψ_k}	N_k
1	3.1416	1.1284
2	-1.5708	0.6515
3	1.0472	0.2379
4	-0.7854	0.0636
5	0.6280	0.0134
6	-0.5088	0.0023
7	0.4491	0.000346
8	-0.3926	0.000044652

The zero area under the wavelet property i.e., $\int_{-\infty}^{\infty} \psi_k(t)dt = 0$ can also easily be verified.The inverse Wavelet transform is guaranteed by the existence of admissibility condition

$$C_{\psi_k} = \int_0^{\infty} \frac{|\widehat{\Psi}_k(f)|^2}{f} df < \infty.$$

where $\widehat{\Psi}_k(f)$is the Fourier transform of $\psi_k(t)$. Table2 provides the values of C_{ψ_k} for different values of k.

Table2. Estimated σusing new wavelet (k= 3).

SNR in db	Estimated σ
100	0.5020
90	0.5019
80	0.5014
70	0.5031
60	0.5020
50	0.5269
40	0.4537
30	0.3548
20	0.2309
10	2.0986

The finiteness of C_{ψ_k}ensures the conservation of signal energy in both time and transform domain with the wavelet function ψ_k.Hence the functions $\psi_k(t)\,\forall\, k = 1,2,3,4,5,6,7,8$ forms a family of one dimensional real continuous admissible wavelets.

Vanishing moments for the proposed wavelet i.e., $\int_{-\infty}^{\infty} t^n \psi_k(t)\, dt = 0$ of order $n = 0,1,2,\dots,k-1$verified for $k = 1,2,3,4,5,6,7,8$.Regularity is infinity for all Wavelets since these continuous wavelets are infinitely differentiable, so by definition of regularity if a function is infinitely differentiable continuous functions belongs to C^n for all n, where C^n is set of infinitely differentiable continuous functions.

The n^{th} derivative of function is

$$\frac{d^n}{dx^n}\left(\frac{x}{1+x^2}\right) = (-1)^n n! \sum_{\alpha=\alpha_i} (x-\alpha_i)^{-1-n}$$

where $\alpha_i$is the root of (z^2+1) in the complex domain. Frequency decay is exponential.

3. METHOD OF SOLUTION

3.1. CONTINUOUS WAVELET TRANSFORM OF GAUSSIAN DISTRIBUTION WITH THE NEW WAVELET

The continuous wavelet transform of a continuous time signal$x(t)$, using the any mother wavelet is defined as

$$CT_{x(t)}^{\psi_k}(a,b) = \frac{1}{\sqrt{a}} \int_{\infty}^{\infty} x(t)\psi_k\left(\frac{t-b}{a}\right) dt$$

where a is the dilation parameter (scale) of the wavelet and$b$ is the location parameter of the wavelet. The Gaussian function is of the form $G(t:\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$with mean$\mu$and standard deviation$\sigma$. For different values of $\mu$and σ, the plots of $G(t:\mu,\sigma)$are given in Fig. 1.

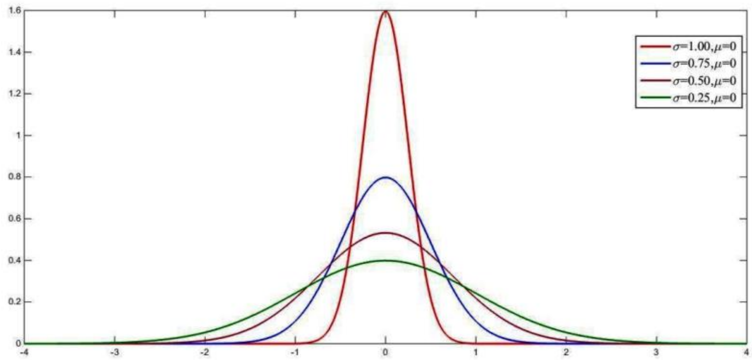

Figure 1. Gaussian curves G(t:μ, σ).

The continuous wavelet transform of Gaussian curve $G(t:\mu,\sigma)$is given by

$$CT_{G(t:\mu,\sigma)}^{\psi_k}(a,b) = \frac{1}{\sigma\sqrt{2a\pi}} \int_{\infty}^{\infty} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \psi_k\left(\frac{t-b}{a}\right) dt$$

substituting

$$\frac{t-\mu}{\sigma} = x \Rightarrow \frac{dt}{\sigma} = dx$$

$$= \frac{1}{\sqrt{2a\pi}} \int_{\infty}^{\infty} e^{-\frac{x^2}{2}} \psi_k\left(\frac{\sigma x-(b-\mu)}{a}\right) dx$$

$$= \frac{1}{\sqrt{\sigma}} \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{a/\sigma}} \psi_k\left(\frac{x-(b-\mu)/\sigma}{a/\sigma}\right) dx$$

Let,

$$\frac{a}{\sigma} = a'$$

and

$$\frac{b-\mu}{\sigma} = b' \Rightarrow a = \sigma a', b = \sigma b' + \mu$$

$$= \frac{1}{\sqrt{\sigma}} \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{a'}} \psi_k \left(\frac{x - b'}{a'} \right) dx$$

$$= \frac{1}{\sqrt{\sigma}} \int_{\infty}^{\infty} G(t: 0, 1) \frac{1}{\sqrt{a'}} \psi_k \left(\frac{x - b'}{a'} \right) dx$$

$$= \frac{1}{\sqrt{\sigma}} CT_{G(t:0,1)}^{\psi_k}(a', b')$$

Therefore the coefficients $CT_{G(t:\mu,\sigma)}^{\psi_k}(a,b)$ and $\sigma$of Gaussian function $G(t: \mu, \sigma)$are related by

$$CT_{G(t:\mu,\sigma)}^{\psi_k}(a'\sigma, \mu + b'\sigma) = \frac{1}{\sqrt{\sigma}} CT_{G(t:0,1)}^{\psi_k}(a', b') \tag{1}$$

A special case when $\mu = 0$, equation (*) provides a relation between scaled wavelet coefficients

$$CT_{G(t:0,\sigma)}^{\psi_k}(a'\sigma,\ b'\sigma) = \frac{1}{\sqrt{\sigma}} CT_{G(t:0,1)}^{\psi_k}(a', b') \tag{2}$$

Figs. 2-3 show the plots of $CT_{G(t:\mu,\sigma)}^{\psi_k}$for $\sigma = 0.25, 0.5, 0.75,1$and μ= 0,1 (k=3).

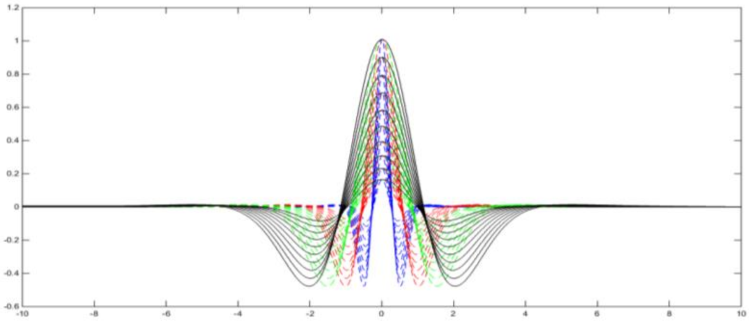

Figure 2. The plot of the coefficients$CT_{G(t:\mu,\sigma)}^{\psi_k}(a,b) for \mu = 0, \sigma = 0.25, 0.5, 0.75, 1.0$where: black lines indicates $CT_{G(t:0,1)}$(a=28:4:64,b= –10:0.001:10)vs. *b*; green doted lines lines indicates $\sqrt{\sigma}\left(CT_{G(t:0,0.75)}(\sigma a, \sigma b)\right)$*vs.b*; red doted lines lines indicates$\sqrt{\sigma}\left(CT_{G(t:0,0.50)}(\sigma a, \sigma b)\right)$*vs.b*;blue doted lines lines indicates $\sqrt{\sigma}\left(CT_{G(t:0,0.25)}(\sigma a, \sigma b)\right)$*vs.b.*

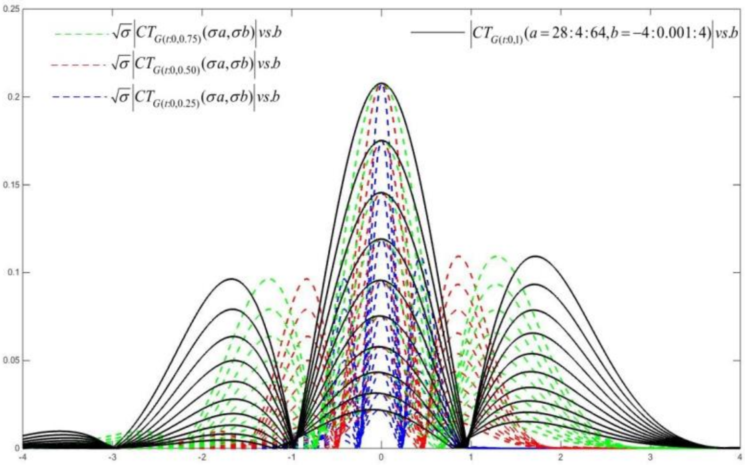

Figure 3. The plot of the coefficients$\left|CT_{G(t:\mu,\sigma)}^{\psi_k}(a,b)\right|$ $for\ \mu = 0, \sigma = 0.25, 0.5, 0.75, 1.0.$

3.2. ESTIMATION OF σ AND μ OF THE GAUSSIAN CURVE USING COEFFICIENT OF ITS WAVELET TRANSFORM

It is clear from the above relation and the coefficient plots that the curves corresponding to each scale a' is symmetric aboutμ. So the value ofμcan be estimated from the point of symmetry.

To compute the value of σ,taking $b' = 0$ in equation (2) we get

$$\sqrt{\sigma}\left|CT^{\psi_k}_{G(t:0,\sigma)}(a'\sigma,0)\right| = \left|CT^{\psi_k}_{G(t:0,1)}(a',0)\right|$$
$$\sqrt{\sigma} = \left|CT^{\psi_k}_{G(t:0,1)}(a',0)\right| / \left|CT^{\psi_k}_{G(t:0,\sigma)}(a'\sigma,0)\right| \quad (3)$$

3.3. EFFECT ON WAVELET COEFFICIENTS ON ADDITION OF WHITE GAUSSIAN NOISE TO THE GAUSSIAN CURVE

By adding white Gaussian noise to the Gaussian function with any $\sigma$and μ(Usingawgn($G(t:\mu,\sigma)$, snr) function in matlab where snr specifies the signal-to-noise ratio per sample, in dB) and computing its wavelet transform, it is found that the PSNR (Peak Signal-to-Noise Ratio)of the computed wavelet coefficient with the coefficient obtain without adding noise of the Gaussian function with respective$\sigma$and μturns out to the approximately the snr of the noise added to the Gaussian function.

3.4. ESTIMATION OF σ OF THE GAUSSIAN CURVE WITH ADDITION OF NOISE USING ITS COEFFICIENT OF ITS WAVELET TRANSFORM

When the Gaussian curve is subjected to the white Gaussian noise with appropriate SNR values in db(shown in Fig. 4) and determine the its wavelet coefficients and then applying the above method to estimate the value σwe have notice that the SNR values greater than or equal to 50 this method is not able to identify approximately the correct value of sigma and for the SNR values less than 50 there is a large change in the value of σ.

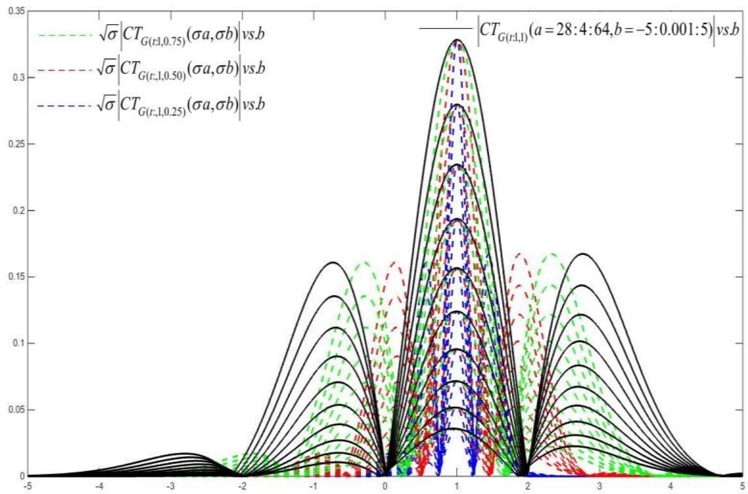

Figure 4. The plot of the coefficients $\left|\left[CT_{G(t:\mu=1,\sigma=0.25,0.5,0.75,1)}(a,b)\right]_{\psi_k}\right|$.

In Table 2, the estimated values of σ with different values of SNR are shown. In that we have taken the Gaussian curve with σ = 0.5 and added white Gaussian noise with different values of snr using awgn function in MATLAB.

In Fig. 5, the plots of the wavelet (k=3) coefficients were shown after adding white Gaussian noise to the Gaussian function with σ = 0.5 and μ = 0 (snr =10,20,…100)

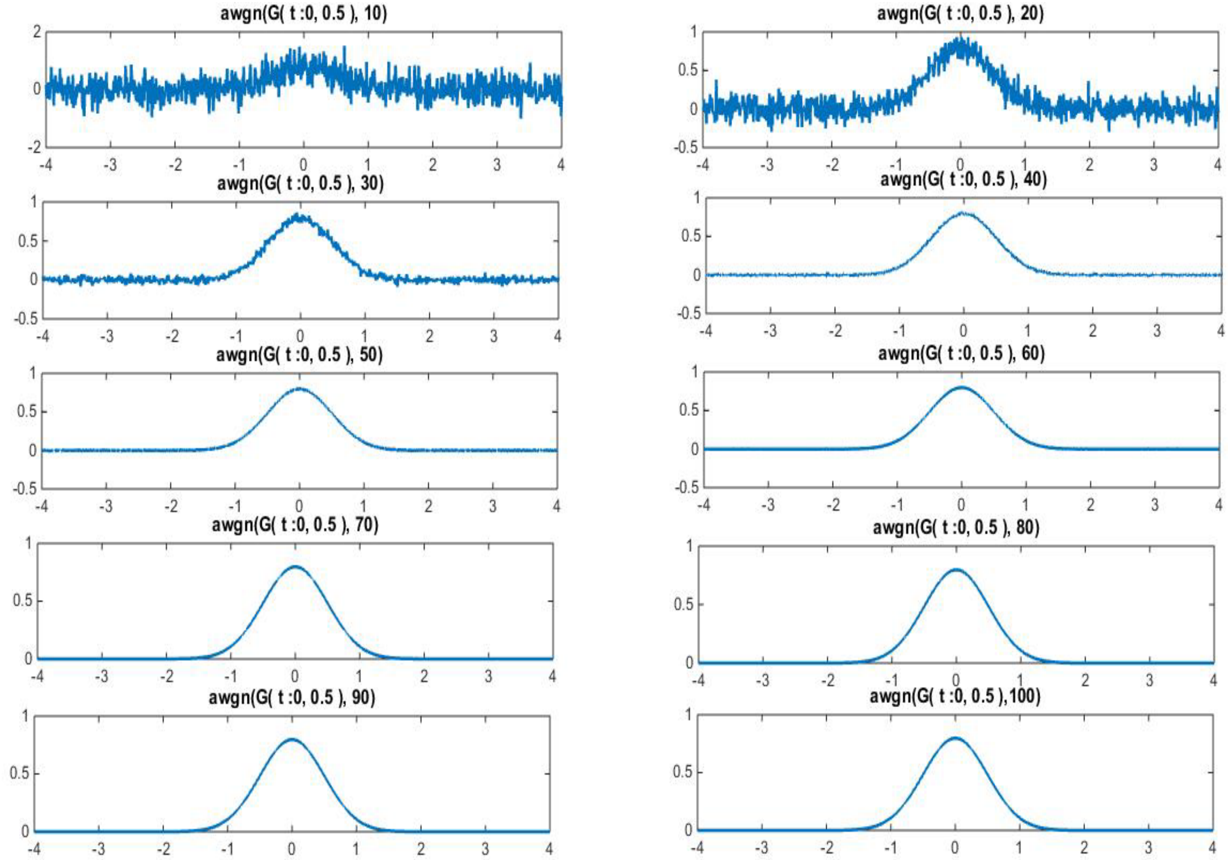

Figure 5. The plots of the wavelet (k=3) coefficients after adding white Gaussian noise to the Gaussian function with σ = 0.5 and μ = 0 (snr =10,20,…100)

4. CONCLUSION

Identify the relationship between the coefficients of wavelet transform obtain after decomposing the Gaussian curve using the new continuous wavelets at different scale and position with the standard deviation and mean of the Gaussian curve. Estimated σ values approximately with and without adding noise to the Gaussian curve.

REFERENCES

[1] Daubechies, I., *IEEE Transactions on Information Theory*, **36**(5), 961, 1990.
[2] Antoine, J.P., Murenzi, R., Vandergheynst, P., Twareque A.S., *Two-Dimensional Wavelets and their Relatives*, Cambridge University Press, New York, 2004.
[3] Brani Vidakovic, *Statistical Modeling with Wavelets*, Wiley, 1999.

[4] Damelin S.B., W. Miller, Jr, *The Mathematics of Signal Processing*, Cambridge University Press, New York. 2012.
[5] Daniel T.L., Lee, A. Y., *Hewlett Packard Journal*, **45**(6), 44, 1994.
[6] Shravani, M., Krishna Reddy, B., *Aryabhatta Journal of Mathematics and Informatics* **07**(01), 78, 2015.
[7] Aguiar-Conrariia, L., Soares, M.J., *The Continuous Wavelet Transform: A Primer,* Working Paper Series, Universidade do Minho, 2011. Available online http://www3.eeg.uminho.pt/economia/nipe/docs/2011/nipe_wp_16_2011.pdf.
[8] Sandowsky, J., *Johns Hopkins APL Technical Digest*, **17**(3), 258, 1996.
[9] Wavelet Toolbox Software, Latest Version, 2020.
[10] Mallat, S., Hwang, W.L., *IEEE Transaction on Information Theory*, **38**, 617, 1992.