ORIGINAL PAPER

HIGHER-ORDER ACCELERATIONS AND POLES UNDER THE ONE-PARAMETER PLANAR DUAL MOTIONS AND THEIR INVERSE MOTIONS

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Abstract. *In this paper, after a brief summary of one-parameter planar dual motion [1], the higher-order accelerations and poles are analyzed under the one-parameter planar dual motion and its inverse.*
Keywords: *Dual numbers; planar dual motion; kinematics.*

1. INTRODUCTION

Similar to complex numbers and hyperbolic numbers (split-complex numbers), dual numbers are extended the real numbers by adjoining one unreal element ϵ, where $\epsilon^2 = 0$. So, we can write the dual numbers set as below: $\mathbb{D} = \left\{ z = x + \epsilon y \mid x, y \in \mathbb{R}, \quad \epsilon^2 = 0, \quad \epsilon \neq 0 \right\}$.

Although the algebra of dual numbers firstly considered by Clifford, W.K. [2], its first application to kinematics and mechanics by Kotelnikov and Study [3]. The set of dual numbers $\mathbb{D}$, is two-dimensional, unit, commutative and associative ring over the real numbers. Dual plane can be formed by composition of all the dual numbers. The dual inner product and the modulus of a dual number $z = x + \epsilon y$ are defined by Yüce and Akar [1] and denoted as $\langle , \rangle_d$, $\|z\|_d$ respectively. Then, if $\|z\|_d = 0$, special modulus $\|z\|_\delta = |y|$ of the dual number z is defined. Also, every dual number z with nonzero modulus can be written as $z = r(cosg\varphi + \epsilon sing\varphi) = re^{\epsilon\varphi}$ and a dual rotation by $e^{\epsilon\varphi}$ corresponds to multiplication by the matrix [1]

$$\begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix} \tag{1}$$

The higher-order accelerations and Poles were investigated under one-parameter planar complex motion by Müller [4] and one-parameter planar hyperbolic motion by Şahin and Yüce [5]. We are going to analyze the higher-order accelerations and Poles under one-parameter planar dual motion in a similar manner. In the paper [6], one-parameter motions on the Galilean plane were defined. Some Holditch-Type Theorems for the polar moments of inertia of the closed orbit curves presented during the 1-parameter closed homothetic motion [7, 8].

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2. ONE-PARAMETER PLANAR DUAL MOTION

Let $\mathbb{D}$ and $\mathbb{D}'$ be moving and fixed dual planes and $\{O, \mathbf{d_1}, \mathbf{d_2}\}$, $\{O', \mathbf{d}'_1, \mathbf{d}'_2\}$ be their orthonormal coordinate systems, respectively. If the vector $\overrightarrow{\mathbf{OO}'}$ is represented by the dual number $\mathbf{u}'$, then the motion can be defined by the transformation below:

$$\mathbf{z}' = \mathbf{u}' + \mathbf{z}e^{\epsilon\varphi} \tag{2}$$

This transformation is called a one-parameter planar dual motion and denoted by $\mathbb{D}/\mathbb{D}'$, where φ is the rotation angle of the motion $\mathbb{D}/\mathbb{D}'$; that is, the dual angle between the vectors $\mathbf{d_1}$ and $\mathbf{d}'_1$ and the dual numbers $\mathbf{z} = x_1 + \epsilon x_2$, $\mathbf{z}' = x'_1 + \epsilon x'_2$ represent the point $Z \in \mathbb{D}$ with respect to the moving and the fixed perpendicular coordinate systems, respectively.
Besides, the rotation angle φ and $\mathbf{z}$, $\mathbf{z}'$, $\mathbf{u}'$ are continuously differentiable functions of a time parameter $t \in \mathbb{I} \subset \mathbb{R}$ and at a initial time $t = 0$ the coordinate systems are coincident [1].

Let the dual number $\mathbf{u} = u_1 + \epsilon u_2$ represent the origin of the fixed point system with respect to the moving system. Then, if we take $Z' = O'$, we obtain $\mathbf{z}' = \mathbf{0}$ and $\mathbf{z} = \mathbf{u}$. Thus, we can obtain $\mathbf{u}'$ from the Eq. (2)

$$\mathbf{u}' = -\mathbf{u}e^{\epsilon\varphi}. \tag{3}$$

If Z is a moving point of $\mathbb{D}$, then the velocity of Z with respect to $\mathbb{D}$ is known as the relative velocity of the motion $\mathbb{D}/\mathbb{D}'$ and is denoted by V_r. This vector can be written as $\mathbf{V_r} = \dfrac{d\mathbf{z}}{dt} = \dot{\mathbf{z}}$. This vector can be expressed with respect to $\mathbb{D}'$ by the equation below:

$$\mathbf{V}'_\mathbf{r} = \mathbf{V_r}e^{\epsilon\varphi} = \dot{\mathbf{z}}e^{\epsilon\varphi}. \tag{4}$$

If we differentiate the Eq. (2) with respect to t, we obtain the absolute velocity of the motion $\mathbb{D}/\mathbb{D}'$ as follows:

$$\mathbf{V}'_\mathbf{a} = \frac{d\mathbf{z}'}{dt} = \dot{\mathbf{z}}' = \dot{\mathbf{u}}' + (\dot{\mathbf{z}} + \epsilon\dot{\varphi}\mathbf{z})e^{\epsilon\varphi}, \tag{5}$$

where

$$\mathbf{V}'_\mathbf{f} = \dot{\mathbf{u}}' + \epsilon\dot{\varphi}\mathbf{z}e^{\epsilon\varphi}, \tag{6}$$

is the sliding velocity of the motion $\mathbb{D}/\mathbb{D}'$. By differentiating the Eq. (3) with respect to t, we also obtain the following equation:

$$\dot{\mathbf{u}}' = -(\dot{\mathbf{u}} + \epsilon\mathbf{u}\dot{\varphi})e^{\epsilon\varphi}. \tag{7}$$

Hence, we can rewrite the sliding velocity as follows:

$$\mathbf{V}'_\mathbf{f} = \epsilon\dot{\varphi}\mathbf{z}e^{\epsilon\varphi} - (\dot{\mathbf{u}} + \epsilon\mathbf{u}\dot{\varphi})e^{\epsilon\varphi}. \tag{8}$$

For a general one-parameter planar dual motions, there is a point that does not move, which means that its coordinates are the same in both reference coordinate systems $\{O;\mathbf{d}_1,\mathbf{d}_2\}$ and $\{O';\mathbf{d}'_1,\mathbf{d}'_2\}$. This point is called the pole point or the instantaneous rotation pole center. In this case, we obtain $\mathbf{V}'_\mathbf{f}=\mathbf{0}$ and use the Eq. (8), then for the pole point $P=p_1+\epsilon p_2$ of the motion, we get

$$\epsilon\mathbf{p}=\epsilon\mathbf{u}+\frac{\dot{\mathbf{u}}}{\dot{\varphi}} \tag{9}$$

so, from the Eqs. (8) and (9), we can rewrite the sliding velocity with the aid of the pole point as below:

$$\mathbf{V}'_\mathbf{f}=\epsilon\dot{\varphi}(\mathbf{z}-\mathbf{p})e^{\epsilon\varphi}. \tag{10}$$

For more properties of one-parameter planar dual motion the reader can see the [1].

3. HIGHER-ORDER ACCELERATIONS AND POLES UNDER ONE-PARAMETER PLANAR DUAL MOTION

3.1. HIGHER-ORDER ACCELERATIONS UNDER THE ONE-PARAMETER PLANAR DUAL MOTIONS

Let the motion $\mathbb{D}/\mathbb{D}'$ be the one-parameter planar Dual motion and let $Z\in\mathbb{D}$ be a fixed point. Then, the absolute velocity and the sliding velocity are equal to each other and this velocity is given as below:

$$\mathbf{V}'_\mathbf{a}=\mathbf{V}'_\mathbf{f}=\dot{\mathbf{z}}'=\dot{\mathbf{u}}'+\epsilon\dot{\varphi}(\mathbf{z}'-\mathbf{u}'). \tag{11}$$

If we differentiate the Eq. (11) with respect to t, then we obtain the second-order velocity (absolute velocity or sliding velocity) as below:

$$\ddot{\mathbf{z}}'=\ddot{\mathbf{u}}'+\epsilon\ddot{\varphi}(\mathbf{z}'-\mathbf{u}')+\epsilon\dot{\varphi}(\dot{\mathbf{z}}'-\dot{\mathbf{u}}'). \tag{12}$$

or from the Eq. (11) and latter equation, we obtain the second-order velocity (or the first-order acceleration) as below:

$$\ddot{\mathbf{z}}'=\ddot{\mathbf{u}}'+\epsilon\ddot{\varphi}(\mathbf{z}'-\mathbf{u}'). \tag{13}$$

Then, if we differentiate the Eq. (13) with respect to t, then we obtain the third-order velocity (or the second-order acceleration) as below:

$$\dddot{\mathbf{z}}'=\dddot{\mathbf{u}}'+\epsilon\dddot{\varphi}(\mathbf{z}'-\mathbf{u}'). \tag{14}$$

If we differentiate the Eq. (14) with respect to t, then we obtain the fourth-order velocity (or the third-order acceleration) as below:

$$\overset{(4)}{\mathbf{z}'} = \overset{(4)}{\mathbf{u}'} + \epsilon \overset{(4)}{\varphi}(\mathbf{z}' - \mathbf{u}'). \tag{15}$$

If we continue to subsequent differentiations, we can get higher-order velocities and accelerations. So, we can give the following theorem:

Theorem 1. During the one-parameter planar dual motion $\mathbb{D}/\mathbb{D}'$, if $Z \in \mathbb{D}$ is a fixed point, then we get

$$\overset{(\mathbf{n})}{\mathbf{z}'} = \overset{(\mathbf{n})}{\mathbf{u}'} + \epsilon \overset{(n)}{\varphi}(\mathbf{z}' - \mathbf{u}') \tag{16}$$

for the velocities from n th order and the acceleration from $(n-1)$ th order for each t.

Proof: We are going to use the induction method. First of all we can see above that the Eq. (16) is true for $n=1$, $n=2$, $n=3$ and $n=4$. Now we assume that the Eq. (16) is true for $n=k$ and prove that it is true for $n=k+1$. Hence, if we assume that

$$\overset{(\mathbf{k})}{\mathbf{z}'} = \overset{(\mathbf{k})}{\mathbf{u}'} + \epsilon \overset{(k)}{\varphi}(\mathbf{z}' - \mathbf{u}') \tag{17}$$

is true, we can obtain

$$\overset{(\mathbf{k+1})}{\mathbf{z}'} = \overset{(\mathbf{k+1})}{\mathbf{u}'} + \epsilon \overset{(k+1)}{\varphi}(\mathbf{z}' - \mathbf{u}') + \epsilon \overset{(k)}{\varphi}(\dot{\mathbf{z}}' - \dot{\mathbf{u}}') \tag{18}$$

by differentiation the latter expression. So, from the Eq. (11) and latter equation we have

$$\overset{(\mathbf{k+1})}{\mathbf{z}'} = \overset{(\mathbf{k+1})}{\mathbf{u}'} + \epsilon \overset{(k+1)}{\varphi}(\mathbf{z}' - \mathbf{u}'). \tag{19}$$

Consequently, we have been proved theorem (11) by the induction method.

3.2. HIGHER-ORDER POLES UNDER THE ONE-PARAMETER PLANAR DUAL MOTIONS

During the one-parameter planar dual motion $\mathbb{D}/\mathbb{D}'$, if $Z \in \mathbb{D}$ is a fixed point, then from the Eq. (11), we can calculate the first-order pole point as follows:

$$\epsilon \mathbf{p}_1' = \epsilon \mathbf{z}' = \epsilon \mathbf{u}' - \frac{\dot{\mathbf{u}}'}{\dot{\varphi}}. \tag{20}$$

By using the Eq. (13), we obtain the second-order pole point as below:

$$\epsilon \mathbf{p}_2' = \epsilon \mathbf{z}' = \epsilon \mathbf{u}' - \frac{\ddot{\mathbf{u}}'}{\ddot{\varphi}}. \tag{21}$$

In a similar way, from the Eq. (16), we can get the $(n-1)$th-order acceleration pole as shown below

$$\epsilon \mathbf{p}'_n = \epsilon \mathbf{z}' = \epsilon \mathbf{u}' - \frac{\overset{(\mathbf{n})}{\mathbf{u}'}}{\overset{(\mathbf{n})}{\boldsymbol{\varphi}}} . \tag{22}$$

Also, from the Eq. (22) we have the following:

$$\overset{(n)}{u'} = \epsilon \overset{(n)}{\varphi} (\mathbf{u}' - \mathbf{p}'_n) . \tag{23}$$

So, if we use the Eq. (16) and the latter equation, we can rewrite $\overset{(\mathbf{n})}{\mathbf{z}'}$ in terms of $\mathbf{p}'_n$ as follows:

$$\overset{(\mathbf{n})}{\mathbf{z}'} = \epsilon \overset{(n)}{\varphi} (\mathbf{z}' - \mathbf{p}'_n) . \tag{24}$$

We can give the following theorems via the Eq. (24):

Theorem 2. The absolute value of higher-order accelerations under the one-parameter planar dual motion is rational with distance between the point Z and corresponding pole point. Higher order accelerations take the same value over the circles centered the pole points.

Theorem 3. The vectors $\overset{(\mathbf{n})}{\mathbf{z}'}$ and $\mathbf{P}'_\mathbf{n}\mathbf{Z}'$ are perpendicular each other.

Example 1. Let us consider the one-parameter planar dual motion with $u(t) = e^{-t} + \epsilon e^{-t}, \varphi(t) = e^{-t}$. In this case, on the plane $\mathbb{D}'$ we show the orbit curve of point $1 + 2\epsilon \in \mathbb{D}$ by using Maple programming for $t \in [-1, 5]$.

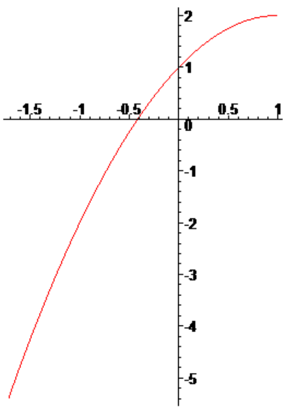

Figure 1. The orbit curve of point $1 + 2\epsilon \in \mathbb{D}$ **.**

We calculate higher-order accelerations and poles under this motion

$$\overset{(\mathbf{n})}{\mathbf{z}'} = e^{-t} + 2^n e^{-2t} \epsilon \text{, if } n \text{ is odd,}$$

$$\overset{(\mathbf{n})}{\mathbf{z}'} = -e^{-t} - 2^n e^{-2t} \epsilon \text{, if } n \text{ is even}$$

and

$$\epsilon \mathbf{p}'_n = 1 + (1 + (2^n - 1) e^{-t}) \epsilon .$$

4. THE INVERSE MOTION OF THE ONE-PARAMETER PLANAR DUAL MOTION

In the inverse motion of the one-parameter planar dual motion $\mathbb{D}$ and $\mathbb{D}'$ fixed and moving dual planes and $\{O,\mathbf{d}_1,\mathbf{d}_2\}$, $\{O',\mathbf{d}_1{}',\mathbf{d}_2{}'\}$ are their orthonormal coordinate systems, respectively. In this motion all of the velocities, accelerations and poles are analyzed in $\mathbb{D}$, instead of $\mathbb{D}'$. This is the main difference from the one-parameter planar dual motion. So, from the Eq. (2), we can obtain the transformation of the inverse motion of the one-parameter planar dual motion as below:

$$\mathbf{z} = (\mathbf{z}' - \mathbf{u}')e^{-\epsilon\varphi}$$

and the Eq. (3) can be written as follows:

$$\mathbf{z} = \mathbf{u} + \mathbf{z}'e^{-\epsilon\varphi}. \tag{25}$$

and denoted by $\mathbb{D}'/\mathbb{D}$, also called one-parameter planar inverse dual motion, where $-\varphi$ is the rotation angle of the motion $\mathbb{D}'/\mathbb{D}$. In analogy to the motion $\mathbb{D}/\mathbb{D}'$, we can get the velocities, accelerations and poles, for the motion $\mathbb{D}'/\mathbb{D}$. But we want to focus on higher-order accelerations and poles.

4.1. HIGHER-ORDER ACCELERATIONS UNDER THE ONE-PARAMETER PLANAR INVERSE DUAL MOTIONS

During the motion $\mathbb{D}'/\mathbb{D}$, Z' be a fixed point of $\mathbb{D}'$ dual plane and denoted by $\mathbf{z}' \in \mathbb{D}'$. Under the motion $\mathbb{D}'/\mathbb{D}$ with the $\mathbf{z}' \in \mathbb{D}'$ is a fixed point, the absolute velocity and the sliding velocity are equal to each other and this velocity is given as below:

$$\dot{\mathbf{z}} = \dot{\mathbf{u}} - \epsilon\dot{\varphi}\mathbf{z}'e^{-\epsilon\varphi}.$$

It is convenient to write the latter equation as follows:

$$\dot{\mathbf{z}} = \dot{\mathbf{u}} - \epsilon\dot{\varphi}(\mathbf{z} - \mathbf{u}). \tag{26}$$

If we differentiate the Eq. (26) with respect to t, then we obtain the second-order velocity (absolute velocity or sliding velocity) as below:

$$\ddot{\mathbf{z}} = \ddot{\mathbf{u}} - \epsilon\ddot{\varphi}(\mathbf{z} - \mathbf{u}) - \epsilon\dot{\varphi}(\dot{\mathbf{z}} - \dot{\mathbf{u}})$$

or from the Eq. (26) and latter equation, we obtain the second-order velocity (or the first-order acceleration) as below:

$$\ddot{\mathbf{z}} = \ddot{\mathbf{u}} - \epsilon\ddot{\varphi}(\mathbf{z} - \mathbf{u}). \tag{27}$$

If we differentiate the Eq. (27) with respect to t, then we obtain the third-order velocity (or the second-order acceleration) such as,

$$\dddot{\mathbf{z}} = \dddot{\mathbf{u}} - \epsilon\dddot{\varphi}(\mathbf{z}-\mathbf{u}) \,. \tag{28}$$

Similarly, if we differentiate the Eq. (28) with respect to t, then we obtain the fourth-order velocity (or the third-order acceleration) as below:

$$\overset{(4)}{\mathbf{z}} = \overset{(4)}{\mathbf{u}} - \epsilon \overset{(4)}{\boldsymbol{\varphi}}(\mathbf{z}-\mathbf{u}) \,. \tag{29}$$

If we continue to subsequent differentiations, we can get higher-order velocities and accelerations. So, we can give the following theorem:

Theorem 4. During the inverse motion of the one-parameter planar dual motion $\mathbb{D}'/\mathbb{D}$, if $Z' \in \mathbb{D}'$ is a fixed point then, we get

$$\overset{(n)}{\mathbf{z}} = \overset{(n)}{\mathbf{u}} - \epsilon \overset{(n)}{\boldsymbol{\varphi}}(\mathbf{z}-\mathbf{u}) \,. \tag{30}$$

for the velocities from n th-order and the acceleration from $(n-1)$ th-order, for each t.

Proof: We are going to use the induction method. First of all we can see above that the Eq. (30) is true for $n=1$, $n=2$, $n=3$ and $n=4$. Now we assume that the Eq. (30) is true for $n=k$ and prove that it is true for $n=k+1$. Hence, if we assume that

$$\overset{(k)}{\mathbf{z}} = \overset{(k)}{\mathbf{u}} - \epsilon \overset{(k)}{\boldsymbol{\varphi}}(\mathbf{z}-\mathbf{u})$$

is true, we can obtain

$$\overset{(k+1)}{\mathbf{z}} = \overset{(k+1)}{\mathbf{u}} - \epsilon \overset{(k+1)}{\boldsymbol{\varphi}}(\mathbf{z}-\mathbf{u}) - \epsilon \overset{(k)}{\boldsymbol{\varphi}}(\dot{\mathbf{z}}-\dot{\mathbf{u}}) \tag{31}$$

by differentiation the latter expression. So, from the last equation and the Eq. (26) we have

$$\overset{(k+1)}{\mathbf{z}} = \overset{(k+1)}{\mathbf{u}} - \epsilon \overset{(k+1)}{\boldsymbol{\varphi}}(\mathbf{z}-\mathbf{u}) .$$

As a result, we have been proved theorem 4 by the induction method.

4.2. HIGHER-ORDER POLES UNDER THE INVERSE MOTION OF THE ONE-PARAMETER PLANAR DUAL MOTIONS

During the inverse motion of the one-parameter planar dual motion $\mathbb{D}'/\mathbb{D}$ and in the case of $Z' \in \mathbb{D}'$ is a fixed point, from the Eq. (26) we can calculate the first order pole point as below:

$$\epsilon\mathbf{q}_1 = \epsilon\mathbf{z} = \epsilon\mathbf{u} + \frac{\dot{\mathbf{u}}}{\dot{\varphi}} \,. \tag{32}$$

Using the Eq. (13), we obtain the second order pole point as below:

$$\epsilon\mathbf{q}_2 = \epsilon\mathbf{z} = \epsilon\mathbf{u} + \frac{\ddot{\mathbf{u}}}{\ddot{\varphi}}. \quad (33)$$

In a similar way, from the Eq. (30), we can get the $(n-1)$ th-order acceleration pole as follows:

$$\epsilon\mathbf{q}_n = \epsilon\mathbf{z} = \epsilon\mathbf{u} + \frac{\overset{(\mathbf{n})}{\mathbf{u}}}{\overset{(\mathbf{n})}{\boldsymbol{\varphi}}}. \quad (34)$$

Also, from the Eq. (34), we can write the following equation:

$$\overset{(\mathbf{n})}{\mathbf{u}} = -\epsilon\,\overset{(\mathbf{n})}{\boldsymbol{\varphi}}(\mathbf{u} - \mathbf{q}_n).$$

So, if we use the Eq. (30), and the latter equation, we can rewrite $\overset{(\mathbf{n})}{\mathbf{z}}$ in terms of $\mathbf{q}_n$ as follows:

$$\overset{(\mathbf{n})}{\mathbf{z}} = -\epsilon\,\overset{(\mathbf{n})}{\boldsymbol{\varphi}}(\mathbf{z} - \mathbf{q}_n). \quad (35)$$

The theorem 2 and the theorem 3 can be also obtained for inverse motion of the one-parameter planar dual motions in similar way.

5. CONCLUSION

The higher-order accelerations and poles are obtained under the one-parameter planar dual motion and its inverse. In addition, on the dual plane under this motion, the orbit curve of a point is sketched by using Maple programming.

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