ORIGINAL PAPER

GAUSSIAN LEONARDO POLYNOMIALS AND APPLICATIONS OF LEONARDO NUMBERS TO CODING THEORY

SELİME BEYZA ÖZÇEVİK[1], ABDULLAH DERTLİ[1]

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Abstract. *In this paper, we firstly introduce the Gaussian Leonardo polynomial sequences* $\{GLe_n(x)\}_{n=0}^{\infty}$ *and we obtain Binet's formula, generating function of this sequence. Moreover, we define the matrix* $Gl(x)$ *in the form of 3 x 3. Finally, we study on the coding and decoding applications of the Leonardo number by using the Leonardo matrix* P.
Keywords: *Leonardo sequence; Gaussian Leonardo polynomial; code matrix.*

1. INTRODUCTION

Number sequences are an important subject for the applications of various fields of mathematics such as algebra, number theory, geometry, cryptography, coding theory. They are based on the Fibonacci number sequence. This sequence was defined by Leonardo Fibonacci in 1202, [1]. Moreover, this sequence satisfies the recurrence relation $F_{n+1} = F_n + F_{n-1}$ with initial conditions $F_0 = 0, F_1 = 1$ for $n \geq 2$. Since then, various number sequences have been introduced by mathematicians based on the Fibonacci sequence like Pell, Lucas and Padovan.

Among the studies in literature, Horadam defined the complex Fibonacci numbers with the recurrence relation $F_n^* = F_n + iF_{n+1}$, where F_n is the nth Fibonacci number. Furthermore, in 1977, Berzsenyi defined complex Fibonacci numbers by a different approach, [2]. He examined them as a set of complex numbers at the Gaussian integers that provided recurrence relation of Fibonacci sequence. These approaches have been applied to other number sequences over time. Also, it has been examined some properties of these sequences. Today, the security of information sharing has become a necessity with the development of technology and the widespread use of communication tools. Coding theory has emerged as a result of these needs. The most important study on coding theory was published by Shannon in 1948, [3]. In his study, he defined a mechanism for the communication system.

P. S. Moharir defined Fibonacci codes and improved single error detection and correction in 1970, [4]. Basu and Prasad established a relationship between code matrix elements and Fibonacci numbers, [5]. It can be given various studies using number sequence in the field of coding theory as examples, [6, 7].

In this study, we introduce Gaussian Leonardo polynomials and study some properties. Moreover, we obtain a new Gaussian Leonardo sequence and examine this sequence in terms of the same properties. Finally, we study the application of Leonardo and Gaussian Leonardo numbers on the field of coding theory.

[1] Ondokuz Mayıs University, Faculty of Sciences, Mathematics Department, 55270 Samsun, Turkey.
E-mail: ozcevikbeyza8@gmail.com; abdullah.dertli@gmail.com.

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2. MATERIALS AND METHODS

Gaussian numbers have an important place among the studies related to number sequences. These numbers are complex numbers $z = a + ib$ in which $a, b \in \mathbb{Z}$ and they were studied by Gauss in 1832, [8]. $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ is the set of these numbers.

In 1963, Horadam defined the Gaussian Fibonacci sequence, [2]. $\{GF_n\}_{n=0}^{\infty}$ Gaussian Fibonacci sequence is defined with the recurrence relation

$$GF_n = F_{n-1} + iF_{n-2}$$

where $GF_0 = i, GF_1 = 1$ and $n \geq 2$.

In addition, Gaussian numbers were defined also for other number sequences such as Pell, Pell-Lucas, Jacobsthal, etc. in the literature and some of their properties were studied [9, 10]. Catalan and Borges defined $\{Le_n\}_{n=0}^{\infty}$ Leonardo sequence in 2019 [11]. Leonardo sequence is defined by

$$Le_n = Le_{n-1} + Le_{n-2} + 1, n \geq 2$$

with the initial conditions $Le_0 = Le_1 = 1$. Moreover, the homogenous form of this recurrence relation is

$$Le_{n+1} = 2Le_n - Le_{n-2}, n \geq 2.$$

The first few terms of Leonardo sequence are

$$1,1,3,5,9,15,25,41, \ldots$$

The third order characteristic equation of this recurrence relation is

$$x^3 - 2x^2 + 1 = 0.$$

Binet's formula for the Leonardo sequence is given as follows

$$Le_n \quad = 2\left(\frac{\Phi^{n+1} - \Psi^{n+1}}{\Phi - \Psi}\right) - 1$$

$$= \frac{\Phi(2\Phi^n - 1) - \Psi(2\Psi^n - 1)}{\Phi - \Psi}, \; n \geq 0$$

where Le_n is the nth Leonardo number, $\Phi = \frac{1+\sqrt{5}}{2}$ and $\Psi = \frac{1-\sqrt{5}}{2}$ are roots of the characteristic equation of Leonardo sequence.

For $1 - 2t + t^3 \neq 0$, the generating function for the Leonardo sequence is given by

$$gl_e(t) = \frac{1 - t + t^2}{1 - 2t + t^3} \; .$$

In 2021, Soykan defined the Leonardo matrix A as follows

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

such that $\det A = -1$ [12].

In 2022, the Gaussian Leonardo numbers are defined in [13]. The Gaussian Leonardo sequence $\{GLe_n\}_{n=0}^{\infty}$ defined by the recurrence relation

$$GLe_n = GLe_{n-1} + GLe_{n-2} + (1+i)$$

with initial conditions $GLe_0 = 1 - i, GLe_1 = 1 + i,\ n \geq 2$.

Also, the homogenous form of this recurrence relation is

$$GLe_{n+1} = 2GLe_n - GLe_{n-2},\ n \geq 2.$$

The first few Gaussian Leonardo numbers are

$$1 - i, 1 + i, 3 + i, 5 + 3i, 9 + 5i, 15 + 9i, \ldots$$

Other studies related to these number sequences can be accessed.

3. RESULTS AND DISCUSSION

3.1. GAUSSIAN LEONARDO POLYNOMIALS

In this section, we define Gaussian Leonardo polynomial sequence and we give some properties of this sequence.

Definition 3.1.1. The Gaussian Leonardo polynomial sequence $\{GLe_n(x)\}_{n=0}^{\infty}$ is defined by

$$GLe_{n+1}(x) = 2xGLe_n(x) - G_L(x)$$

for $n \geq 2$, with initial conditions $GLe_0(x) = 1 - i, GLe_1(x) = x + i$.

We give the first few terms of this sequence in the following table:

Table 1. Some Gaussian Leonardo Polynomials

n	$GLe_n(x)$
0	$1 - i$
1	$x + i$
2	$3x + i$
3	$6x^2 + 2xi - 1 + i$
⋮	⋮

We introduce the generating function for the Gaussian Leonardo polynomial sequence with the power series.

Theorem 3.1.2. The generating function of the Gaussian Leonardo polynomial sequence is

$$gl_e x(t) = \frac{t^2 GLe_2(x) + (1-2xt)[tGLe_1(x) + GLe_0(x)]}{1 - 2\mathrm{x}t + t^3}.$$

Proof: Gaussian Leonardo polynomial sequence is

$$GLe_{n+1}(x) = 2xGLe_n(x) - GLe_{n-2}(x)$$

We have

$$\sum_{t=0}^{\infty} GLe_{n+3}t^n = 2\mathrm{x}\sum_{t=0}^{\infty} GLe_{n+2}t^n - \sum_{t=0}^{\infty} GLe_n t^n.$$

This equation can be written as

$$\sum_{t=3}^{\infty} GLe_n t^{n-3} = 2\mathrm{x}\sum_{t=2}^{\infty} GLe_n t^{n-2} - \sum_{t=0}^{\infty} GLe_n t^n$$

and the proof is completed with the help of necessary operations.

Now, we give the relation between Gaussian Leonardo numbers and the Leonardo matrix A.

Theorem 3.1.3. For $n \geq 0$, we have

$$[3+i \quad 1+i \quad 1-i]A^n = [G\mathrm{Le}_{n+2} \quad GLe_{n+1} \quad G\mathrm{Le}_n]$$

where A is the Leonardo matrix.

Proof: The proof can be seen by induction on n.

Theorem 3.1.4. For $n \geq 0$, we have

$$Gl^n = \begin{bmatrix} GLe_{n+2} & GLe_{n+1} & GLe_n \\ GLe_{n+1} & GLe_n & GLe_{n-1} \\ GLe_n & GLe_{n-1} & GLe_{n-2} \end{bmatrix}.$$

Moreover, $det|Gl^n| = 4(3+i)(-1)^{n+1}$.

Theorem 3.1.5. (Simson's identity): For $n \geq 0$, we have

$$GLe_{n-2}(GLe_{n+2}GLe_n - GLe_{n+1}^2) + GLe_n(GLe_{n+1}GLe_{n-1} - GLe_n^2)$$
$$+GLe_{n-1}(GLe_nGLe_{n+1} - GLe_{n-1}GLe_{n+2}) = 4(3+i)(-1)^{n+1}.$$

Let's give similar properties of the Gaussian Leonardo polynomial sequence.

Theorem 3.1.6. For n ≥ 0, we have

$$[3x+i \quad x+i \quad 1-i]A^n = [GLe_{n+2}(x) \quad GLe_{n+1}(x) \quad GLe_n(x)]$$

where A is the Leonardo matrix.

Proof: The proof can be seen by induction on n.

Theorem 3.1.7. For $n \geq 0$, we have

$$Gl(x)^n = \begin{bmatrix} GLe_{n+2}(x) & GLe_{n+1}(x) & GLe_n(x) \\ GLe_{n+1}(x) & GLe_n(x) & GLe_{n-1}(x) \\ GLe_n(x) & GLe_{n-1}(x) & GLe_{n-2}(x) \end{bmatrix}.$$

Moreover,

$$det|Gl(x)^n| = -12x^5 + (36-28i)x^4 - (4-58i)x^3 - (33+19i)x^2 - (4+14i)x + 5 - i.$$

Proof: The proof can be seen by induction on n.

Theorem 3.1.8. Binet's formula of the Gaussian Leonardo polynomial sequence is

$$GLe_n(x) = A\alpha^n(x) + B\beta^n(x) + C\gamma^n(x)$$

where

$$\alpha = \frac{2x}{3} + A + B$$

$$\beta = \frac{2x}{3} + wA + w^2B$$

$$\gamma = \frac{2x}{3} + w^2A + wB$$

and α, β, γ are roots of the characteristic equation $GLe_n(x) = 2xGLe_{n-1}(x) - GLe_{n-3}(x)$. In addition, we have the following identities

$$\alpha + \beta + \gamma = 2x$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = 0$$

$$\alpha\beta\gamma = 1.$$

Proof: The proof of this theorem can be proven as in [14].

3.2. APPLICATIONS OF LEONARDO NUMBER AND POLYNOMIAL SEQUENCES ON CODING THEORY

In this section, the applications of Leonardo numbers and polynomial sequences on the field of coding theory are introduced and examples are given. Let's represent an initial message with a square matrix M in the form of 3×3. When M and A^n are multiplied, we get the code matrix C where A is the Leonardo matrix the size of 3×3 and $n \geq 2$. That is, the code matrix is obtained as $MA^n = C$. To get the initial message from the code, C and $(A^{-1})^n$ are multiplied, where A^{-1} is the inverse matrix of A and is called the decoding matrix.

Let the message matrix M be as follows

$$M = \begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix}$$

where m_i are positive integers for $i = 1, \dots, 9$.

Consequently, we can get the following equations for the coding and decoding methods. For the encoding message M, we have

$$\begin{aligned} MA^n &= \begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix} \begin{bmatrix} Le_{n+2} & Le_{n+1} & Le_n \\ Le_{n+1} & Le_n & Le_{n-1} \\ Le_n & Le_{n-1} & Le_{n-2} \end{bmatrix} \\ &= \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}. \end{aligned}$$

Then,

$$c_1 = m_1 Le_{n+2} + m_2 Le_{n+1} + m_3 Le_n$$

$$c_2 = m_1 Le_{n+1} + m_2 Le_n + m_3 Le_{n-1}$$

$$c_3 = m_1 Le_n + m_2 Le_{n-1} + m_3 Le_{n-2}$$

$$c_4 = m_4 Le_{n+2} + m_5 Le_{n+1} + m_6 Le_n$$

$$c_5 = m_4 Le_{n+1} + m_5 Le_n + m_6 Le_{n-1}$$

$$c_6 = m_4 Le_n + m_5 Le_{n-1} + m_6 Le_{n-2}$$

$$c_7 = m_7 Le_{n+2} + m_8 Le_{n+1} + m_9 Le_n$$

$$c_8 = m_7 Le_{n+1} + m_8 Le_n + m_9 Le_{n-1}$$

$$c_9 = m_7 Le_n + m_8 Le_{n-1} + m_9 Le_{n-2}.$$

Since the matrix A and message matrix consist of non-negative integers, all elements of the matrix C are non-negative integers. We can characterize the matrix $(A^{-1})^n$ for the decoding method as follows

$$A^n = \frac{1}{4(-1)^{n+1}}\begin{bmatrix} Le_{n-1}^2 - Le_nLe_{n-2} & -Le_nLe_{n-1} + Le_{n+1}Le_{n-2} & -Le_{n+1}Le_{n-1} + Le_n^2 \\ -Le_nLe_{n-1} + Le_{n+1}Le_{n-2} & Le_n^2 - Le_{n+2}Le_{n-2} & Le_{n+2}Le_{n-1} - Le_nLe_{n+1} \\ Le_n^2 - Le_{n+1}Le_{n-1} & Le_{n+2}Le_{n-1} - Le_nLe_{n+1} & Le_{n+1}^2 - Le_{n+2}Le_n \end{bmatrix}.$$

For example,

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 2 & 1 \\ 2 & -3 & -1 \\ 1 & -1 & -2 \end{bmatrix},$$

$$A^{-2} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -3 & -1 \end{bmatrix}.$$

Example 3.2.1. Let $n = 3$. We take message matrix M as follows

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 4 \\ 1 & 3 & 2 \end{bmatrix}.$$

The coding matrix is

$$A^3 = \begin{bmatrix} Le_5 & Le_4 & Le_3 \\ Le_4 & Le_3 & Le_2 \\ Le_3 & Le_2 & Le_1 \end{bmatrix} = \begin{bmatrix} 15 & 9 & 5 \\ 9 & 5 & 3 \\ 5 & 3 & 1 \end{bmatrix}.$$

We can get the code matrix C by multiplying M and A^3.

$$C = MA^3 = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 4 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 15 & 9 & 5 \\ 9 & 5 & 3 \\ 5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 34 & 20 & 10 \\ 74 & 44 & 22 \\ 52 & 30 & 16 \end{bmatrix}.$$

Then, the decoding matrix is

$$A^{-3} = \frac{1}{2}\begin{bmatrix} -2 & 3 & 1 \\ 3 & -5 & 0 \\ 1 & 0 & -3 \end{bmatrix}.$$

The decoding of the message transmitted to the receiver by the decoding matrix A^{-3} is as follows

$$CA^{-3} = \begin{bmatrix} 34 & 20 & 10 \\ 74 & 44 & 22 \\ 52 & 30 & 16 \end{bmatrix}\begin{bmatrix} -1 & \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{5}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{3}{2} \end{bmatrix} = M.$$

Now, we will give encoding and decoding method and a numerical example by using the terms of Leonardo polynomial sequence. The Leonardo polynomial matrix P is given by

$$P=\begin{bmatrix} Le_{n+2}(x) & Le_{n+1}(x) & Le_n(x) \\ Le_{n+1}(x) & Le_n(x) & Le_{n-1}(x) \\ Le_n(x) & Le_{n-1}(x) & Le_{n-2}(x) \end{bmatrix}$$

where $\det P=4(-1)^{n+1}(3x^2-11x+9)$.

Let the initial message be M. Then, the code matrix C is obtained by multiplying M and P matrices. So, we have

$$C=\begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix}\begin{bmatrix} \mathrm{Le}_{n+2}(x) & Le_{n+1}(x) & Le_n(x) \\ Le_{n+1}(x) & Le_n(x) & Le_{n-1}(x) \\ \mathrm{Le}_n(x) & Le_{n-1}(x) & Le_{n-2}(x) \end{bmatrix} =\begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}.$$

Then,

$$c_1=m_1Le_{n+2}(x)+m_2Le_{n+1}(x)+m_3Le_n(x)$$

$$c_2=m_1Le_{n+1}(x)+m_2Le_n(x)+m_3Le_{n-1}(x)$$

$$c_3=m_1Le_n(x)+m_2Le_{n-1}(x)+m_3Le_{n-2}(x)$$

$$c_4=m_4Le_{n+2}(x)+m_5Le_{n+1}(x)+m_6Le_n(x)$$

$$c_5=m_4Le_{n+1}(x)+m_5Le_n(x)+m_6Le_{n-1}(x)$$

$$c_6=m_4Le_n(x)+m_5Le_{n-1}(x)+m_6Le_{n-2}(x)$$

$$c_7=m_7Le_{n+2}(x)+m_8Le_{n+1}(x)+m_9Le_n(x)$$

$$c_8=m_7Le_{n+1}(x)+m_8Le_n(x)+m_9Le_{n-1}(x)$$

$$c_9=m_7Le_n(x)+m_8Le_{n-1}(x)+m_9Le_{n-2}(x).$$

Let's take the inverse of the matrix P as the decoding matrix to decode the received message. Then, we get,

$$M=CP^{-1}$$

where $\det P^{-1}=\frac{1}{4(-1)^{n+1}}\frac{1}{3x^2-11x+9}$.

Example 3.2.2. Let $n=2$. The initial message be M and the coding matrix P are as follows, respectively.

$$M = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 12x^2 - 2x - 1 & 6x - 1 & 3 \\ 6x - 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

We get the code matrix C

$$C = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 12x^2 - 2x - 1 & 6x - 1 & 3 \\ 6x - 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 12x^2 - 2x + 5 & 6x + 1 & 5 \\ 24x^2 + 14x - 2 & 12x + 8 & 10 \\ 12x^2 - 2x + 8 & 6x + 2 & 6 \end{bmatrix}.$$

Now, let's decode the matrix C by using P^{-1}. For $n = 2$, the decoding matrix P^{-1} is as follows

$$P^{-1} = \frac{1}{6x^2 - 22x + 18} \begin{bmatrix} -1 & 3x - 2 & -3x + 5 \\ 3x - 2 & -6x^2 + x + 5 & 6x^2 - 10x + 1 \\ -3x + 5 & 6x^2 - 10x + 1 & -3x + 2 \end{bmatrix}.$$

Then, the initial message M is obtained from

$$M = CP^{-1}.$$

3.3. ERROR DETECTION AND CORRECTION

Error detection and correction are one of the main problems of coding theory. So, in this section, we study on the necessary information to detect and correct the error which transmitted message by Leonardo numbers.

The coding transform is

$$\begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix} \begin{bmatrix} Le_{n+2} & Le_{n+1} & Le_n \\ Le_{n+1} & Le_n & Le_{n-1} \\ Le_n & Le_{n-1} & Le_{n-2} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}.$$

Also, we have

$$\det C = \det(MA) = \det \mathrm{M} \det A$$
$$= 4(-1)^{n+1} \det M.$$

In this equation, the relationship between the message matrix and the code matrix can be seen. Therefore, error detection can be done according to the received n values as follows;

i. If n is odd, then $\det C = 4 \det M$.
ii. If n is even, then $\det C = -4 \det M$.

Now, let's consider the errors that can occur in each component of the message matrix.

$$\begin{bmatrix} x & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}, \begin{bmatrix} c_1 & y & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}, \begin{bmatrix} c_1 & c_2 & z \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}$$
$$\begin{bmatrix} c_1 & c_2 & c_3 \\ t & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}, \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & u & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}, \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & v \\ c_7 & c_3 & c_9 \end{bmatrix}$$
$$\begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ k & c_8 & c_9 \end{bmatrix}, \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_2 & c_9 \end{bmatrix}, \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & n \end{bmatrix}.$$

Then, we have the following equations since $4(-1)^{n+1}\det M = \det C$.

$$\mathrm{x} = \frac{4 \cdot (-1)^{n+1} \cdot \det M}{c_5 c_9 - c_6 c_8}$$

$$\mathrm{y} = \frac{4 \cdot (-1)^{n+1} \cdot \det M}{c_4 c_9 - c_6 c_7}$$

$$\mathrm{z} = \frac{4 \cdot (-1)^{n+1} \cdot \det M}{c_4 c_8 - c_5 c_7}$$

$$\mathrm{t} = \frac{4 \cdot (-1)^{n+1} \cdot \det M}{c_2 c_9 - c_3 c_8}$$

$$\mathrm{u} = \frac{4 \cdot (-1)^{n+1} \cdot \det M}{c_1 c_9 - c_3 c_7}$$

$$\mathrm{v} = \frac{4 \cdot (-1)^{n+1} \cdot \det M}{c_1 c_8 - c_2 c_7}$$

$$\mathrm{k} = \frac{4 \cdot (-1)^{n+1} \cdot \det M}{c_2 c_6 - c_3 c_5}$$

$$\mathrm{l} = \frac{4 \cdot (-1)^{n+1} \cdot \det M}{c_1 c_6 - c_3 c_4}$$

$$\mathrm{n} = \frac{4 \cdot (-1)^{n+1} \cdot \det M}{c_1 c_5 - c_2 c_4}.$$

If the encoding transformation is done with the help of Leonardo polynomial sequence, error detection can be done as follows. If n is even, then $\det C = (-12x^2 + 44x - 36)\det M$. If n is odd, then $\det C = (12x^2 - 44x + 36)\det M$. Then, if we name each error in the M matrix as $x', y', z', t', u', v', k', l', n'$, respectively, the following equations can be used to correct these errors.

$$x' = \frac{4 \cdot (-1)^{n+1} \cdot (3x^2 - 11x + 9) \cdot \det M}{c_5 c_9 - c_6 c_8}$$

$$y' = \frac{4 \cdot (-1)^{n+1} \cdot (3x^2 - 11x + 9) \cdot \det M}{c_4 c_9 - c_6 c_7}$$

$$z' = \frac{4 \cdot (-1)^{n+1} \cdot (3x^2 - 11x + 9) \cdot \det M}{c_4 c_8 - c_5 c_7}$$

$$t' = \frac{4 \cdot (-1)^{n+1}(3x^2 - 11x + 9) \cdot \det M}{c_2 c_9 - c_3 c_8}$$

$$u' = \frac{4 \cdot (-1)^{n+1}(3x^2 - 11x + 9) \cdot \det M}{c_1 c_9 - c_3 c_8}$$

$$v' = \frac{4 \cdot (-1)^{n+1}(3x^2 -| 1x + 9) \det N |}{c_1 c_8 - c_2 c_7}$$

$$k' = \frac{4 \cdot (-1)^{n+1}(3x^2 - 11x + 9) \cdot \det M}{c_2 c_6 - c_3 c_5}$$

$$l' = \frac{4 \cdot (-1)^{n+1}(3x^2 - 11x + 9) \cdot \det M}{c_1 c_6 - c_3 c_4}$$

$$n' = \frac{4 \cdot (-1)^{n+1} \cdot (3x^2 - 11x + 9) \cdot \det M}{c_1 c_5 - c_2 c_4}.$$

4. CONCLUSION

This study presents the Gaussian Leonardo polynomial sequence firstly. We obtain the basic properties of this sequence such as generating function, Binet's formula, matrix form. Moreover, we study the coding end decoding method using the Leonardo numbers. Finally, we examine the error detection and correction in case of sending an incorrect to the receiver.

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