ORIGINAL PAPER

ON A WEIGHTED ALGEBRA UNDER FRACTIONAL CONVOLUTION

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Abstract. *In this study, we describe a linear space $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ of functions $f \in L^1_w(\mathbb{R}^d)$ whose fractional Fourier transforms $\mathcal{F}_\alpha f$ belong to $L^{p(.)}_\nu(\mathbb{R}^d)$ for $p^+ < \infty$. We show that $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ becomes a Banach algebra with the sum norm $\|f\|_{A^{w,\nu}_{\alpha,p(.)}} = \|f\|_{1,w} + \|\mathcal{F}_\alpha f\|_{p(.),\nu}$ and under Θ (fractional convolution) convolution operation. Besides, we indicate that the space $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ is an abstract Segal algebra, where w is weight function of regular growth. Moreover, we find an approximate identity for $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$. We also discuss some other properties of $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$. Finally, we investigate some inclusions of this space.*

Keywords: *Fractional Fourier transform; weighted variable exponent Lebesgue space; convolution.*

1. INTRODUCTION

In this work, we study on $\mathbb{R}^d$. Let g be any function from $\mathbb{R}^d$ into $\mathbb{C}$. Then the translation and character (modulation) operators are defined by $T_y g(x) = g(x-y)$ and $M_\omega g(x) = e^{i\omega x} g(x)$ for all $x, \omega \in \mathbb{R}^d$, respectively, see [1]. $C_c(\mathbb{R}^d)$ indicates the space of continuous complex-valued functions on $\mathbb{R}^d$ whose support is compact, see [2]. Also, we note the Lebesgue space $\left(L^p(\mathbb{R}^d), \|.\|_p\right)$, for $1 \le p < \infty$. For $p = \infty$, $L^\infty(\mathbb{R}^d)$ is the class of complex-valued measurable and essentially bounded functions (equivalence classes). The space $L^\infty(\mathbb{R}^d)$ is a Banach space with norm $\|.\|_\infty$ defined by

$$\|g\|_\infty = \operatorname*{esssup}_{x\in\mathbb{R}^d} |g(x)|$$

for all $g \in L^\infty(\mathbb{R}^d)$. If w is a measurable and locally bounded function on $\mathbb{R}^d$ which satisfies $w(x) \ge 1$ and $w(x+y) \le w(x)w(y)$ (i.e, submultiplicative, see [3]) for all $x, y \in \mathbb{R}^d$, then it is called weight function (Beurling weight). During this paper, we take the Beurling weights. A weight function w is weight function of regular growth if $w\left(\frac{x}{\rho}\right) \le w(x)$ $(\rho \ge 1)$ and there are constants $c \ge 1$ and $\lambda > 0$ such that $w(\rho x) \le c\rho^\lambda w(x)$ $(\rho \ge 1)$ for all $x \in \mathbb{R}^d$. We denote the weighted Lebesgue space $L^p_w(\mathbb{R}^d)$ as

$$L^p_w(\mathbb{R}^d) = \{f \mid fw \in L^p(\mathbb{R}^d)\},$$

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for $1 \leq p < \infty$. It is known that $L_w^p(\mathbb{R}^d)$ is a Banach space under the norm $\|f\|_{p,w} = \|fw\|_p$. The dual space of $L_w^1(\mathbb{R}^d)$ is $L_w^\infty(\mathbb{R}^d)$ that consists of complex-valued measurable functions (equivalence classes) ϕ on $\mathbb{R}^d$ such that $\frac{\phi}{w} \in L^\infty(\mathbb{R}^d)$. The space $L_w^\infty(\mathbb{R}^d)$ is a Banach space under the norm $\|\phi\|_{\infty,w} = \left\|\frac{\phi}{w}\right\|_\infty$, see [4].

Let w_1 and w_2 are two weight functions. We say that $w_1 \prec w_2$ if there exists $c > 0$, such that $w_1(x) \leq c w_2(x)$ for all $x \in \mathbb{R}^d$, see [4, 5].

We denote the set of all measurable functions $p(.)$ from $\mathbb{R}^d$ into $[1, +\infty)$ by the symbol $\mathrm{P}(\mathbb{R}^d)$. Let

$$p_- = \underset{x\in\mathbb{R}^d}{\operatorname{essinf}} p(x), \qquad p^+ = \underset{x\in\mathbb{R}^d}{\operatorname{esssup}} p(x)$$

for $p(.) \in \mathrm{P}(\mathbb{R}^d)$. We define the function

$$\varrho_{p(.)} = \int_{\mathbb{R}^d} |f(x)|^{p(x)} dx$$

for every measurable functions f on $\mathbb{R}^d$. The function $\varrho_{p(.)}$ is convex modular (see [6]). The generalized Lebesgue space (or the variable exponent Lebesgue space) $L^{p(.)}(\mathbb{R}^d)$ is the set of all measurable functions (equivalence classes) f such that $\varrho_{p(.)}(\lambda f) < \infty$ for some $\lambda > 0$. This space is normed space if it is endowed with the Luxemburg norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 : \varrho_{p(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

Let $p^+ < \infty$. Then $f \in L^{p(.)}(\mathbb{R}^d)$ if and only if $\varrho_{p(.)}(f) < \infty$. The space $L^{p(.)}(\mathbb{R}^d)$ is a Banach space under the Luxemburg norm $\|.\|_{p(.)}$. If $p(.) = p$ is a constant function, then the Luxemburg norm $\|.\|_{p(.)}$ corresponds to the Lebesgue norm $\|.\|_p$, see [7]. The weighted variable exponent Lebesgue space $L_w^{p(.)}(\mathbb{R}^d)$ is the set of all measurable functions (equivalence classes) f , where

$$\|f\|_{p(.),w} = \|fw\|_{p(.)} < \infty.$$

The space $L_w^{p(.)}(\mathbb{R}^d)$ is a Banach space with the norm $\|.\|_{p(.),w}$, see [8, 9].

The Fourier transform $\hat{f}$ (or Ff) of $f \in L^1(\mathbb{R})$ is given by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt.$$

The fractional Fourier transform is a generalization of the Fourier transform with a parameter α and can be interpreted as a rotation by an angle α in the time-frequency plane. Let δ be Dirac delta function. The fractional Fourier transform with angle α of a function $f \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}_\alpha f(x) = \int_{-\infty}^{+\infty} K_\alpha(x,t) f(t) dt,$$

where,

$$K_\alpha(x,t) = \begin{cases} \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\, e^{i\left(\frac{x^2+t^2}{2}\right)\cot\alpha - ixt\operatorname{cosec}\alpha}, & \text{if } \alpha \text{ is not multiple of } \pi \\ \delta(t-x), & \text{if } \alpha = 2k\pi, k \in \mathbb{Z} \\ \delta(t+x), & \text{if } \alpha = (2k+1)\pi, k \in \mathbb{Z}. \end{cases}$$

The fractional Fourier transform with $\alpha = \frac{\pi}{2}$ corresponds to the Fourier transform, see [10, 11, 12, 13, 14].

The fractional Fourier transform is defined for higher dimensions as in [15]:

$$\left(\mathcal{F}_{\alpha_1,\dots,\alpha_d} f\right)(x) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} K_{\alpha_1,\dots,\alpha_d}(x_1,\dots,x_d;t_1,\dots,t_d) f(t_1,\dots,t_d) dt_1 \dots dt_d,$$

or briefly

$$\mathcal{F}_\alpha f(x) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} K_\alpha(x,t) f(t) dt,$$

such that

$$K_\alpha(x,t) = K_{\alpha_1,\dots,\alpha_d}(x_1,\dots,x_d;t_1,\dots,t_d) = K_{\alpha_1}(x_1,t_1) K_{\alpha_2}(x_2,t_2) \dots K_{\alpha_n}(x_d,t_d).$$

Throughout this study, unless otherwise indicated, we get $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \neq k\pi$ for $i = 1,2,\dots,d$ and $k \in \mathbb{Z}$. Let $z = (-y_1 \cot\alpha_1, \dots, -y_d \cot\alpha_d)$ for all $y = (y_1, \dots, y_d) \in \mathbb{R}^d$. The Θ convolution operation is

$$\begin{aligned}(f\Theta g)(x) &= \int_{\mathbb{R}^d} f(y) g(x-y) e^{\sum_{j=1}^{d} i y_j (y_j - x_j)\cot\alpha_j} dy \\ &= \int_{\mathbb{R}^d} f(y) T_y M_z g(x) dy\end{aligned} \quad (1)$$

for all $f, g \in L^1(\mathbb{R}^d)$, see [16, 17].

Let A and B be commutative Banach algebras and $B \subseteq A$. Then the space B is a Banach ideal of A if $fg \in B$ and the inequalities $\|f\|_A \leq \|f\|_B$, $\|fg\|_B \leq \|f\|_B \|g\|_A$ holds for all $f \in B$, $g \in A$, see [18]. A normed space $(A, \|.\|_A)$ of measurable function is called solid, if for any measurable function h and every $g \in A$, $h \in A$ and $\|h\|_A \leq \|g\|_A$ hold when $|h(x)| \leq |g(x)|$ almost everywhere, see [19].

$(Y, \|.\|_Y)$ is called an abstract Segal algebra with respect to a Banach algebra $(X, \|.\|_X)$ if it ensures the following properties (see [20]):

1. Y is a Banach algebra under the norm $\|.\|_Y$ and Y is a dense ideal in X.
2. There exists $k > 0$ such that $\|g\|_X \leq k\|g\|_Y$ for all $g \in Y$.
3. There exists $l > 0$ such that $\|gh\|_Y \leq l\|g\|_X \|h\|_Y$ for all $g, h \in Y$.

Let w be a weight function on $\mathbb{R}^d$. The space $S_w(\mathbb{R}^d)$ is subalgebra of $L^1_w(\mathbb{R}^d)$ satisfying the following properties (see [21]):

1. The space $S_w(\mathbb{R}^d)$ is dense in $L^1_w(\mathbb{R}^d)$.
2. The subalgebra $S_w(\mathbb{R}^d)$ is a Banach algebra under some norm $\|.\|_{S_w}$ and the inequality $\|g\|_{1,w} \leq \|g\|_{S_w}$ holds for all $g \in S_w(\mathbb{R}^d)$.
3. $S_w(\mathbb{R}^d)$ is translation invariant and for each $g \in S_w(\mathbb{R}^d)$ and all $y \in \mathbb{R}^d$, the inequality $\|T_y g\|_{S_w} \leq w(y)\|g\|_{S_w}$ holds.
4. The function $y \to T_y g$ from $\mathbb{R}^d$ into $S_w(\mathbb{R}^d)$ is continuous.

Let X and Y be Banach spaces and $X \subset Y$. If the identity operator I from X into Y is bounded i.e, there exist some constant c where

$$\|I(x)\|_Y \leq c\|x\|_X$$

for all $x \in X$, then we write $X \hookrightarrow Y$.

In this study, we define the function spaces with fractional Fourier transform in the weighted variable exponent Lebesgue spaces. Throughout this paper, we use Θ convolution operator as a multiplication operator. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$. If we choose $\alpha_i = \frac{\pi}{2}$ for all

$i = 1,2,\ldots,d$, then the Θ convolution operator and fractional Fourier transform coincide with the ordinary convolution and Fourier transform, respectively. Therefore we extend the results of [22] to the fractional Fourier transform. Since weighted variable exponent Lebesgue spaces $L_w^{p(.)}(\mathbb{R}^d)$ corresponds to weighted Lebesgue spaces $L_w^p(\mathbb{R}^d)$, where $p(.) = p$ is a constant function, then we also extend the results of [17]. Moreover, this work correlates to the results from studies [5] and [23].

2. THE SPACE $A_{\alpha,p(.)}^{w,\nu}(\mathbb{R}^d)$

Definition 1. Let w and ν be weight functions on $\mathbb{R}^d$ and $p^+ < \infty$. The space $A_{\alpha,p(.)}^{w,\nu}(\mathbb{R}^d)$ consists of all $f \in L_w^1(\mathbb{R}^d)$ such that $\mathcal{F}_\alpha f \in L_\nu^{p(.)}(\mathbb{R}^d)$. The norm on vector space $A_{\alpha,p(.)}^{w,\nu}(\mathbb{R}^d)$ is

$$\|f\|_{A_{\alpha,p(.)}^{w,\nu}} = \|f\|_{1,w} + \|\mathcal{F}_\alpha f\|_{p(.),\nu}.$$

Theorem 2. Let $p^+ < \infty$. $\left(A_{\alpha,p(.)}^{w,\nu}(\mathbb{R}^d), \|.\|_{A_{\alpha,p(.)}^{w,\nu}}\right)$ is a Banach space.

Proof: Let $(g_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $A_{\alpha,p(.)}^{w,\nu}(\mathbb{R}^d)$. Therefore $(g_n)_{n\in\mathbb{N}}$ and $(\mathcal{F}_\alpha g_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $L_w^1(\mathbb{R}^d)$ and $L_\nu^{p(.)}(\mathbb{R}^d)$, respectively. Since $L_w^1(\mathbb{R}^d)$ and $L_\nu^{p(.)}(\mathbb{R}^d)$ are Banach spaces, then there exist $g \in L_w^1(\mathbb{R}^d)$ and $h \in L_\nu^{p(.)}(\mathbb{R}^d)$ such that $\|g_n - g\|_{1,w} \to 0$, $\|\mathcal{F}_\alpha f_n - h\|_{p(.),\nu} \to 0$. By using Proposition 2.3 (i) in [8], we have $\|g_n - g\|_1 \to 0$ and $\|\mathcal{F}_\alpha g_n - h\|_{p(.)} \to 0$. Since $\|\mathcal{F}_\alpha g_n - h\|_{p(.)} \to 0$, then the sequence $(\mathcal{F}_\alpha g_n)_{n\in\mathbb{N}}$ converges to h in measure and hence has a subsequence $\left(\mathcal{F}_\alpha g_{n_k}\right)_{n_k\in\mathbb{N}}$ that converges pointwise to h almost everywhere, see [7]. Also it is quite obvious that $\left\|g_{n_k} - g\right\|_1 \to 0$. Then we have

$$\begin{aligned}
|\mathcal{F}_\alpha g(u) - h(u)| &\le \left|\mathcal{F}_\alpha\left(g_{n_k} - g\right)(u)\right| + \left|\mathcal{F}_\alpha g_{n_k}(u) - h(u)\right| \\
&\le \prod_{j=1}^d \left|\sqrt{\frac{1-i\cot\alpha_j}{2\pi}}\right| \\
&\times \int_{\mathbb{R}^d} \left|\left(g_{n_k} - g\right)(t)\right| \left|e^{\sum_{j=1}^d \left(\frac{i}{2}(u_j{}^2 + t_j{}^2)\cot\alpha_j - iu_jt_j\mathrm{cosec}\alpha_j\right)}\right| dt \\
&+ \left|\mathcal{F}_\alpha g_{n_k}(u) - h(u)\right| \\
&= \prod_{j=1}^d \left|\sqrt{\frac{1-i\cot\alpha_j}{2\pi}}\right| \left\|g_{n_k} - g\right\|_1 + \left|\mathcal{F}_\alpha g_{n_k}(u) - h(u)\right|
\end{aligned}$$

By using this inequality, we get $\mathcal{F}_\alpha g = h$ almost everywhere. Then $\|g_n - g\|_{A_{\alpha,p(.)}^{w,\nu}} \to 0$ and $g \in A_{\alpha,p(.)}^{w,\nu}(\mathbb{R}^d)$. Hence $\left(A_{\alpha,p(.)}^{w,\nu}(\mathbb{R}^d), \|.\|_{A_{\alpha,p(.)}^{w,\nu}}\right)$ is a Banach space.

Theorem 3. Let $p^+ < \infty$. $\left(A_{\alpha,p(.)}^{w,\nu}(\mathbb{R}^d), \|.\|_{A_{\alpha,p(.)}^{w,\nu}}\right)$ is a Banach algebra under Θ convolution operation.

Proof: It is shown that $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ is a Banach space by Theorem 2. Let $f, g \in A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$. Then $f, g \in L^1_w(\mathbb{R}^d)$ and $\mathcal{F}_\alpha f, \mathcal{F}_\alpha g \in L^{p(.)}_\nu(\mathbb{R}^d)$. It is known that the space $L^1_w(\mathbb{R}^d)$ is a Banach algebra under Θ convolution operation, see [17]. Thus we have

$$\|f\Theta g\|_{1,w} \leq \|g\|_{1,w}\|f\|_{1,w}. \tag{2}$$

By using Theorem 7 in [17], we get

$$\begin{aligned} |\mathcal{F}_\alpha(f\Theta g)(u)| &= \left|\prod_{j=1}^{d} \sqrt{\frac{2\pi}{1-i\cot\alpha_j}}\right| \left|e^{\sum_{j=1}^{d} -\frac{i}{2}u_j^2\cot\alpha_j}\right| |\mathcal{F}_\alpha f(u)||\mathcal{F}_\alpha g(u)| \\ &\leq |\mathcal{F}_\alpha g(u)| \int_{\mathbb{R}^d} |f(t)|dt \\ &\leq |\mathcal{F}_\alpha g(u)|\|f\|_{1,w}. \end{aligned} \tag{3}$$

Since $\mathcal{F}_\alpha(f\Theta g)$ is continuous, then it is measurable. Also, it is well known that $L^{p(.)}_\nu(\mathbb{R}^d)$ is a solid space by Lemma 2.1 (b) in [22]. Since $L^{p(.)}_\nu(\mathbb{R}^d)$ is a solid space, by using (3) we obtain $\mathcal{F}_\alpha(f\Theta g) \in L^{p(.)}_\nu(\mathbb{R}^d)$ and

$$\|\mathcal{F}_\alpha(f\Theta g)\|_{p(.),\nu} \leq \left\|\mathcal{F}_\alpha g\|f\|_{1,w}\right\|_{p(.),\nu} = \|\mathcal{F}_\alpha g\|_{p(.),\nu}\|f\|_{1,w}. \tag{4}$$

By using (2) and (4) we get

$$\begin{aligned} \|f\Theta g\|_{A^{w,\nu}_{\alpha,p(.)}} &= \|f\Theta g\|_{1,w} + \|\mathcal{F}_\alpha(f\Theta g)\|_{p(.),\nu} \\ &\leq \|f\|_{1,w}\|g\|_{1,w} + \|\mathcal{F}_\alpha g\|_{p(.),\nu}\|f\|_{1,w} \\ &= \|f\|_{1,w}\|g\|_{A^{w,\nu}_{\alpha,p(.)}} \\ &\leq \|f\|_{A^{w,\nu}_{\alpha,p(.)}}\|g\|_{A^{w,\nu}_{\alpha,p(.)}}. \end{aligned} \tag{5}$$

It is easy to show that the other properties of Banach algebra are satisfied.

The following theorem is obvious from the inequality (5).

Theorem 4. Let $p^+ < \infty$. $\left(A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d), \|.\|_{A^{w,\nu}_{\alpha,p(.)}}\right)$ is a Banach ideal on $L^1_w(\mathbb{R}^d)$ under Θ convolution operation.

Proposition 5 Let $p^+ < \infty$ and w be a weight function of regular growth on $\mathbb{R}^d$. Then $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ is dense in $L^1_w(\mathbb{R}^d)$.

Proof: Let w be a weight function of regular growth on $\mathbb{R}^d$. Let us take the set $F^\alpha_{0,w}(\mathbb{R}^d) = \{f \in L^1_w(\mathbb{R}^d) | \mathcal{F}_\alpha f \in C_c(\mathbb{R}^d)\}$. Thus $F^\alpha_{0,w}(\mathbb{R}^d)$ is dense in $L^1_w(\mathbb{R}^d)$ by Corollary 2.4 in [24]. Since $C_c(\mathbb{R}^d) \subset L^{p(.)}_\nu(\mathbb{R}^d)$ by Proposition 2.3 in [8], then we may write

$$F^\alpha_{0,w}(\mathbb{R}^d) \subset A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d) \subset L^1_w(\mathbb{R}^d)$$

by the definition of the space $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$. By this inclusion, it is clear that $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ is dense in $L^1_w(\mathbb{R}^d)$.

Proposition 6. Let $p^+ < \infty$ and w be a weight function of regular growth on $\mathbb{R}^d$. Then $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ is an abstract Segal algebra with respect to $L^1_w(\mathbb{R}^d)$.

Proof: Let w be a weight function of regular growth on $\mathbb{R}^d$. It is shown that $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ is a Banach algebra and is a Banach ideal on $L^1_w(\mathbb{R}^d)$, also the inequality $\|g\Theta h\|_{A^{w,\nu}_{\alpha,p(.)}} \leq \|g\|_{A^{w,\nu}_{\alpha,p(.)}}\|h\|_{1,w}$ holds for all $g,h \in A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ by Theorem 3 and Theorem 4. Besides, by the definition of the norm $\|.\|_{A^{w,\nu}_{\alpha,p(.)}}$, we have $\|g\|_{1,w} \leq \|g\|_{A^{w,\nu}_{\alpha,p(.)}}$. Furthermore, $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ is dense in $L^1_w(\mathbb{R}^d)$ by Proposition 5. Therefore, $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ is an abstract Segal algebra with respect to $L^1_w(\mathbb{R}^d)$.

To working on function spaces for which the fractional Fourier transform is defined, we take the Θ convolution operation as the multiplication operator. If the definition of Θ convolution operation is examined as (1), we see that the translation operator in the ordinary convolution operation is replaced by the operator $y \to T_y M_z g$ from $\mathbb{R}^d$ into $L^1(\mathbb{R}^d)$ for all $g \in L^1(\mathbb{R}^d)$ and $z = (-y_1\cot\alpha_1, \dots, -y_d\cot\alpha_d)$ for all $y = (y_1, \dots, y_d) \in \mathbb{R}^d$. Therefore, the following theorem is very important for us.

Theorem 7. Let $p^+ < \infty$ and $z = (-y_1\cot\alpha_1, \dots, -y_d\cot\alpha_d)$ for all $y = (y_1, \dots, y_d) \in \mathbb{R}^d$.

1. $T_y M_z g \in A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ and
$$\|T_y M_z g\|_{A^{w,\nu}_{\alpha,p(.)}} \leq w(y)\|g\|_{A^{w,\nu}_{\alpha,p(.)}}$$

for all $g \in A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$.

2. The mapping $y \to T_y M_z g$ from $\mathbb{R}^d$ into $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ is continuous.

Proof:

1. Let $g \in A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$. Then $g \in L^1_w(\mathbb{R}^d)$ and $\mathcal{F}_\alpha g \in L^{p(.)}_\nu(\mathbb{R}^d)$. Let $z = (-y_1\cot\alpha_1, \dots, -y_d\cot\alpha_d)$ for all $y = (y_1, \dots, y_d) \in \mathbb{R}^d$. It is easy to show that $\|M_z g\|_{1,w} = \|g\|_{1,w}$ and $M_z g \in L^1_w(\mathbb{R}^d)$. Also it is well known that the space $L^1_w(\mathbb{R}^d)$ is translation invariant and holds $\|T_y g\|_{1,w} \leq w(y)\|g\|_{1,w}$ for all $y \in \mathbb{R}^d$, see [25]. Hence we have

$$\|T_y M_z g\|_{1,w} \leq w(y)\|g\|_{1,w}. \tag{6}$$

Let $\tau = (-y_1\operatorname{cosec}\alpha_1, \dots, -y_d\operatorname{cosec}\alpha_d)$. By the proof of Theorem 2.17 (1) in [24], we may write

$$\mathcal{F}_\alpha(T_y M_z g)(u) = e^{\sum_{j=1}^d \frac{i}{2} y_j^2 \cot\alpha_j} M_\tau \mathcal{F}_\alpha g(u). \tag{7}$$

Since $L^{p(.)}_\nu(\mathbb{R}^d)$ is strongly character invariant by Proposition 2.4 in [8], then we get

$$\begin{aligned}
\|\mathcal{F}_\alpha(T_y M_z g)\|_{p(.),\nu} &= \left\| e^{\sum_{j=1}^d \frac{i}{2} y_j^2 \cot\alpha_j} M_\tau \mathcal{F}_\alpha g \right\|_{p(.),\nu} && (8)\\
&= \left| e^{\sum_{j=1}^d \frac{i}{2} y_j^2 \cot\alpha_j} \right| \|M_\tau \mathcal{F}_\alpha g\|_{p(.),\nu} && (9)\\
&= \|M_\tau \mathcal{F}_\alpha g\|_{p(.),\nu} = \|\mathcal{F}_\alpha g\|_{p(.),\nu}
\end{aligned}$$

and

$$e^{\sum_{j=1}^{d}\frac{i}{2}y_j^2\cot\alpha_j}M_\tau\mathcal{F}_\alpha g\in L_\nu^{p(.)}(\mathbb{R}^d).$$

Hence, by using (6) and (8), we obtain

$$\left\|T_yM_zg\right\|_{A_{\alpha,p(.)}^{w,\nu}}\le w(y)\|g\|_{A_{\alpha,p(.)}^{w,\nu}}.$$

2. Firstly, we will show continuity at 0. Let $g\in A_{\alpha,p(.)}^{w,\nu}(\mathbb{R}^d)$ and $\lim_{n\to\infty}y_n=0$ for any sequence $(y_n)_{n\in\mathbb{N}}\subset\mathbb{R}^d$. Let $z=(-y_1\cot\alpha_1,\dots,-y_d\cot\alpha_d)$ for all $y=(y_1,\dots,y_d)\in\mathbb{R}^d$. Then the mapping $y\to T_yM_zg$ is continuous from $\mathbb{R}^d$ into $L_w^1(\mathbb{R}^d)$ by Theorem 2.1 in [24]. Let us take the sequences $(z_n)_{n\in\mathbb{N}}$ and $(\tau_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ where j sequences of coordinates $z_{nj}=-y_{nj}\cot\alpha_j$ and $\tau_{nj}=-y_{nj}\operatorname{cosec}\alpha_j$. Thus, we have

$$\left\|T_{y_n}M_{z_n}g-g\right\|_{1,w}\to 0 \tag{10}$$

as $n\to\infty$. By using (7), we obtain

$$\begin{aligned}\left\|\mathcal{F}_\alpha\left(T_{y_n}M_{z_n}g-g\right)\right\|_{p(.),\nu}&=\left\|\mathcal{F}_\alpha\left(T_{y_n}M_{z_n}g\right)-F_\alpha g\right\|_{p(.),\nu}\\&=\left\|e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\cot\alpha_j}M_{\tau_n}\mathcal{F}_\alpha g-e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\cot\alpha_j}\mathcal{F}_\alpha g\right.\\&\quad\left.+e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\cot\alpha_j}\mathcal{F}_\alpha g-\mathcal{F}_\alpha g\right\|_{p(.),\nu}\\&\le\left|e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\cot\alpha_j}\right|\left\|\left(M_{\tau_n}\mathcal{F}_\alpha g-\mathcal{F}_\alpha g\right)\right\|_{p(.),\nu}\\&\quad+\left|e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\cot\alpha_j}-1\right|\|\mathcal{F}_\alpha g\|_{p(.),\nu}\\&=\left\|\left(M_{\tau_n}\mathcal{F}_\alpha g-\mathcal{F}_\alpha g\right)\right\|_{p(.),\nu}\\&\quad+\left|e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\cot\alpha_j}-1\right|\|\mathcal{F}_\alpha g\|_{p(.),\nu}.\end{aligned}\tag{11}$$

Let $\tau=(-y_1\operatorname{cosec}\alpha_1,\dots,-y_1\operatorname{cosec}\alpha_d)$ for all $y=(y_1,\dots,y_d)\in\mathbb{R}^d$. It is easy to show that the mapping $y\to\tau$ from $\mathbb{R}^d$ into $\mathbb{R}^d$ is continuous. Since the mapping $y\to M_y\mathcal{F}_\alpha g$ from $\mathbb{R}^d$ into $L_\nu^{p(.)}(\mathbb{R}^d)$ is continuous by Proposition 2.4 in [8], then the composition mapping $y\to M_\tau\mathcal{F}_\alpha g$ from $\mathbb{R}^d$ into $L_\nu^{p(.)}(\mathbb{R}^d)$ is continuous. Therefore we get

$$\left\|\left(M_{\tau_n}\mathcal{F}_\alpha g-\mathcal{F}_\alpha g\right)\right\|_{p(.),\nu}\to 0 \tag{12}$$

as $n\to\infty$. Let $k_n=e^{\sum_{j=1}^{d}\frac{i}{2}y_{nj}^2\cot\alpha_j}-1$ for all $n\in\mathbb{N}$. Since $\lim_{n\to\infty}y_n=0$, then $|k_n|\to 0$ as $n\to\infty$. Thus by combining (10), (11) and (12), we have

$$\begin{aligned}\left\|T_{y_n}M_{z_n}g-g\right\|_{A_{\alpha,p(.)}^{w,\nu}}&=\left\|T_{y_n}M_{z_n}g-g\right\|_{1,w}+\left\|\mathcal{F}_\alpha\left(T_{y_n}M_{z_n}g\right)-\mathcal{F}_\alpha g\right\|_{p(.),\nu}\\&\le\left\|T_{y_n}M_{z_n}g-g\right\|_{1,w}+\left\|\left(M_{\tau_n}\mathcal{F}_\alpha g-\mathcal{F}_\alpha g\right)\right\|_{p(.),\nu}\\&\quad+|k_n|\|\mathcal{F}_\alpha g\|_{p(.),\nu}\to 0\end{aligned}$$

as $n \to \infty$. This proves that the mapping $y \to T_y M_z g$ is continuous at 0. By using the same technique in the proof of Theorem 2.17 (2) in [24], it is easy to show that the mapping $y \to T_y M_z g$ is continuous on $\mathbb{R}^d$.

Definition 8. Let w be a weight function on $\mathbb{R}^d$. The space $S_{w,\Theta}(\mathbb{R}^d)$ is subalgebra of $L_w^1(\mathbb{R}^d)$ under Θ convolution operation that satisfies the following conditions:

1. The space $S_{w,\Theta}(\mathbb{R}^d)$ is dense in $L_w^1(\mathbb{R}^d)$.
2. The subalgebra $S_{w,\Theta}(\mathbb{R}^d)$ is a Banach algebra under some norm $\|.\|_{S_{w,\Theta}}$ and the inequality $\|g\|_{1,w} \leq \|g\|_{S_{w,\Theta}}$ holds for all $g \in S_{w,\Theta}(\mathbb{R}^d)$.
3. Let $z = (-y_1 \cot\alpha_1, \dots, -y_d \cot\alpha_d)$ for all $y = (y_1, \dots, y_d) \in \mathbb{R}^d$. Then $T_y M_z g \in S_{w,\Theta}(\mathbb{R}^d)$ and the inequality $\|T_y M_z g\|_{S_{w,\Theta}} \leq w(y)\|g\|_{S_{w,\Theta}}$ holds for each $g \in S_{w,\Theta}(\mathbb{R}^d)$.
4. Let $z = (-y_1 \cot\alpha_1, \dots, -y_d \cot\alpha_d)$ for all $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ and $g \in S_{w,\Theta}(\mathbb{R}^d)$. Then the mapping $y \to T_y M_z g$ from $\mathbb{R}^d$ into $S_{w,\Theta}(\mathbb{R}^d)$ is continuous.

Example 9. Let $1 < p < \infty$ and w be a weight function on $\mathbb{R}^d$. The space $L_w^1(\mathbb{R}^d) \cap L_w^p(\mathbb{R}^d)$ with the norm $\|.\|_{L_w^1 \cap L_w^p} = \|.\|_{1,w} + \|.\|_{p,w}$ is a Banach algebra under Θ convolution operation by Example 2.4 in [26]. By the definition of $\|.\|_{L_w^1 \cap L_w^p}$, the inequality $\|.\|_{1,w} \leq \|.\|_{L_w^1 \cap L_w^p}$ holds. Since

$$C_c(\mathbb{R}^d) \subset L_w^1(\mathbb{R}^d) \cap L_w^p(\mathbb{R}^d) \subset L_w^1(\mathbb{R}^d),$$

then $L_w^1(\mathbb{R}^d) \cap L_w^p(\mathbb{R}^d)$ is dense in $L_w^1(\mathbb{R}^d)$. Let $z = (-y_1 \cot\alpha_1, \dots, -y_d \cot\alpha_d)$ for all $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ and $g \in L_w^1(\mathbb{R}^d) \cap L_w^p(\mathbb{R}^d)$. Since the spaces $L_w^1(\mathbb{R}^d)$ and $L_w^p(\mathbb{R}^d)$ are invariance under translation and modulation operators, then $T_y M_z g \in L_w^1(\mathbb{R}^d) \cap L_w^p(\mathbb{R}^d)$. Also, since $\|T_y M_z g\|_{1,w} = \|T_y g\|_{1,w}$, $\|T_y M_z g\|_{p,w} = \|T_y g\|_{p,w}$ and inequalities $\|T_y g\|_{1,w} \leq w(y)\|g\|_{1,w}$, $\|T_y g\|_{1,w} \leq w(y)\|g\|_{S_w}$ hold (see [25]), then we have $\|T_y M_z g\|_{L_w^1 \cap L_w^p} \leq w(y)\|g\|_{L_w^1 \cap L_w^p}$. Futhermore, it is well known that the mapping $y \to T_y M_z g$ from $\mathbb{R}^d$ into $L_w^1(\mathbb{R}^d)$ is continuous by Theorem 2.1 (2) in [24]. By using the same methods in the proof of Theorem 2.1 (2) in [24] and the proof of Theorem 2.9 in [26], it is easy to show that the mapping $y \to T_y M_z g$ from $\mathbb{R}^d$ into $L_w^p(\mathbb{R}^d)$ is continuous. Hence the mapping $y \to T_y M_z g$ from $\mathbb{R}^d$ into $L_w^1(\mathbb{R}^d) \cap L_w^p(\mathbb{R}^d)$ is continuous. Thus $L_w^1(\mathbb{R}^d) \cap L_w^p(\mathbb{R}^d)$ is the space $S_{w,\Theta}(\mathbb{R}^d)$.

The following example is clear from the definition of $\|.\|_{A_{\alpha,p(.)}^{w,\nu}}$, Theorem 3, Theorem 4, Proposition 5 and Theorem 7.

Example 10. Let $p^+ < \infty$ and w be a weight function of regular growth on $\mathbb{R}^d$. Then the space $A_{\alpha,p(.)}^{w,\nu}(\mathbb{R}^d)$ is the space $S_{w,\Theta}(\mathbb{R}^d)$.

Remark 11. If we take $\alpha_i = \frac{\pi}{2}$ for all $i = 1,2,\dots,d$, then the Θ convolution and fractional Fourier transform coincide ordinary convolution and Fourier transform, respectively. Then the spaces $S_{w,\Theta}(\mathbb{R}^d)$ and $A_{\alpha,p(.)}^{w,\nu}(\mathbb{R}^d)$ coincide the spaces $S_w(\mathbb{R}^d)$ (see [21]) and $A_{w,\nu}^{1,p(.)}(\mathbb{R}^d)$ (see [22]), respectively. It is well known that the space $A_{w,\nu}^{1,p(.)}(\mathbb{R}^d)$ is the space $S_w(\mathbb{R}^d)$ by Proposition 2.7 in [22].

Proposition 12. Let $p^+ < \infty$ and w be a weight function of regular growth on $\mathbb{R}^d$.Then $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ has an approximate identity with compactly supported fractional Fourier transforms.

Proof: Let w be a weight function of regular growth on $\mathbb{R}^d$ and also A be a finite subset of $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ where $A = \{g_1, \ldots, g_n\}$. Let $g \in A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ and $z = (-y_1 \cot\alpha_1, \ldots, -y_d \cot\alpha_d)$ for all $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$. It is shown that the mapping $y \to T_y M_z g$ from $\mathbb{R}^d$ into $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ is continuous by Theorem 7. Let $\varepsilon > 0$ be given. There exist $\delta_i > 0$ such that

$$\left\|T_y M_z g_i - g_i\right\|_{A^{w,\nu}_{\alpha,p(.)}} < \frac{\varepsilon}{2}$$

where $\|y\| < \delta_i$ for all $i = 1,2, \ldots, n$. Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$. Hence, we write

$$\left\|T_y M_z g_i - g_i\right\|_{A^{w,\nu}_{\alpha,p(.)}} < \frac{\varepsilon}{2} \tag{13}$$

where $\|y\| < \delta$ for all $i = 1,2, \ldots, n$. Let us take a positive function $f \in C_c(\mathbb{R}^d)$ such that $supp\ \ f \subset B(0, \delta)$ and $\int_{\mathbb{R}^d} f(x)dx = 1$. Then we get

$$\begin{aligned}(f\Theta g_i)(x) - g_i(x) &= \int_{\mathbb{R}^d} f(y) T_y M_z g_i(x) dy - g_i(x) \\ &= \int_{\mathbb{R}^d} f(y)\big(T_y M_z g_i(x) - g_i(x)\big)dy\end{aligned}$$

for all $x \in \mathbb{R}^d$ and $i = 1, \ldots, n$. Thus by using (13), we have

$$\begin{aligned}\|(f\Theta g_i) - g_i\|_{A^{w,\nu}_{\alpha,p(.)}} &= \left\|\int_{\mathbb{R}^d} f(y)\big(T_y M_z g_i - g_i\big)dy\right\|_{A^{w,\nu}_{\alpha,p(.)}} \\ &\leq \int_{supp\ f} |f(y)| \left\|T_y M_z g_i - g_i\right\|_{A^{w,\nu}_{\alpha,p(.)}} dy \\ &< \frac{\varepsilon}{2} \int_{supp\ f} f(y)dy = \frac{\varepsilon}{2}\end{aligned} \tag{14}$$

for all $i = 1, \ldots, n$. Let $M = \max\left\{\|g_1\|_{A^{w,\nu}_{\alpha,p(.)}}, \ldots, \|g_n\|_{A^{w,\nu}_{\alpha,p(.)}}\right\}$. Thus there exists a function $h \in F^{\alpha}_{0,w}(\mathbb{R}^d)$ such that

$$\|f - h\|_{1,w} < \frac{\varepsilon}{2M} \tag{15}$$

by Corollary 2.14 in [24]. Since $C_c(\mathbb{R}^d) \subset L^{p(.)}_{\nu}(\mathbb{R}^d)$ by Proposition 2.3 in [8], then $h \in A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$. Combining (14) and (15), we obtain

$$\begin{aligned}\|(h\Theta g_i) - g_i\|_{A^{w,\nu}_{\alpha,p(.)}} &\leq \|(h\Theta g_i) - (f\Theta g_i)\|_{A^{w,\nu}_{\alpha,p(.)}} + \|(f\Theta g_i) - g_i\|_{A^{w,\nu}_{\alpha,p(.)}} \\ &\leq \|f - h\|_{1,w} \|g_i\|_{A^{w,\nu}_{\alpha,p(.)}} + \|(f\Theta g_i) - g_i\|_{A^{w,\nu}_{\alpha,p(.)}} \\ &\leq \|f - h\|_{1,w} M + \|(f\Theta g_i) - g_i\|_{A^{w,\nu}_{\alpha,p(.)}} \\ &< \frac{\varepsilon}{2M} M + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

for all $i = 1, \ldots, n$. Therefore, $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ has an approximate identity with compactly supported fractional Fourier transforms by 1.3. Proposition in [27]

Now we will show the dual space of $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$.

Let the mapping $\Phi: A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d) \to L^1_w(\mathbb{R}^d) \times L^{p(.)}_\nu(\mathbb{R}^d)$ be defined by $\Phi(g) = (g, \mathcal{F}_\alpha g)$ for $p^+ < \infty$ and let $H = \Phi\left(A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)\right)$. Then

$$\|\Phi(g)\| = \|(g, \mathcal{F}_\alpha g)\| = \|g\|_{1,w} + \|\mathcal{F}_\alpha f\|_{p(.),\nu}$$

is a norm on H for all $g \in A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$. Hence, the map Φ is a linear isometry. Furthermore, we define a set K as

$$K = \left\{ \begin{matrix} (\varphi,\psi): \left((\varphi,\psi) \in L^\infty_w(\mathbb{R}^d) \times L^{p'(.)}_{\nu^{-1}}(\mathbb{R}^d)\right), \\ \int_{\mathbb{R}^d} g(x)\varphi(x)dx + \int_{\mathbb{R}^d} \mathcal{F}_\alpha g(y)\psi(y)dy = 0, \text{ for all } (g, \mathcal{F}_\alpha g) \in H \end{matrix} \right\},$$

where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$.

The following proposition is proved by Duality Theorem 1.7 in [28].

Proposition 13. Let $\mathrm{p}^+ < \infty$. The dual space of $\mathrm{A}^{\mathrm{w},\nu}_{\alpha,\mathrm{p}(.)}(\mathbb{R}^\mathrm{d})$ is isomorphic to $\mathrm{L}^\infty_{\mathrm{w}^{-1}}(\mathbb{R}^\mathrm{d}) \times \mathrm{L}^{\mathrm{p}'(.)}_{\nu^{-1}}(\mathbb{R}^\mathrm{d})/\mathrm{K}$, where $\frac{1}{\mathrm{p}(.)} + \frac{1}{\mathrm{p}'(.)} = 1$.

3. INCLUSION PROPERTIES OF THE SPACE $A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$

In this section, unless otherwise stated, we take $p^+ < \infty$. We begin with a basic proposition to be used as a tool to prove next findings.

Proposition 14. For every $0 \neq \mathrm{g} \in \mathrm{A}^{\mathrm{w},\nu}_{\alpha,\mathrm{p}(.)}(\mathbb{R}^\mathrm{d})$ there exists a constant $\mathrm{c(g)} > 0$ such that

$$\mathrm{c(g)w(y)} \le \left\|\mathrm{T_y M_z g}\right\|_{\mathrm{A}^{\mathrm{w},\nu}_{\alpha,\mathrm{p}(.)}} \le \mathrm{w(y)}\|\mathrm{g}\|_{\mathrm{A}^{\mathrm{w},\nu}_{\alpha,\mathrm{p}(.)}} \tag{16}$$

where $\mathrm{z} = (-\mathrm{y}_1\cot\alpha_1, \dots, -\mathrm{y}_\mathrm{d}\cot\alpha_\mathrm{d})$ for all $\mathrm{y} = (\mathrm{y}_1, \dots, \mathrm{y}_\mathrm{d}) \in \mathbb{R}^\mathrm{d}$.

Proof: Let us take $z = (-y_1\cot\alpha_1, \dots, -y_d\cot\alpha_d)$ for all $y = (y_1, \dots, y_d) \in \mathbb{R}^d$. Let $0 \neq g \in A^{w,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ be given. Thus $g \in L^1_w(\mathbb{R}^d)$ and $\mathcal{F}_\alpha g \in L^{p(.)}_\nu(\mathbb{R}^d)$. By [25, Proposition 1.7], there exists $c(g) > 0$ such that

$$c(g)w(y) \le \left\|T_y g\right\|_{1,w} \le w(y)\|g\|_{1,w}.$$

Hence, by using (8), we may write

$$\begin{aligned} c(g)w(y) &\le \left\|T_y g\right\|_{1,w} \le \left\|T_y M_z g\right\|_{1,w} + \left\|\mathcal{F}_\alpha\left(T_y M_z g\right)\right\|_{p(.),\nu} \\ &\le w(y)\|g\|_{1,w} + w(y)\|\mathcal{F}_\alpha g\|_{p(.),\nu} = w(y)\|g\|_{A^{w,\nu}_{\alpha,p(.)}}. \end{aligned}$$

This is the desired result.

Lemma 15. Let w_1, w_2 and ν be weight functions on $\mathbb{R}^d$. If $A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d) \subset L^1_{w_2}(\mathbb{R}^d)$, then $A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d) \hookrightarrow L^1_{w_2}(\mathbb{R}^d)$.

Proof: Let $A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d) \subset L^1_{w_2}(\mathbb{R}^d)$ and $g \in A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$. Let us define the sum norm $|||g||| = \|g\|_{A^{w_1,\nu}_{\alpha,p(.)}} + \|g\|_{1,w_2}$ on $A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$. It is easy to show that $A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ is a Banach space under this norm (see [17, Lemma 11]). Let us take the identity map from $\left(A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d), \|.\|_{A^{w_1,\nu}_{\alpha,p(.)}}\right)$ into $\left(A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d), |||.|||\right)$. Then by the closed graph theorem this identity map is bounded. Hence, there exists $c > 0$ such that $|||g||| \le c\|g\|_{A^{w_1,\nu}_{\alpha,p(.)}}$ for all $g \in A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$. Then by the definition of the sum norm $|||.|||$, we may write

$$\|g\|_{1,w_2} \le |||g||| \le c\|g\|_{A^{w_1,\nu}_{\alpha,p(.)}}$$

for all $g \in A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$. Therefore the identity map from $\left(A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d), \|.\|_{A^{w_1,\nu}_{\alpha,p(.)}}\right)$ into $\left(L^1_{w_2}(\mathbb{R}^d), \|.\|_{1,w_2}\right)$ is continuous.

Theorem 16. Let w_1, w_2 and ν be weight functions on $\mathbb{R}^d$. Then $A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d) \subset L^1_{w_2}(\mathbb{R}^d)$ if and only if $w_2 \prec w_1$.

Proof: Assume that $w_2 \prec w_1$. Therefore, there exists $c_1 > 0$ such that $w_2(x) \le c_1 w_1(x)$ for all $x \in \mathbb{R}^d$. Let $g \in A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d)$ be given. Then $g \in L^1_w(\mathbb{R}^d)$ and $\mathcal{F}_\alpha g \in L^{p(.)}_\nu(\mathbb{R}^d)$. Hence, we write

$$\|gw_2\|_1 \le c_1\|gw_1\|_1$$

and so

$$\|g\|_{1,w_2} \le c_1\|g\|_{1,w_1}.$$

By the definition of the norm $\|.\|_{A^{w_1,\nu}_{\alpha,p(.)}}$, we have

$$\|g\|_{1,w_2} \le c_1\|g\|_{1,w_1} + c_1\|\mathcal{F}_\alpha g\|_{p(.),\nu} = c_1\|g\|_{A^{w_1,\nu}_{\alpha,p(.)}} < \infty.$$

Consequently, $A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d) \subset L^1_{w_2}(\mathbb{R}^d)$.

Conversely, suppose that $A^{w_1,\nu}_{\alpha,p(.)}(\mathbb{R}^d) \subset L^1_{w_2}(\mathbb{R}^d)$. Let us take $z = (-y_1\cot\alpha_1, \ldots, -y_d\cot\alpha_d)$ for all $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$. By Proposition 14, there exist constants $c_2, c_3 > 0$ such that

$$c_2 w_1(y) \le \left\|T_y M_z g\right\|_{A^{w_1,\nu}_{\alpha,p(.)}} \le c_3 w_1(y). \tag{17}$$

By [5, Lemma 2.2] and using the equality $\left\|T_y g\right\|_{1,w_2} = \left\|T_y M_z g\right\|_{1,w_2}$, there exists constants $c_4, c_5 > 0$ such that

$$c_4 w_2(y) \le \left\|T_y M_z g\right\|_{1,w_2} \le c_5 w_2(y). \tag{18}$$

It is shown that the embedding I from $\left(A_{\alpha,p(.)}^{w_1,\nu}(\mathbb{R}^d), \|.\|_{A_{\alpha,p(.)}^{w_1,\nu}}\right)$ into $\left(L_{w_2}^1(\mathbb{R}^d), \|.\|_{1,w_2}\right)$ is continuous by Lemma 15. Then there exists a constant $c_6 > 0$ such that $\|g\|_{1,w_2} \leq c_6\|g\|_{A_{\alpha,p(.)}^{w_1,\nu}}$ for all $g \in A_{\alpha,p(.)}^{w_1,\nu}(\mathbb{R}^d)$. Since $T_y M_z g \in A_{\alpha,p(.)}^{w_1,\nu}(\mathbb{R}^d)$ by Theorem 7, we may write

$$\left\|T_y M_z g\right\|_{1,w_2} \leq c_6\left\|T_y M_z g\right\|_{A_{\alpha,p(.)}^{w_1,\nu}}. \tag{19}$$

Therefore, combining (17), (18) and (19), we get

$$c_4 w_2(y) \leq \left\|T_y M_z g\right\|_{1,w_2} \leq c_6\left\|T_y M_z g\right\|_{A_{\alpha,p(.)}^{w_1,\nu}} \leq c_6 c_3 w_1(y).$$

Let $c = \frac{c_6 c_3}{c_4}$. Then we have $w_2(y) \leq c w_1(y)$ for all $y \in \mathbb{R}^d$.

The proof of following lemma is very similiar to the proof of Lemma 15 and then we omit details.

Lemma 17. Let w_1, w_2 and ν be weight functions on $\mathbb{R}^{\mathrm{d}}$. If $\mathrm{A}_{\alpha,\mathrm{p}(.)}^{\mathrm{w}_1,\nu}(\mathbb{R}^{\mathrm{d}}) \subset \mathrm{A}_{\alpha,\mathrm{p}(.)}^{\mathrm{w}_2,\nu}(\mathbb{R}^{\mathrm{d}})$, then $\mathrm{A}_{\alpha,\mathrm{p}(.)}^{\mathrm{w}_1,\nu}(\mathbb{R}^{\mathrm{d}}) \hookrightarrow \mathrm{A}_{\alpha,\mathrm{p}(.)}^{\mathrm{w}_2,\nu}(\mathbb{R}^{\mathrm{d}})$.

Theorem 18. Let w_1, w_2 and ν be weight functions on $\mathbb{R}^{\mathrm{d}}$. Then $\mathrm{A}_{\alpha,\mathrm{p}(.)}^{\mathrm{w}_1,\nu}(\mathbb{R}^{\mathrm{d}}) \subset \mathrm{A}_{\alpha,\mathrm{p}(.)}^{\mathrm{w}_2,\nu}(\mathbb{R}^{\mathrm{d}})$ if and only if $\mathrm{w}_2 \prec \mathrm{w}_1$.

Proof: Assume that $w_2 \prec w_1$. Then it is easy to see that $A_{\alpha,p(.)}^{w_1,\nu}(\mathbb{R}^d) \subset A_{\alpha,p(.)}^{w_2,\nu}(\mathbb{R}^d)$.

Now, suppose that $A_{\alpha,p(.)}^{w_1,\nu}(\mathbb{R}^d) \subset A_{\alpha,p(.)}^{w_2,\nu}(\mathbb{R}^d)$. Let $g \in A_{\alpha,p(.)}^{w_1,\nu}(\mathbb{R}^d)$, then $g \in A_{\alpha,p(.)}^{w_2,\nu}(\mathbb{R}^d)$. By Proposition 14, there exist constants $c_1, c_2, c_3, c_4 > 0$ such that

$$c_1 w_1(y) \leq \left\|T_y M_z g\right\|_{A_{\alpha,p(.)}^{w_1,\nu}} \leq c_2 w_1(y) \tag{20}$$

and

$$c_3 w_2(y) \leq \left\|T_y M_z g\right\|_{A_{\alpha,p(.)}^{w_2,\nu}} \leq c_4 w_2(y), \tag{21}$$

where $z = (-y_1\cot\alpha_1, \ldots, -y_d\cot\alpha_d)$ for all $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$. Also, it is known that the embedding I from $\left(A_{\alpha,p(.)}^{w_1,\nu}(\mathbb{R}^d), \|.\|_{A_{\alpha,p(.)}^{w_1,\nu}}\right)$ into $\left(A_{\alpha,p(.)}^{w_2,\nu}(\mathbb{R}^d), \|.\|_{A_{\alpha,p(.)}^{w_2,\nu}}\right)$ is continuous by Lemma 17. Therefore, there exists a constant $c_5 > 0$ such that $\|g\|_{A_{\alpha,p(.)}^{w_2,\nu}} \leq c_5\|g\|_{A_{\alpha,p(.)}^{w_1,\nu}}$ for all $g \in A_{\alpha,p(.)}^{w_1,\nu}(\mathbb{R}^d)$. Since $T_y M_z g \in A_{\alpha,p(.)}^{w_1,\nu}(\mathbb{R}^d)$ by Theorem 7, we get

$$\left\|T_y M_z g\right\|_{A_{\alpha,p(.)}^{w_2,\nu}} \leq c_5\left\|T_y M_z g\right\|_{A_{\alpha,p(.)}^{w_1,\nu}}. \tag{22}$$

Therefore, combining (20), (21) and (22), we obtain

$$c_3 w_2(y) \leq \left\|T_y M_z g\right\|_{A_{\alpha,p(.)}^{w_2,\nu}} \leq c_5\left\|T_y M_z g\right\|_{A_{\alpha,p(.)}^{w_1,\nu}} \leq c_5 c_2 w_1(y).$$

Let $c = \frac{c_5 c_2}{c_3}$. Then we write $w_2(y) \leq c w_1(y)$ for all $y \in \mathbb{R}^d$.

Theorem 19. Let ν_1, ν_2 and w be weight functions on $\mathbb{R}^d$. If $\nu_2 \prec \nu_1$, then $A^{w,\nu_1}_{\alpha,p(.)}(\mathbb{R}^d) \subset A^{w,\nu_2}_{\alpha,p(.)}(\mathbb{R}^d)$.

Proof: Let $\nu_2 \prec \nu_1$, then it is well known from Proposition 2.6 (i) in [8] that $L^{p(.)}_{\nu_1}(\mathbb{R}^d) \hookrightarrow L^{p(.)}_{\nu_2}(\mathbb{R}^d)$. Let $g \in A^{w,\nu_1}_{\alpha,p(.)}(\mathbb{R}^d)$. Thus $g \in L^1_w(\mathbb{R}^d)$ and $\mathcal{F}_\alpha g \in L^{p(.)}_{\nu_1}(\mathbb{R}^d)$. Since $L^{p(.)}_{\nu_1}(\mathbb{R}^d) \subset L^{p(.)}_{\nu_2}(\mathbb{R}^d)$, then $\mathcal{F}_\alpha g \in L^{p(.)}_{\nu_2}(\mathbb{R}^d)$. Hence $g \in A^{w,\nu_2}_{\alpha,p(.)}(\mathbb{R}^d)$.

4. CONCLUSIONS

This article expands upon the research presented in reference [22] by applying it to the fractional Fourier transform. It is important to note that the fractional Fourier transform is an extended version of the Fourier transform that includes a parameter α. The weighted variable exponent Lebesgue spaces $L^{p(.)}_w(\mathbb{R}^d)$ can be associated with the weighted Lebesgue spaces $L^p_w(\mathbb{R}^d)$, in which the exponent $p(.) = p$ is a constant function. In this particular context, we attempt to provide an expanded discussion of the findings that were previously presented in the citation [17]. Furthermore, this research aligns with the conclusions drawn from previous studies referenced in [5] and [23].

REFERENCES

[1] Rudin, W., *Functional Analysis*, Mc Graw-Hill, New York, 1973.
[2] Rudin, W., *Real and Complex Analysis*, McGraw-Hill, New York, 1966.
[3] Gröchenig, K., *Weight functions in time-frequency analysis*, in: Pseudodifferential Operators: Partial Differential Equations and Time-Frequency Analysis, Fields Inst. Commun. 52, 343-366, Amer. Math. Soc., Providence, RI, 2007.
[4] Reiter, H., Stegeman, J. D., *Classical Harmonic Analysis and Locally Compact Group*, Clarendon Press, Oxford, 2000.
[5] Feichtinger, H. G., Gürkanlı, A, T., *Internat. J. Math. Sci.*, **13**, 517, 1990.
[6] Musielak, J., *Orlicz Spaces and Modular Spaces*, Springer-Verlag, Berlin, 1983.
[7] Kováčik, O., Rákosník, J., *Czechoslovak Mathematical Journal*, **41**(116), 592, 1991.
[8] Aydın, I., *Analele Stiintifice Ale Universitatii Ovidius Constanta-Seria Matematica,* **Vol. XX**(3), 2012.
[9] Kokilashvili, V., Samko, S., *Georgian Math. J.*, **10**(1), 145, 2003.
[10] Namias, V., *J. Inst. Math. Appl.*, **25**, 241, 1980.
[11] Ozaktas, H. M., Kutay, M. A., Zalevsky, Z., *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, Wiley, Chichester, 2001.
[12] Almeida, L. B., *IEEE Trans. Sig. Proc.*, **42**, 3084, 1994.
[13] Almeida, L. B., *IEEE Signal Proc. Let.*, **4**(1), 15, 1997.
[14] Zayed, A. I., *IEEE Signal Proc. Let.*, **3**, 310, 1996.
[15] Bultheel, A., Martinez, H., *A shattered survey of the fractional Fourier transform,* Department of Computer Science, K.U.Leuven, Report TW337, 2002.
[16] Singh, A. K., Saxena, R., *Wirel. Pers. Commun.*, **65**, 189, 2012.
[17] Toksoy E., Sandıkçı, A., *J. Inequal. Appl.*, **2015(**1), 2015.
[18] Feichtinger, H. G., Graham, C., Lakien, E., *Pacific Journal of Mathematics*, **80**(1), 117, 1979.

[19] Feichtinger, H. G., *Annales de l'Institut Fourier*, **27**(3), 135, 1977.
[20] Burnham, J. T., *Proceedings of the American Mathematical Society,* **32**(2), 551, 1972.
[21] Cigler J., *Indagationes Mathematicae,* **72**(3), 273, 1969.
[22] Unal, C., Aydın, I., *Proceedings of the Jangjeon Mathematical Society*, **18**(1), 109, 2015.
[23] Aydın, I., Gürkanlı, A. T., *Proceedings of the Jangjeon Mathematical Society*, **12**(2), 141, 2009.
[24] Sandıkçı, A., Toksoy, E., *Montes Taurus Journal of Pure and Applied Mathematics,* **4**(1), 1, 2022.
[25] Fischer, R. H., Gürkanlı, A. T,, Liu, T. S., *Math. Slovaca*, **46**(1), 71, 1996.
[26] Toksoy E., Sandıkçı, A., *Erzincan Üniversitesi Fen Bilimleri Enstitüsü Dergisi*, **13**(2), 923, 2020.
[27] Doran, R. S., Wichmann, J., *Approximate Identities and Factorization in Banach Modules*, 768, Springer-Verlag, New York, 1979.
[28] Liu, T., Rooij, A. V., *Math. Nachr.,* **42**, 29, 1969.