ORIGINAL PAPER

RT-CONVEX FUNCTIONS AND THEIR APPLICATIONS

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Abstract. *Convex functions play a crucial role in various fields of mathematics, optimization, economics, and machine learning due to their distinctive properties and applications. In this paper, a new class of convex functions, called the RT-convex functions, is presented. Moreover, Hermite-Hadamard-type inequalities for the RT-convex functions are discussed. A number of applications of the RT-convex functions is also discussed.*

Keywords: *Convex functions; Hermite-Hadamard type inequality; Fejer type inequality; RT-convex functions.*

1. INTRODUCTION

A positive function $\varphi : S \to R$ is called a convex function if the inequality

$$\varphi(cv+(1-c)w) \le c\,\varphi(v)+(1-c)\,\varphi(w)$$

holds for all $v, w \in S$ and $c \in [0,1]$.

Geometrically this means that the chord $\overline{PQ}$ lies on above side of the corresponding arc PQ for all distinct points P and Q lying on the curve $y = \varphi(x)$. Large dedicated literature is available for the study of the convex functions. We refer some of them; e.g. [1-4]. We also refer some work regarding properties, extensions, generalizations and applications of convex functions and the related Hermite-Hadamard type inequalities; e.g. [5-11].

In [12], a weighted version of Hermite-Hadamard type inequality for functions whose derivative's absolute values are $m-$convex and Fejér type inequalities for quasi-convex functions have also been developed. In [13], a variant of discrete Jensen-type inequality for harmonic convex functions has been presented. Jensen-type inequality and a variant of Jensen-type inequality for harmonic $p-$convex functions have also been found. In [14], Hermite-Hadamard type, various discrete Schur's and Jensen's inequalities for LR-$p-$convex interval-valued functions and Hermite-Hadamard type integral inequalities for the product of $p-$convex have been established. Results have also been verified by non-trivial examples. In [15], geometric-arithmetic-$F-$convex functions have been introduced and several integral inequalities of H-H type have also been established for these functions. In [16], various fractional convex inequalities of the Hermite–Hadamard type in addition to many other inequalities have been established. Many consequences of fractional inequalities

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and generalizations have also been obtained. In [17], Hermite–Hadamard inequality, fejer inequality and several additional inequalities for pre-invex interval-valued functions have been developed. In [18], some new Newton's type inequalities for differentiable convex functions through the well-known Riemann–Liouville fractional integrals and some inequalities of Riemann–Liouville fractional Newton's type have been proved. Moreover, validity of these inequalities has also been shown with examples. In [19], several integral inequalities for q and $h-$integrals in implicit form and for symmetric functions using certain types of convex functions have been established. It has also been shown that Hadamard-type inequalities are deductible for $q-$integrals, $h-$convex, $m-$convex and convex functions defined on the non-negative part of the real line. In [20], a new class of convex functions associated with strong $\eta-$convexity and Hermite Hadamard type inequality for this new class has been established. For this new class some examples, some inequalities and results for strong $\eta-$convex function have also been derived. In [21-24], some new classes of convex functions, strongly $h-$convex functions, log harmonically convex functions, and coordinate strongly $s-$convex functions have been introduced. Moreover, Hermite Hadamard type inequality for these new functions and for the product of two $(\alpha,\ m)-$convex functions have also been established.

2. MATERIALS AND METHODS

2.1. MATERIALS

A function defined in [25] is called HT -convex function, denoted by $\varphi \in HT(S)$ if the inequality

$$\varphi\left(\frac{vw}{cv+(1-c)w}\right) \le \frac{\sqrt{c}}{2\sqrt{1-c}}\varphi(w) + \frac{\sqrt{1-c}}{2\sqrt{c}}\varphi(v)$$

holds for all $v, w \in S$ and $c \in]0,1[$.

If $S \subseteq R \setminus \{0\}$ is a nonempty set, $\varphi \in HT(S)$ and φ is Lebesgue integrable on $[v,\ u]$, then

$$\varphi\left(\frac{2vw}{v+w}\right) \le \frac{vw}{w-v}\int_v^w \left(\frac{\varphi(x)}{x^2}dx \le \frac{\pi}{4}\{\varphi(v)+\varphi(w)\}.\right.$$

holds for all $v,\ w \in S$ with $v < w$ (see [25]).

For a convex function $\varphi : S \to R$; the inequality

$$\varphi\left(\frac{v+w}{2}\right) \le \frac{1}{w-v}\int_v^w \varphi(x)dx \le \frac{\varphi(v)+\varphi(w)}{2}$$

is called Hermite-Hadamard inequality where $v, w \in S$ and $v < w$ (see [26]).

A positive function $\varphi : S \to R$, defined in [27], is known as MT -convex function if

$$\varphi(cv+(1-c)w) \le \frac{\sqrt{c}}{2\sqrt{1-c}}\varphi(v) + \frac{\sqrt{1-c}}{2\sqrt{c}}\varphi(w)$$

holds for all $v, w \in S$ and $c \in]0,1[$.

A positive function φ, defined in [28], is known as r-convex function on $[a, b]$, if the inequality

$$\varphi(cv+(1-c)w) \leq \begin{cases} [c\varphi^r(v)+(1-c)\varphi^r(w)]^{1/r}, & r \neq 0, \\ [\varphi(v)]^c[\varphi(w)]^{1-c}, & r=0 \end{cases}$$

holds for each $v, u \in [a, b]$ and $c \in [0,1]$.

2.2. METHODS

The method contains functional analysis, convex analysis, special function, and calculus techniques. Hermite Hadamard type inequalities are supportive to get initial results.

3. RESULTS AND DISCUSSION

A positive valued function $\varphi: S \to R$ is called $RT-$convex function, denoted by $\varphi \in RT(S)$ if the inequality

$$\varphi(cv+(1-c)w) \leq \begin{cases} [\frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(w)]^{1/r}, & r \neq 0, \\ [\varphi(v)]^{\frac{\sqrt{c}}{2\sqrt{1-c}}}[\varphi(w)]^{\frac{\sqrt{1-c}}{2\sqrt{c}}}, & r=0 \end{cases}$$

holds for all $v, w \in S$ and $c \in]0,1[$.

Remarks 3.1: For $r=0$; if $c=\frac{1}{2}$, then the function φ becomes Jensen *log*-convex function.

For $r \neq 0$; if $c=\frac{1}{2}$, then the function φ becomes Jensen $r-$convex function.

Theorem 3.2: Let $\varphi: S \to R$ be a $RT-$convex function and $\varphi \in L_1[v,w]$. Then the following inequality holds

$$\frac{1}{w-v}\int_v^w \varphi(x)dx \leq \{\frac{\pi}{4}(\varphi^r(v)+\varphi^r(w))\}^{1/r}. \tag{1}$$

Proof: Let $\varphi: S \to R$ be $RT-$convex function and $r \geq 1$. Then by Jensen inequality, we have

$$(\frac{1}{w-v}\int_v^w \varphi(x)dx)^r \leq \frac{1}{w-v}\int_v^w \varphi^r(x)dx.$$

Using convexity of $\varphi^r(x)$, we obtain

$$\frac{1}{w-v}\int_v^w \varphi^r(x)dx \le \frac{1}{w-v}\int_v^w [\frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(w)]dx.$$

After evaluation, the above inequality becomes

$$\frac{1}{w-v}\int_v^w \varphi^r(x)dx \le \frac{\pi}{4}(\varphi^r(v)+\varphi^r(w)).$$

Using Jensen inequality, we have

$$(\frac{1}{w-v}\int_v^w \varphi(x)dx)^r \le \frac{\pi}{4}(\varphi^r(v)+\varphi^r(w)),$$

which leads to the required result (1).

2nd Method: Let $\varphi : S \to R$ be a non-negative $RT-$convex function and $\varphi \in L_1[v,w]$. Then the following inequality holds

$$\frac{1}{w-v}\int_v^w \varphi(x)dx \le \{\frac{\pi}{4}(\varphi^r(v)+\varphi^r(w))\}^{1/r}. \tag{2}$$

Proof: Since $\varphi : S \to R$ is $RT-$convex function, we have

$$\varphi(vc+(1-c)w) \le [\frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(w)]^{1/r} \tag{3}$$

and

$$\varphi(wc+(1-c)v) \le [\frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(w)+\frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(v)]^{1/r}. \tag{4}$$

From (3) and (4), it follows that

$$\varphi^r(vc+(1-c)w) \le \frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(w) \tag{5}$$

and

$$\varphi^r(wc+(1-c)v) \le \frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(w)+\frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(v). \tag{6}$$

Adding (5) and (6), we get

$$\varphi^r(cv+(1-c)w)+\varphi^r(cw+(1-c)v)\le \frac{1}{2\sqrt{c(1-c)}}(\varphi^r(v)+\varphi^r(w)). \tag{7}$$

Integrating (7) with respect to c on $[0,1]$, we have

$$\int_0^1\{\varphi^r(cv+(1-c)w)+\varphi^r(cw+(1-c)w)\}dc\le \int_0^1\frac{1}{2\sqrt{c(1-c)}}(\varphi^r(v)+\varphi^r(w))dc. \tag{8}$$

Since

$$\int_0^1\varphi^r(cv+(1-c)w)dc=\int_0^1\varphi^r(cw+(1-c)w)dc, \tag{9}$$

$$\int_0^1\frac{1}{2\sqrt{c(1-c)}}\{\varphi^r(v)+\varphi^r(w)\}=\frac{\pi}{2}\{\varphi^r(v)+\varphi^r(w)\}, \tag{10}$$

$$\int_0^1\varphi^r(cv+(1-c)w)dc=\frac{1}{w-v}\int_v^w\varphi^r(x)dx \tag{11}$$

and

$$(\frac{1}{w-v}\int_v^w\varphi(x)dx)^r\le\frac{1}{w-v}\int_v^w\varphi^r(x)dx. \tag{12}$$

Using (9) to (12), the inequality (8) leads to the result (2).

Corollary 3.3: Suppose $\varphi:S\to R$ be $RT-$convex function and $r=1$. Then the inequality (1) becomes

$$\frac{1}{w-v}\int_v^w\varphi(x)dx\le\frac{\pi}{4}\{\varphi(v)+\varphi(w)\}.$$

Theorem 3.4: Suppose $\varphi,\ \psi:S\to R$ be two non-negative $RT-$convex functions and $\varphi,\ \psi\in L_1[v,w]$. Then the following estimate holds

$$\begin{aligned}&\varphi^r(\frac{v+w}{2})\{\psi^r(v)+\psi^r(w)\}+\psi^r(\frac{v+w}{2})\{\varphi^r(v)+\varphi^r(w)\}\\ &\le\frac{16}{3\pi}\varphi^r\psi^r(\frac{v+w}{2})+\frac{2}{\pi}\{\varphi^r(v)+\varphi^r(w))(\psi^r(v)+\psi^r(w)\}.\end{aligned} \tag{13}$$

Proof: Since $\varphi,\ \psi$ are $RT-$convex functions, we have

$$\varphi^r(\frac{v+w}{2})\le\frac{1}{2}(\frac{\sqrt{c}}{2\sqrt{1-c}}+\frac{\sqrt{1-c}}{2\sqrt{c}})\{\varphi^r(v)+\varphi^r(w)\}$$

and

$$\psi^r(\frac{v+w}{2})\le\frac{1}{2}(\frac{\sqrt{c}}{2\sqrt{1-c}}+\frac{\sqrt{1-c}}{2\sqrt{c}})\{\psi^r(v)+\psi^r(w)\}.$$

Using $er+fp\le ep+fr,$ where $e,\ r,\ f,\ p\in R^{+},$ it follows from the above inequalities

$$\begin{aligned}&\frac{1}{2}\varphi^{r}(\frac{v+w}{2})(\frac{\sqrt{c}}{2\sqrt{1-c}}+\frac{\sqrt{1-c}}{2\sqrt{c}})\{\psi^{r}(v)+\psi^{r}(w)\}\\&+\frac{1}{2}\psi^{r}(\frac{v+w}{2})(\frac{\sqrt{c}}{2\sqrt{1-c}}+\frac{\sqrt{1-c}}{2\sqrt{c}})\{\varphi^{r}(v)+\varphi^{r}(w)\}\\&\le\varphi^{r}\psi^{r}(\frac{v+w}{2})+[\frac{1}{4}(\frac{\sqrt{c}}{\sqrt{1-c}}+\frac{\sqrt{1-c}}{\sqrt{c}})]^{2}\{(\varphi^{r}(v)+\varphi^{r}(w)\times(\psi^{r}(v)+\psi^{r}(w)\}.\end{aligned}$$

Simplifying the above inequality, we obtain

$$\begin{aligned}&\frac{1}{4}\varphi^{r}(\frac{v+w}{2})\{\psi^{r}(v)+\psi^{r}(w)\}(\frac{\sqrt{c}}{\sqrt{1-c}}+\frac{\sqrt{1-c}}{\sqrt{c}})\\&+\frac{1}{4}\psi^{r}(\frac{v+w}{2})\{\varphi^{r}(v)+\varphi^{r}(w)\}(\frac{\sqrt{c}}{\sqrt{1-c}}+\frac{\sqrt{1-c}}{\sqrt{c}})\\&\le\varphi^{r}\psi^{r}(\frac{v+w}{2})+\frac{1}{16v(1-w)}\{(\varphi^{r}(v)+\varphi^{r}(w))(\psi^{r}(v)+\psi^{r}(w))\}.\end{aligned}$$

Multiplying by $c(1-c)$ on both sides of above inequality, we obtain

$$\begin{aligned}&\frac{1}{4}\varphi^{r}(\frac{v+w}{2})\{(\psi^{r}(v)+\psi^{r}(w))(c\sqrt{c}\sqrt{1-c})+(1-c)\sqrt{c}\sqrt{1-c})\}\\&+\frac{1}{4}\psi^{r}(\frac{v+w}{2})\{\varphi^{r}(v)+\varphi^{r}(w)\}\{(c\sqrt{c}\sqrt{1-c})+(1-c)\sqrt{c}\sqrt{1-c})\}\\&\le\varphi^{r}\psi^{r}(\frac{v+w}{2})c(1-c)+\frac{1}{16}\{(\varphi^{r}(v)+\varphi^{r}(w))(\psi^{r}(v)+\psi^{r}(w))\}.\end{aligned}\tag{14}$$

Integrating the inequality (14) with respect to c on $[0,1]$, we have

$$\begin{aligned}&\int_{0}^{1}[\frac{1}{4}\varphi^{r}(\frac{v+w}{2})\{\psi^{r}(v)+\psi^{r}(w)\}((1-c)\sqrt{c}\sqrt{1-c}+c\sqrt{c}\sqrt{1-c}))\\&+\frac{1}{4}\psi^{r}(\frac{v+w}{2})\{\varphi^{r}(v)+\varphi^{r}(w)\}(c\sqrt{c}\sqrt{1-c})+(1-c)\sqrt{c}\sqrt{1-c})]dc\\&\le\int_{0}^{1}[\varphi^{r}\psi^{r}(\frac{v+w}{2})c(1-c)+\frac{1}{16}\{(\varphi^{r}(v)+\varphi^{r}(w))(\psi^{r}(v)+\psi^{r}(w))\}]dc.\end{aligned}$$

After evaluation, it becomes

$$\frac{1}{4}\varphi^{r}(\frac{v+w}{2})\{\psi^{r}(v)+\psi^{r}(w)\}\int_{0}^{1}[(c\sqrt{c}\sqrt{1-c})+(1-c)\sqrt{c}\sqrt{1-c})]dc$$

$$+\frac{1}{4}\psi^r(\frac{v+w}{2})\{\varphi^r(v)+\varphi^r(w)\}\int_0^1[(c\sqrt{c}\sqrt{1-c})+(1-c)\sqrt{c}\sqrt{1-c})]dc$$
$$\leq \varphi^r\psi^r(\frac{v+w}{2})\int_0^1 c(1-c)dc+\frac{1}{16}\{(\varphi^r(v)+\varphi^r(w))(\psi^r(v)+\psi^r(w))\}\int_0^1 1dc.$$

Finally, we obtain

$$\frac{1}{4}\varphi^r(\frac{v+w}{2})\{\psi^r(v)+\psi^r(w)\}[\int_0^1((c\sqrt{c}\sqrt{1-c}))dc+\int_0^1((1-c)\sqrt{c}\sqrt{1-c}))dc]$$
$$+\frac{1}{4}\psi^r(\frac{v+w}{2})(\varphi^r(v)+\varphi^r(w))[\int_0^1((c\sqrt{c}\sqrt{1-c}))dc+\int_0^1((1-c)\sqrt{c}\sqrt{1-c}))dc] \tag{15}$$
$$\leq \varphi^r\psi^r(\frac{v+w}{2})\int_0^1(c(1-c))dc+\frac{1}{16}\{(\varphi^r(v)+\varphi^r(w))(\psi^r(v)+\psi^r(w))\}.$$

Since,

$$\int_0^1(c\sqrt{c(1-c)})dc=\frac{\pi}{16},$$
$$\int_0^1((1-c)\sqrt{c(1-c)})dc=\frac{\pi}{16}$$

and

$$\int_0^1 c(1-c)dc=\frac{1}{6},$$

the inequality (15) leads to the result (13).

Theorem 3.5: Assume that $\varphi,\ \psi: S\to R$ be $RT-$convex and $ST-$convex functions respectively and $v<w$. Then the following inequality holds if $r>v$ and $\frac{1}{r}+\frac{1}{r}=1$

$$\frac{1}{w-v}\int_v^w\varphi(x)\psi(x)dx\leq(\frac{\pi}{2}(\frac{\varphi^r(v)+\varphi^r(w)}{2}))^{1/r}(\frac{\pi}{2}(\frac{\psi^s(v)+\psi^S(w)}{2}))^{1/s}. \tag{16}$$

Proof: Since φ and ψ are $RT-$convex and $ST-$convex functions, we have

$$\varphi(cv+(1-c)w)\leq(\frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(w))^{1/r} \tag{17}$$

and

$$\psi(cv+(1-c)w)\leq(\frac{\sqrt{c}}{2\sqrt{1-c}}\psi^s(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\psi^s(w))^{1/s}. \tag{18}$$

Using (17) and (18), we obtain

$$\frac{1}{w-v}\int_v^w \varphi(x)\psi(x)dx = \int_0^1 \varphi(cv+(1-c)w)\psi(cv+(1-c)w)dc$$
$$\le \int_0^1 \{(\frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(w))^{1/r}$$
$$(\frac{\sqrt{c}}{2\sqrt{1-c}}\psi^s(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\psi^s(w))^{1/s})\}dc.$$

Using Holders inequality, we have

$$\frac{1}{w-v}\int_v^w \varphi(x)\psi(x)dx \le \int_0^1 (\frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(w))^{1/r}(\frac{\sqrt{c}}{2\sqrt{1-c}}\psi^s(v)$$
$$+\frac{\sqrt{1-c}}{2\sqrt{c}}\psi^s(w))^{1/s}dc$$
$$=\{(\varphi^r(v)\int_0^1 \frac{\sqrt{c}}{2\sqrt{1-c}}dc+\varphi^r(w)\int_0^1 \frac{\sqrt{1-c}}{2\sqrt{c}}dc)^{1/r}\}\{(\psi^s(v)$$
$$\int_0^1 \frac{\sqrt{c}}{2\sqrt{1-c}}dc+\psi^s(w)\int_0^1 \frac{\sqrt{1-c}}{2\sqrt{c}}dc)^{1/s}\}$$
$$=[\frac{\pi}{2}\{\frac{\varphi^r(v)+\varphi^r(w)}{2}\}]^{1/r}[\frac{\pi}{2}\{\frac{\psi^s(v)+\psi^s(w)}{2}\}]^{1/s},$$

which leads to the result (16).

Corollary 3.6: In above theorem (3.5), if $r=s=2$. Then

$$\frac{1}{w-v}\int_v^w \varphi(x)\psi(x)dx \le \sqrt{\frac{\pi}{4}\{\varphi^2(v)+\varphi^2(w)\}}\sqrt{\frac{\pi}{4}\{\psi^2(v)+\psi^2(w)\}}.$$

Corollary 3.7: In above theorem (3.5), if $r=s=2$ and $\varphi(x)=\psi(x)$. Then

$$\frac{1}{w-v}\int_v^w (\varphi(x))^2 dx \le \frac{\pi}{2}\{\frac{(\varphi(v))^2+(\varphi(w))^2}{2}\}.$$

Theorem 3.8: Suppose that φ and ψ be $RT-$convex functions and $v=w.$ Then the following inequality holds

$$\frac{1}{w-v}\int_v^w \varphi(x)\psi(x)dx \le \frac{1}{4^{1/r}(w-v)^{1/r}}[(w-v)\left\{p^r(v,w)+p^r(w,v)\right\}$$
$$+\varphi^r(w)\varphi^r(w)\int_v^w (\frac{v-x}{x-w})dx+\varphi^r(v)\psi^r(v)\int_v^w (\frac{x-w}{v-x})dx]^{1/r}. \tag{19}$$

Proof: Since φ and ψ are $RT-$convex functions, we have

$$\varphi^r(cv+(1-c)w) \le \frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(v) + \frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(w)$$

and

$$\psi^r(cv+(1-c)w) \le \frac{\sqrt{c}}{2\sqrt{1-c}}\psi^r(v) + \frac{\sqrt{1-c}}{2\sqrt{c}}\psi^r(w).$$

Using the Jensen inequality, we have

$$\left(\frac{1}{w-v}\int_v^w \varphi(x)\psi(x)dx\right)^r \le \frac{1}{w-v}\int_v^w \varphi^r(x)\psi^r(x)dx.$$

Using convexity of φ^r and ψ^r, we have

$$\begin{aligned}
&\left(\frac{1}{w-v}\int_v^w \varphi(x)\psi(x)dx\right)^r \\
&\le \frac{1}{w-v}\int_v^w \varphi^r(cv+(1-c)w)\psi^r(cv+(1-c)w)dx \\
&= \frac{1}{w-v}\int_v^w \left(\frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(v) + \frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(w)\right)\left(\frac{\sqrt{c}}{2\sqrt{1-c}}\psi^r(v) + \frac{\sqrt{1-c}}{2\sqrt{c}}\psi^r(w)\right)dc\,.
\end{aligned}$$

After evaluation, we obtain

$$\begin{aligned}
&= \frac{1}{w-v}\Big[\frac{\varphi^r(v)\psi^r(v)}{4}\int_v^w \left(\frac{c}{(1-c)}\right)dc + \frac{\varphi^r(v)\psi^r(w)}{4}\int_v^w dc + \frac{\varphi^r(w)\psi^r(v)}{4}\int_v^w dc \\
&\quad + \frac{\varphi^r(w)\psi^r(w)}{4}\int_v^w \left(\frac{1-c}{c}\right)dc\Big].
\end{aligned}$$

Applying limits and changing the variables, we have

$$\begin{aligned}
\left(\frac{1}{w-v}\int_v^w \varphi(x)\psi(x)dx\right)^r &\le \frac{\varphi^r(v)\psi^r(w)}{4} + \frac{\varphi^r(w)\psi^r(v)}{4} + \frac{\varphi^r(v)\psi^r(v)}{4}\int_0^1 \left(\frac{c}{1-c}\right)dc \\
&\quad + \frac{\varphi^r(w)\psi^r(w)}{4}\int_0^1 \left(\frac{1-c}{c}\right)dc \\
&= \frac{1}{4}\{\varphi^r(v)\psi^r(w) + \varphi^r(w)\psi^r(v)\} + \frac{\varphi^r(v)\psi^r(v)}{4(w-v)}\int_v^w \left(\frac{x-w}{v-x}\right)dx \\
&\quad + \frac{\varphi^r(w)\psi^r(w)}{4(w-v)}\int_v^w \left(\frac{v-x}{x-w}\right)dx\,.
\end{aligned}$$

Finally, we obtain

$$
\begin{aligned}
&(\frac{1}{w-v}\int_v^w \varphi(x)\psi(x)dx)^r \leq \frac{1}{4(w-v)}[(w-v)\{p^r(v,w)+p^r(w,v)\} \\
&+\varphi^r(v)\psi^r(v)\int_v^w(\frac{x-w}{v-x})dx+\varphi^r(w)\psi^r(w)\int_v^w(\frac{v-x}{x-w})dx].
\end{aligned}
\tag{20}
$$

Taking $1/r^{th}$ power of (20), leads to the result (19).

Theorem 3.9: Let $\varphi,\ \psi\ :\ S\subseteq R\rightarrow R$ two non-negative $RT-$convex functions and $\varphi,\ \psi\in L_1[v,\ w].$ Then the following inequalit1y holds

$$
\begin{aligned}
&\varphi^r(v)\frac{(w-x)\sqrt{(w-x)(x-v)}}{(w-v)^3}\int_v^w\psi^r(x)dx+\varphi^r(w)\frac{(x-v)\sqrt{(w-x)(x-v)}}{(w-v)^3}\int_v^w\psi^r(x)dx \\
&+\psi^r(v)\frac{(w-x)\sqrt{(w-x)(x-v)}}{(w-v)^3}\int_v^w\varphi^r(x)dx+\psi^r(w)\frac{(x-v)\sqrt{(w-x)(x-v)}}{(w-v)^3}\int_v^w\varphi^r(x)dx \\
&\leq\frac{1}{2}\{\frac{1}{3}M(v,w))+\frac{1}{6}N(v,w)+\int_v^w(w-x)(x-v)\varphi^r(x)\psi^r(x)d\}.
\end{aligned}
\tag{21}
$$

where

$$M\ (v,\ w)=\varphi^r(v)\psi^r(v)+\varphi^r(w)\psi^r(w)\ and\ N(v,\ w)=\varphi^r(v)\psi^r(w)+\varphi^r(w)\psi^r(v).$$

Proof: Since φ and ψ are $RT-$convex functions, we have

$$\varphi^r(cv+(1-c)w)\leq\frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(w) \tag{22}$$

and

$$\psi^r(cv+(1-c)w)\leq\frac{\sqrt{c}}{2\sqrt{1-c}}\psi^r(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\psi^r(w). \tag{23}$$

Using $er\ +\ fp\ \leq\ ep\ +\ fr,$ where $e,\ f,\ r,\ p\in R^+,$ (22) and (23) follows that

$$
\begin{aligned}
&\varphi^r(cv+(1-c)w)\{\frac{\sqrt{c}}{2\sqrt{1-c}}\psi^r(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\psi^r(w)\} \\
&+\psi^r(cv+(1-c)w)\{\frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(w)\} \\
&\leq\{\frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(w)\}\{\frac{\sqrt{c}}{2\sqrt{1-c}}\psi^r(v)+\frac{\sqrt{1-c}}{2\sqrt{c}}\psi^r(w)\} \\
&+\varphi^r(cv+(1-c)w)\psi^r(cv+(1-c)w).
\end{aligned}
$$

After evaluation the above inequality, we obtain

$$=(\frac{\sqrt{c}}{2\sqrt{1-c}})^2\varphi^r(v)\psi^r(v)+\frac{\sqrt{c(1-c)}}{4\sqrt{c(1-c)}})\varphi^r(v)\psi^r(w)+(\frac{\sqrt{c(1-c)}}{4\sqrt{c(1-c)}})\varphi^r(w)\psi^r(v)$$
$$+(\frac{\sqrt{1-c}}{2\sqrt{c}})^2\varphi^r(w)\psi^r(w)+\varphi^r(cv+(1-c)w)\psi^r(cv+(1-c)w)$$

Finally, we obtain

$$\psi^r(v)\frac{\sqrt{c}}{2\sqrt{1-c}}\varphi^r(cv+(1-c)w)+\psi^r(w)\frac{\sqrt{1-c}}{2\sqrt{c}}\varphi^r(cv+(1-c)w)$$
$$+\varphi^r(v)\frac{\sqrt{c}}{2\sqrt{1-c}}\psi^r(cv+(1-c)w)+\varphi^r(w)\frac{\sqrt{1-c}}{2\sqrt{c}}\psi^r(cv+(1-c)w)$$
$$\leq\frac{c}{4(1-c)}\varphi^r(v)\psi^r(v)+\frac{1-c}{4c}\varphi^r(w)\psi^r(w)+\frac{1}{4}\varphi^r(v)\psi^r(w)+\frac{1}{4}\varphi^r(w)\psi^r(v)$$
$$+\varphi^r(cv+(1-c)w)\psi^r(cv+(1-c)w).$$

Multiplying by $c(1-\mathrm{c})$ on both sides of above inequality, we obtain

$$c\sqrt{c}\sqrt{1-c})\psi^r(v)\varphi^r(cv+(1-c)w)+(1-c)(\sqrt{c}\sqrt{1-c})\psi^r(w)\varphi^r(cv+(1-c)w)$$
$$+c\sqrt{c}\sqrt{1-c})\varphi^r(v)\psi^r(cv+(1-c)w)+(1-c)(\sqrt{c}\sqrt{1-c})\varphi^r(w)\psi^r(cv+(1-c)w)$$
$$\leq\frac{1}{2}[c^2\varphi^r(v)\psi^r(v)+(1-c)^2\varphi^r(w)\psi^r(w)+c(1-c)\{\varphi^r(v)\psi^r(w)+\varphi^r(w)\psi^r(v)\}$$
$$+c(1-c)\{\varphi^r(cv+(1-c)w)+\psi^r(cv+(1-c)w)\}].$$

Integrating above inequality on $[0,1]$ with respect to c, we obtain

$$\int_0^1[c\sqrt{c}\sqrt{1-c})\psi^r(v)\varphi^r(cv+(1-c)w)+(1-c)(\sqrt{c}\sqrt{1-c})\psi^r(w)\varphi^r(cv+(1-c)w)$$
$$+c\sqrt{c}\sqrt{1-c})\varphi^r(v)\psi^r(cv+(1-c)w)+(1-c)(\sqrt{c}\sqrt{1-c})\varphi^r(w)\psi^r(cv+(1-c)w)]dc$$
$$\leq\int_0^1[\frac{1}{2}\{c^2\varphi^r(v)\psi^r(v)+(1-c)^2\varphi^r(w)\psi^r(w)\}+c(1-c)\{\varphi^r(v)\psi^r(w)+\varphi^r(w)\psi^r(v)\}$$
$$+c(1-c)\{\varphi^r(cv+(1-c)w)+\psi^r(cv+(1-c)w)\}]dc.$$

After evaluation, we obtain

$$\psi^r(v)\int_0^1(c\sqrt{c}\sqrt{1-c})\varphi^r(cv+(1-c)w)dc+\psi^r(w)\int_0^1(1-c)(\sqrt{c}\sqrt{1-c})\varphi^r(cv+(1-c)w)dc$$
$$+\varphi^r(v)\int_0^1(c\sqrt{c}\sqrt{1-c})\times\psi^r(cv+(1-c)w)dc+\varphi^r(w)\int_0^1(1-c)(\sqrt{c}\sqrt{1-c}) \tag{24}$$
$$\times\psi^r(cv+(1-c)w)dc$$

$$\leq \frac{1}{2}(\varphi^r(v)\psi^r(v)\int_0^1 c^2 dc + \varphi^r(w)\psi^r(w)\int_0^1 (1-c)^2 dc + \{\varphi^r(v)\psi^r(w) + \varphi^r(w)\psi^r(v)\}\int_0^1 c(1-c)dc$$
$$+\int_0^1 c(1-c)\{\varphi^r(cv+(1-c)w + \psi^r(cv+(1-c)w\}dc.$$

Substituting $cv+(1-c)w = x,$ in (24), we get

$$\int_0^1 (c\sqrt{c}\sqrt{1-c})\varphi^r(cv+(1-c)w)dc = \frac{w-x}{w-v}\frac{\sqrt{(w-x)(x-v)}}{(w-v)}\frac{1}{w-v}\int_v^w \varphi^r(x)dx, \tag{25}$$

$$\int_0^1 (1-c)(\sqrt{c}\sqrt{1-c})\varphi^r(cv+(1-c)w)dc = \frac{x-v}{w-v}\frac{\sqrt{(w-x)(x-v)}}{(w-v)}\frac{1}{w-v}\int_v^w \varphi^r(x)dx, \tag{26}$$

$$\int_0^1 (c\sqrt{c}\sqrt{1-c})\psi^r(cv+(1-c)w)dc = \frac{w-x}{w-v}\frac{\sqrt{(w-x)(x-v)}}{(w-v)}\frac{1}{w-v}\int_v^w \psi^r(x)dx, \tag{27}$$

$$\int_0^1 (1-c)(\sqrt{c}\sqrt{1-c})\psi^r(cv+(1-c)w)dc = \frac{x-v}{w-v}\frac{\sqrt{(w-x)(x-v)}}{(w-v)}\frac{1}{w-v}\int_v^w \psi^r(x)dx, \tag{28}$$

$$\int_0^1 c^2 dc = \int_0^1 (1-c)^2 dc = \frac{1}{3} \tag{29}$$

and

$$\int_0^1 c(1-c)dc = \frac{1}{6}. \tag{30}$$

Using (25) to (30) in (24), leads to the required result (21).

4. APPLICATIONS

In this part, we set some applications by using special means. Means are significant in pure and applied mathematics particularly they are applied in mathematical approximation. They are arranged as follows:

$$H \leq G \leq L \leq I \leq A$$

The special means i.e., arithmetic mean, Geometric mean, Logarithmic mean, and Identic mean of two numbers v, w are defined as

$$A(v,w) = \frac{w+v}{2} \tag{31}$$

$$G(v,w) = (\sqrt{vw}) \tag{32}$$

$$L_p(v,w) = \left(\frac{w^{p+1} - v^{p+1}}{(p+1)(w-v)}\right)^{\frac{1}{p}} \tag{33}$$

$$I(v,w) = \frac{1}{e}\left(\frac{w^p}{v^p}\right)^{\frac{1}{w-v}} \tag{34}$$

Proposition 4.1: Assume that $1 < v < w$ and $p \in [0,\ 1].$ Then the following inequality holds

$$L_1^r(v,w) \le \frac{\pi}{2}\ A(v^r, w^r). \tag{35}$$

Proof: Putting $\varphi(x) = x$ in theorem $(3.2),$ we get

$$\frac{1}{w-v}\int_v^w x dx \le \{\frac{\pi}{4}(v^r + w^r)\}^{1/r}. \tag{36}$$

Solving the left side of above inequality, we get

$$\frac{1}{w-v}\int_v^w x dx = \frac{w^2 - v^2}{2(w-v)}.$$

Putting the value of $\frac{1}{(w-v)}\int_v^w x dx$ in (36), we have

$$\frac{w^2 - v^2}{2(w-v)} \le \{\frac{\pi}{2}(\frac{v^r + w^r}{2})\}^{1/r}.$$

Using (31) and (33), it follows that

$$L_1(v,w) \le \{\frac{\pi}{2}\ A(v^r, w^r)\}^{1/r}. \tag{37}$$

Taking r^{th} power on both sides of (37), leads to the result (35).

Proposition 4.2: suppose $v, w > 0$ and $v < w.$ Then the following result holds

$$(-1)^{r+1} I^r(v,w) \ge G^{\frac{\pi r}{2}}(v,w). \tag{38}$$

Proof: Putting $\varphi(x) = -\ln x$ in theorem $(3.2),$ we obtain

$$-\frac{1}{w-v}\int_v^w \ln x \le (-\frac{\pi}{4}(\ln v^r + \ln w^r))^{1/r}$$

After evaluation, we obtain

$$-\frac{1}{w-v}\int_v^w \ln x \le (-\frac{\pi r}{4}(\ln(vw))^{1/r}.$$

Using the equation (32) and (34), we get

$$-\ln I(v,w) \le \{-\frac{\pi r}{4}(\ln(G^2(vw)\}^{1/r}$$
$$=\{-(\ln(G^{2(\frac{\pi r}{4})}(vw)\}^{1/r}.$$

Finally, we obtain

$$-\ln I(v,w) \le \{-(\ln(G^{\frac{\pi r}{2}}(vw)\}^{1/r}.$$

Taking r^{th} power on both sides of above inequality, we obtain

$$(-1)^r \ln I^r(v,w) \le -\ln G^{\frac{\pi r}{2}}(v,w).$$

Multiplying both sides by (-1), we have

$$(-1)^{r+1} \ln I^r(v,w) \ge \ln G^{\frac{\pi r}{2}}(v,w),$$

which leads to the required result (38).

Proposition 4.3: Assume that $1 < v < w$ and $p \in [0,\ 1/100]$. Then we have

$$L_p{}^{rp}(v,w) \le \frac{\pi}{2}\ A(v^{rp}, w^{rp}). \tag{39}$$

Proof: Putting $\varphi(x) = x^p$ in theorem (3.2), we get

$$\frac{1}{w-v}\int_v^w x^p dx \le \{\frac{\pi}{4}(v^{rp}+w^{rp}\}^{1/r}. \tag{40}$$

Solving left side of above inequality, we get

$$\frac{1}{w-v}\int_v^w x^p dx = \frac{w^{p+1}-v^{p+1}}{(p+1)(w-v)}.$$

Substituting the value of $\frac{1}{w-v}\int_v^w x^p dx$ in (40), we get

$$\frac{w^{p+1}-v^{p+1}}{(p+1)(w-v)} \le \{\frac{\pi}{4}(v^{rp}+w^{rp})\}^{1/r}$$
$$=\{\frac{\pi}{2}(\frac{v^{rp}+w^{rp}}{2})\}^{1/r}.$$

Using (31) and (33), and then taking r^{th} power on both sides, leads to the result (39).

5. CONCLUSION

In this study, using both the $r-$convex functions and $MT-$convex functions, a new class $RT-$convex functions is introduced. Some inequalities of Hermite-Hadamard type for this new class are established. Based on these inequalities, it is concluded that these inequalities provide useful tools to analyze and estimate integrals involving various classes of functions. Moreover, these inequalities have a wide range of applications in different fields of mathematics and computational sciences.

In addition, this article highlights the importance of $RT-$convexity as it plays a crucial role in deriving and establishing these inequalities. $RT-$convex functions are particularly simple to optimize. For this reason, there is a very rich theory to solve convex optimization problems with numerous practical applications. Finally, it stimulates additional exploration and investigation in this domain, such as the development of new inequalities, the study of their applications in different fields, and to examine their connections with other areas of mathematics.

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