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SEMI-INVARIANT HYPERSURFACES IN RIEMANNIAN PRODUCT MANIFOLDS

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Abstract. *Semi-invariant hypersurfaces of almost product Riemannian manifolds are introduced. Using the fact that this kind of hypersurfaces admits a para-contact structure, some basic geometric properties are investigated. Some examples of semi-invariant are presented. With the aid of the Lie derivative and Riemannian curvature tensors, various characterizations are obtained.*

Keywords: *Hypersurface; almost product Riemannian manifold; para-contact structure; Lie derivative; curvature.*

1. INTRODUCTION

The theory of hypersurfaces of Riemannian manifolds includes comprehensive geometric properties by admitting various differentiable structures. For example, every hypersurface of an almost complex space form possesses a contact structure and every hypersurface of an almost contact space form possesses a natural f – structure. With such reduced structures, this frame of hypersurfaces includes quite interesting and complicated geometric properties. For some remarkable applications, we refer to [1-9], etc.

Besides this facts, the theory of product manifolds includes worthwhile physical and geometrical models. In physics, there exist significant applications of product manifolds on the general theory of relativity, the Kaluza--Klein theory, the brane theory, and the gauge theory (see [10-12]). Hypersurfaces of almost product manifolds were firstly studied by S. Deshmukh, A. Sharfuddin and S.I. Husain in [13]. Some of the main formulas for hypersurfaces of almost product manifolds were obtained by T. Adati in [14]. Considering these studies, one can notice the fact that a hypersurface of almost product Riemannian manifolds includes a para-contact structure under some conditions.

The main purpose of this study is to investigate some basic geometrical properties of hypersurfaces in a Riemannian product manifold with para-contact structures. Taking the concept of semi-invariant submanifolds examined by B. Şahin and M. Atçeken in [15], we introduce a new type of hypersurface, which we will call semi-invariant hypersurfaces. Furthermore, we obtain some characterizations on semi-invariant hypersurfaces in terms of the Lie derivative and Riemannian curvature tensors.

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2. PRELIMINARIES

In this section, we shall primarily recall some basic properties of para-contact manifolds, (see [16]). Let M be a smooth manifold, φ be a tensor field of type $(1,1)$, ξ be a vector field, and η be a 1-form on M. Then (M,g,φ,η,ξ) is called a para-contact manifold if

$$\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \tag{1}$$

are satisfied, where I denotes the identity map. We also have from (1) that

$$\eta \circ \varphi = 0 \tag{2}$$

is satisfied. A para-contact manifold (M,g,φ,η,ξ) is called an almost para-contact metric manifold if there exists a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{3}$$

and

$$g(X, \varphi Y) = g(\varphi X, Y) \tag{4}$$

are satisfied for any $X,Y \in \Gamma(TM)$. We note that one of eqs. (3) and (4) requires the other.

Let (M,g,φ,η,ξ) be an almost para-contact metric manifold and ∇ be the Levi-Civita connection on *M*. The manifold is called a para-Sasakian manifold if the following relations are satisfied

$$d\eta = 0, \quad \nabla_X \xi = \varphi X \tag{5}$$

and

$$(\nabla_X \varphi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{6}$$

for any $X,Y \in \Gamma(TM)$.

An almost para-contact metric manifold is called a para-cosymplectic manifold if $\nabla\varphi = 0$. For more details on para-contact manifolds and their submanifolds, we refer to [17-21], etc.

Now we shall recall some basic notations about product manifolds. Let $\tilde{M}$ be a smooth manifold. If there exists a non-trivial tensor field F of type $(1,1)$ satisfying the condition

$$F^2 = I, \tag{7}$$

where I denotes the identity map, then $(\tilde{M},F)$ is called an almost product manifold and F is called an almost product structure on $\tilde{M}$. An almost product manifold $(\tilde{M},F)$ is called an almost product Riemannian manifold if there exists a Riemannian metric $\tilde{g}$ satisfying

$$\tilde{g}(FX,FY) = \tilde{g}(X,Y), \tag{8}$$

for any $X,Y \in \Gamma(T\tilde{M})$.

Let $(\tilde{M},\tilde{g},F)$ be an almost product Riemannian manifold and $\tilde{\nabla}$ be the Levi-Civita connection of $(\tilde{M},\tilde{g},F)$. In particular, if

$$\tilde{\nabla}_X F = 0 \tag{9}$$

is satisfied for any $X\in\Gamma(T\tilde{M})$, then $(\tilde{M},\tilde{g},F)$ is called a locally product Riemannian manifold.

Let $\tilde{M}$ be a locally product of two manifolds $\tilde{M}_1$ and $\tilde{M}_2$. Suppose P and Q are the projections of $\Gamma(T\tilde{M})$ onto $\Gamma(T\tilde{M}_1)$ and $\Gamma(T\tilde{M}_2)$, respectively. If we put

$$P=\frac{1}{2}(I+F),\quad Q=\frac{1}{2}(I-F), \tag{10}$$

then we obtain

$$P+Q=I,\quad P^2=P,\quad Q^2=Q,\quad PQ=QP=0,\quad F=P-Q. \tag{11}$$

Consider $\tilde{M}_1$ and $\tilde{M}_2$ to be of constant sectional curvatures λ_1 and λ_2, respectively. If $\tilde{M}$ is a locally product Riemannian manifold of $\tilde{M}_1$ and $\tilde{M}_2$, then the curvature tensor $\tilde{R}$ of $\tilde{M}$ has the following form:

$$\begin{aligned}\tilde{g}(\tilde{R}(X,Y)Z,W) = a\{&g(X,W)g(Y,Z)-g(X,Z)g(Y,W)+g(X,FW)g(Y,FZ)\\ &-g(X,FZ)g(Y,FW)\}+b\{g(X,FW)\,g(Y,Z)-g(X,FZ)\,g(Y,W)\\ &+g(X,W)g(Y,FZ)-g(X,Z)\,g(Y,FW)\}\end{aligned} \tag{12}$$

for all $X,Y,Z,W\in\Gamma(T\tilde{M})$ and

$$a=\frac{1}{4}(\lambda_1+\lambda_2),\qquad b=\frac{1}{4}(\lambda_1-\lambda_2). \tag{13}$$

A locally product Riemannian manifold is called a manifold of almost constant curvature if its curvature tensor $\tilde{R}$ satisfies the equation (12). These manifolds are denoted by $\tilde{M}(a,b)$. For more details on locally product manifolds and manifolds of almost constant curvatures, we refer to [21-23].

Now, we shall recall some basic facts on hypersurfaces of Riemannian manifolds. Let (M,g) be an $m-$dimensional orientable hypersurface in a Riemannian manifold $(\tilde{M},\tilde{g})$. For any $X,Y\in\Gamma(TM)$ and a local normal unit vector field $N\in\Gamma(TM^{\perp})$, the Gauss and Weingarten formulas for (M,g) are given by

$$\tilde{\nabla}_X Y=\nabla_X Y+g(A_N X,Y)N, \tag{14}$$

$$\tilde{\nabla}_X N=-A_N X, \tag{15}$$

where $\tilde{\nabla}$ and ∇ are the Levi-Civita connections of $\tilde{M}$ and M, respectively, A_N is the shape operator with respect to N.

The hypersurface (M,g) is called totally geodesic in $(\tilde{M},\tilde{g})$ if $A_N=0$ and it is called totally umbilical if there exists a smooth function λ on M such that $A_N X=\lambda X$ for all $X\in\Gamma(TM)$ [24]. A hypersurface is called totally $\eta-$umbilical if there exist two functions c_1 and c_2 such that the shape operator A_N takes the form

$$A_N X=c_1 X+c_2\eta(X)\xi \tag{16}$$

for any X on M [25]. Here ξ is called the structure vector field on $\Gamma(TM)$.

Let $\tilde{R}$ and R be the curvature tensors of $\tilde{M}$ and M, respectively. For any $X,Y,Z\in\Gamma(TM)$ and $N\in\Gamma(TM)^{\perp}$, the Gauss and Codazzi equations are given by

$$R(X,Y)Z=\tilde{R}(X,Y)Z+g(A_N Y,Z)A_N X-g(A_N X,Z)A_N Y \tag{17}$$

and

$$R^{\perp}(X,Y)N=\tilde{R}(X,Y)N+\left[g(X,A_N Y)-g(Y,A_N X)\right]N+(\nabla_X A)_N Y-(\nabla_Y)A_N X\ , \tag{18}$$

respectively. Here $R^{\perp}(X,Y)N$ denotes the normal part of $\tilde{R}(X,Y)N$ [24].

3. HYPERSURFACES OF RIEMANNIAN PRODUCT MANIFOLDS

Let (M,g) be a hypersurface immersed in an almost product Riemannian manifold $(\tilde{M},\tilde{g},F)$. For any vector field $X\in\Gamma(TM)$, we can write

$$FX=\phi X+\eta(X)N\ , \tag{19}$$

where ϕX and $\eta(X)N$ are the tangential and normal parts of FX, respectively, η is a $1-$form. In addition, for the unit normal vector field $N\in\Gamma(TM^{\perp})$, we can also write

$$FN=\xi+sN\ , \tag{20}$$

where ξ and sN are the tangential and normal parts of FN, respectively.

Now, we shall recall the following lemma (see Lemma 1.1 in [14]):

Lemma 3.1. Let (M,g) be a hypersurface immersed in an almost product Riemannian manifold $(\tilde{M},\tilde{g},F)$. For any $X,Y\in\Gamma(TM)$ and $N\in\Gamma(TM^{\perp})$, we have

$$\phi^2 X+\eta(X)\xi=X\ , \tag{21}$$

$$\eta(\phi X)+\eta(X)sN=0\ , \tag{22}$$

$$s^2 + \eta(\xi) = 1, \tag{23}$$

$$\phi\xi + s\xi = 0, \tag{24}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{25}$$

$$g(X, \phi Y) = g(\phi X, Y). \tag{26}$$

We note that the proof of Lemma 3.1 is straightforward by (7), (8), (19) and (20). In view of (21), (22), (23), and (24), we get the following:

Corollary 3.2. Let (M, g) be a hypersurface immersed in an almost product Riemannian manifold $(\tilde{M}, \tilde{g}, F)$. If FN is tangent to M, then we have the following relations for any $X \in \Gamma(TM)$:

$$\phi^2 X + \eta(X)\xi = X, \tag{27}$$

$$\eta(\phi X) = 0, \tag{28}$$

$$\eta(\xi) = 1, \tag{29}$$

$$\phi\xi = 0. \tag{30}$$

Remark 3.3. In [15], any submanifold M of almost product Riemannian manifolds is called a semi-invariant submanifold if the tangent bundle TM has two distributions D and $D^\perp$ such that

i. $TM = D \oplus D^\perp$,

ii. $F(D) = D$ and $F(D^\perp) \subset TM^\perp$.

We note that if the equation $FN = \xi$ such that $D^\perp = \mathrm{Span}\{\xi\}$ is satisfied for a hypersurface of an almost product Riemannian manifold, then this hypersurface becomes a semi-invariant submanifold. Based on Corollary 3.2 and Remark 3.3, it is possible to give the following definition, which is a new special classification of hypersurfaces of Riemannian product manifolds.

Definition 3.4. Let (M, g) be a hypersurface of an almost product Riemannian manifold $(\tilde{M}, \tilde{g}, F)$ and N be the unit normal vector field on M. If FN is a tangential vector field on M, then M is called a semi-invariant hypersurface.

From Corollary 3.2 and Definition 3.4, we see that any semi-invariant hypersurface (M, g, ϕ, η, ξ) of $(\tilde{M}, \tilde{g}, F)$ is a para-contact metric manifold.

Remark 3.5. We note that the tensor field ϕ denotes a projection on $(\tilde{M}, \tilde{g}, F)$ and it is not a para-contact structure for any hypersurface (M, g) of $(\tilde{M}, \tilde{g}, F)$. If (M, g) is semi-invariant, then ϕ becomes a para-contact structure. Since we consider the tensor ϕ as a projection in (19), we prefer to use the ϕ symbol instead of the φ symbol which is defined in (1).

Example 3.6. Let $\mathbb{E}^{2n+2}$ be the $(2n+2)-$dimensional Euclidean space with the coordinate neighborhood $(x^1,\ldots,x^{2n+2})$ of a point p and the natural basis $\{\delta_1,\ldots,\delta_{2n+2}\}$. Let F be an almost product structure on $\mathbb{E}^{2n+2}$ defined by

$$F(x^1,x^2,\ldots,x^{2n+1},x^{2n+2})=(x^2,x^1,\ldots,x^{2n+2},x^{2n+1}).$$

From (10), we have

$$P(x^1,x^2,\ldots,x^{2n+1},x^{2n+2})=\frac{1}{2}(x^1+x^2,x^1+x^2,x^3+x^4,\ldots,x^{2n+1}+x^{2n}),$$
$$Q(x^1,x^2\ldots,x^{2n+1},x^{2n+2})=\frac{1}{2}(x^1-x^2,x^1-x^2,x^3-x^4,\ldots,x^{2n+1}-x^{2n}).$$

These show that $\mathbb{E}^{2n+2}=\mathbb{E}^{n+1}\times\mathbb{E}^{n+1}$. Thus, it is clear from (13) that $\mathbb{E}^{2n+2}$ is an almost constant curvature manifold with $a=b=0$. Now, we consider a hyperplane M of $\mathbb{E}^{2n+2}$ defined by the following immersion:

$$x(x^1,x^2,\ldots,x^{2n+1},x^{2n+2})=(x^1,x^2,\ldots,x^{2n+1},0).$$

Then we get $TM=\mathrm{Span}\{\delta_1,\delta_2,\ldots,\delta_{2n+1}\}$ and $TM^{\perp}$=Span$\{N\}$ such that $\xi=\delta_{2n+1}$ and $N=\delta_{2n+2}$. In this case, we see that $F\delta_1=\delta_2,\ F\delta_3=\delta_4,\ldots,F\xi=N$, which imply that M is a semi-invariant hypersurface of $\mathbb{E}^{2n+1}$.

Example 3.7. Let us consider

$$S^3\times\mathbb{R}^2=\left\{\ (z_1,z_2,x_1,x_2):z_1,z_2\in\mathbb{C}\ \text{and}\ x_1,x_2\in\mathbb{R}\ \right\},$$

where S^3 is the unit sphere in $\mathbb{C}^2$ defined by $|z_1|^2+|z_2|^2=1$ for $(z_1,z_2)\in\mathbb{C}^2$. It is known that a hypersurface $S^3\times\mathbb{R}$ is defined by the intersection of a plane and $S^3\times\mathbb{R}^2$. Thus, we write

$$S^3\times\mathbb{R}=\left\{\ (z_1,z_2,x_1,0):(z_1,z_2)\in S^3\ \text{and}\ x_1\in\mathbb{R}\ \right\}.$$

Let F be an almost product structure on $S^3\times\mathbb{R}^2$ defined by

$$F(z_1,z_2,x_1,x_2)=(z_2,z_1,x_2,x_1).$$

Then it is clear that $S^3\times\mathbb{R}$ is a semi-invariant hypersurface of $S^3\times\mathbb{R}$.

We note that every hypersurface of almost product manifold does not need to be semi-invariant.

Now we shall state an example of non-semi-invariant hypersurfaces as follows:

Example 3.8. Let us consider a submanifold $\tilde{M}$ in $\mathbb{E}^9$ given by

$$\tilde{M} = \left\{ \left(t, -t, 0, t, -t, \cos u \cos v \cos w, \cos u \cos v \sin w, \cos u \sin v, \sin u\right) : t \in \mathbb{R} \quad \text{and } u, v, w \in [0, \frac{\pi}{2}) \right\}$$

Let F be an almost product structure on $\mathbb{R}^9$ defined by

$$FX = (x^2, x^1, x^3, x^5, x^4, x^6, x^7, x^8, x^9)$$

where $X = (x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9)$. Then we have

$$PX = \frac{1}{2}(x^1 + x^2, x^1 + x^2, 2x^3, x^4 + x^5, x^4 + x^5, 2x^6, 2x^7, 2x^8, 2x^9)$$

and

$$QX = \frac{1}{2}(x^1 - x^2, x^1 - x^2, 0, x^4 - x^5, x^5 - x^4, 0, 0, 0, 0)\,,$$

which show that $\tilde{M}$ is a locally product of the unit $3-$sphere S^3 and a plane section M_1. Here S^3 is given by the spherical coordinates in $\mathbb{E}^9$ as follows:

$$(\cos u \cos v \cos w, \cos u \cos v \sin w, \cos u \sin v, \sin u, 0, 0, 0, 0, 0)$$

for u ranges over $[0, \frac{\pi}{2})$ and the other coordinates range over $[0, \frac{\pi}{2}]$ and M_1 is given by

$$M_1 = \{\ (t, -t, 0, t, t, 0, 0, 0, 0) : t \in \mathbb{R}\ \}.$$

Thus, it follows from (13) that $\tilde{M}$ is an almost constant curvature manifold with $a = b = \frac{1}{4}$. By a straightforward computation, we have

$$\begin{aligned}
e_1 &= -\sin u \cos v \cos w\, \delta_6 - \sin u \cos v \sin w\, \delta_7 - \sin u \sin v\, \delta_8 + \cos u\, \delta_9,\\
e_2 &= -\cos u \sin v \cos w\, \delta_6 - \cos u \sin v \sin w\, \delta_7 + \cos u \cos v\, \delta_8,\\
e_3 &= -\cos u \cos v \sin w\, \delta_6 - \cos u \cos v \cos w\, \delta_7,\\
N &= \delta_1 - \delta_2 + \delta_4 - \delta_5.
\end{aligned}$$

Since FN is not a tangential vector field on S^3, it is clear that S^3 is not a semi-invariant hypersurface of $\tilde{M}$.

Further examples of semi-invariant and non-semi-invariant hypersurfaces could be derived.

Lemma 3.9. Let (M, g) be a semi-invariant hypersurface immersed in a locally product Riemannian manifold $(\tilde{M}, \tilde{g}, F)$. Then the following equalities are satisfied for any $X, Y \in \Gamma(TM)$:

$$\eta(\nabla_X \xi) = 0\,, \tag{31}$$

$$\nabla_X \xi = -\phi A_N X\,, \tag{32}$$

$$(\nabla_X \phi)Y = \eta(Y) A_N X + g(A_N X, Y)\xi\,. \tag{33}$$

Proof: Since ξ is a unit vector field, we get $X[g(\xi,\xi)] = 0$ for any $X \in \Gamma(TM)$. Using the fact that ∇ is the Levi-Civita connection on M, the proof of (31) is straightforward.

From (15), we write

$$\tilde{\nabla}_X N = \tilde{\nabla}_X (F\xi) = -A_N X,$$

which is equivalent to

$$(\tilde{\nabla}_X F)\xi + F\tilde{\nabla}_X \xi = -A_N X\,.$$

Since $\tilde{M}$ is a locally product Riemannian manifold, it follows from (9) that

$$F\tilde{\nabla}_X \xi = -A_N X\,.$$

Putting (14) in the last equation, we get

$$F\nabla_X \xi + g(A_N X, \xi)\xi = -A_N X\,.$$

Using (19) and (31), we obtain

$$\phi\nabla_X \xi + g(A_N X, \xi)\xi = -A_N X\,.$$

Applying ϕ to both sides of the last equation and considering Lemma 3.1, we obtain

$$\phi^2 \nabla_X \xi = -\phi A_N X\,,$$

which is the proof of (32).

For any $Y \in \Gamma(TM)$, we write

$$\phi Y = FY - \eta(Y)N\,.$$

If the derivative of the last equation is taken from both sides in the direction of a vector field $X \in \Gamma(TM)$, then we have

$$\tilde{\nabla}_X (\phi Y) = \tilde{\nabla}_X \left(FY - \eta(Y)N\right),$$

which implies that

$$(\tilde{\nabla}_X \phi)Y = (\tilde{\nabla}_X F)Y + F\tilde{\nabla}_X Y - \eta(Y)\tilde{\nabla}_X N - \phi\tilde{\nabla}_X Y - X[g(Y,\xi)]N\,.$$

Using (14) and (32), we get

$$(\tilde{\nabla}_X \phi)Y = g(A_N X, \phi Y)N + g(A_N X, Y)\xi + \eta(Y) A_N X\,. \tag{34}$$

On the other hand, using (14) in

$$\tilde{\nabla}_X \phi Y = (\tilde{\nabla}_X \phi)Y + \phi \nabla_X Y ,$$

we have

$$(\tilde{\nabla}_X \phi)Y = \nabla_X \phi Y + g(A_N X, \phi Y)N - \phi \nabla_X Y \tag{35}$$

for any $X, Y \in \Gamma(TM)$. In view of (34) and (35), we obtain

$$\nabla_X \phi Y + g(A_N X, \phi Y)N - \phi \nabla_X Y = g(A_N X, \phi Y)N + g(A_N X, Y)\xi + \eta(Y) A_N X .$$

Considering tangential parts of the last equation from both sides, the proof of (33) is straightforward. □

Proposition 3.10. Let (M, g) be a semi-invariant hypersurface of a locally product Riemannian manifold with a para-cosymplectic structure, that is, $\nabla_X \xi = 0$ is satisfied for any $X \in \Gamma(TM)$. Then the following relation is satisfied

$$(\nabla_X \phi)Y = 2\eta(A_N X)\eta(Y)\xi \tag{36}$$

for any $X, Y \in \Gamma(TM)$.

Proof: If $\nabla_X \xi = 0$ then we write from (32) that $A_N X = \eta(A_N X)\xi$. Puttting this fact in (33), we obtain (36). □

Considering (32) and (36), we get the following corollary:

Corollary 3.11. A semi-invariant hypersurface (M, g) of a locally product Riemannian manifold is satisfied $\nabla_X \xi = \nabla_X \phi = 0$ for any $X \in \Gamma(TM)$ if and only if (M, g) is totally geodesic.

Theorem 3.12. Let (M, g) be a totally umbilical semi-invariant hypersurface of a locally product Riemannian manifold $\left(\tilde{M}, \tilde{g}, F\right)$. Then M does not admit a para-Sasakian structure.

Proof: By the assumption, if (M, g) is a totally umbilical semi-invariant hypersurface, then there exists a smooth function λ on M such that the shape operator could be written as $A_N X = \lambda X$ for any $X \in \Gamma(TM)$. Now, we suppose that (M, g) includes a para-Sasakian structure. From (5) and (32), we see that $\lambda = 1$, i.e., $A_N X = -X$ for any $X \in \Gamma(TM)$. Considering these facts in (14) and (15), we get

$$\tilde{\nabla}_X \xi = \phi X - \eta(X)N , \tag{37}$$

and

$$\tilde{\nabla}_X N = X \tag{38}$$

for any $X \in \Gamma(TM)$. Since $\left(\tilde{M}, \tilde{g}, F\right)$ is a locally product Riemannian manifold, we have from (38) that

$$\tilde{\nabla}_X N = \tilde{\nabla}_X F\xi = F\tilde{\nabla}_X \xi = X \ . \tag{39}$$

In view of (19) and (39), we get

$$\tilde{\nabla}_X \xi = FX = \phi X + \eta(X)N \tag{40}$$

for any $X \in \Gamma(TM)$. Comparing (37) and (40), we get $\eta(X) = 0$ for any $X \in \Gamma(TM)$. This result gives us a contradiction. Thus, (M, g) does not admit a para-contact structure. □

Theorem 3.13. Let (M, g) be a semi-invariant hypersurface of a locally product Riemann manifold. If (M, g) is also a para-Sasakian manifold, then it is a totally $\eta-$umbilical hypersurface.

Proof: From (5) and (32), we get

$$\phi(A_N X + X) = 0,$$

which implies that there exists a smooth function γ on M such that

$$A_N X = -X + \gamma\xi. \tag{41}$$

is satisfied. On the other hand, we have from (6) and (33) that

$$\eta(Y)A_N X + g(A_N X, Y)\xi = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi. \tag{42}$$

If we put (41) in (42), then we obtain

$$\eta(Y)\gamma\xi = \eta(X)\eta(Y)\xi ,$$

which shows that

$$\gamma = \eta(X) \tag{43}$$

for any $X \in \Gamma(TM)$. From (41) and (43), we get

$$A_N X = -X + \eta(X)\xi , \tag{44}$$

which implies that M is totally $\eta-$umbilical with $c_1 = -1$ and $c_2 = 1$. □

Proposition 3.14. Let (M, g) be a semi-invariant hypersurface of a locally product Riemann manifold $(\tilde{M}, \tilde{g}, F)$. If $A\phi = 0$ then one of the following situations occurs:

i. ξ is a principal direction, i.e., there exists a function α on M such that $A_N\xi = \alpha\xi$.

ii. M is totally geodesic.

Proof: It is known that the Lie derivative of a tensor field K is given by

$$(L_X K)Y = \nabla_X KY - \nabla_{KY} X - K\nabla_X Y + K\nabla_Y X$$

for any $X, Y \in \Gamma(TM)$. If we take $X = \xi$, $K = A_N$ and $Y = X$, then we write

$$\begin{aligned}(L_\xi A_N)X &= \nabla_\xi A_N X - \nabla_{A_N X}\xi - A_N \nabla_\xi X + A_N \nabla_X \xi \\ &= (\nabla_\xi A_N)X - \nabla_{A_N X}\xi + A_N \nabla_X \xi.\end{aligned}$$

Using $\nabla_X \xi = -\phi A_N X$, we obtain

$$\begin{aligned}(L_\xi A_N)X &= (\nabla_\xi A_N)X - \left(-\phi A_N (A_N X)\right) + A_N\left(-\phi A_N X\right) \\ &= (\nabla_\xi A_N)X + \phi A_N^2 X - A_N \phi A_N X.\end{aligned}$$

Hence we can write

$$g\left((L_\xi A_N)X, Y\right) = g\left((\nabla_\xi A_N)X, Y\right) + g(\phi A_N^2 X, Y) - g(A_N \phi A_N X, Y) \tag{45}$$

for any $X, Y \in \Gamma(TM)$. If the roles of X and Y are changed in (45), then we have

$$g\left((L_\xi A_N)Y, X\right) = g\left((\nabla_\xi A_N)Y, X\right) + g(\phi A_N^2 Y, X) - g(A_N \phi A_N Y, X). \tag{46}$$

By subtracting (45) and (46), we have

$$\begin{aligned}g\left((L_\xi A_N)X, Y\right) - g\left((L_\xi A_N)Y, X\right) &= g(\phi A_N^2 X, Y) - g(\phi A_N^2 Y, X) - g(A_N \phi A_N X, Y) \\ &\quad + g(A_N \phi A_N Y, X) \\ &= g(\phi A_N^2 X, Y) - g(A_N^2 \phi X, Y).\end{aligned}$$

Therefore, we have

$$g\left((L_\xi A_N)X, Y\right) - g\left((L_\xi A_N)Y, X\right) = g(\phi A_N^2 X - A_N^2 \phi X, Y). \tag{47}$$

Now assume that ξ is not principal, that is, we write $A_N \xi = \alpha\xi + \beta U$ where U is perpendicular to ξ, α and β are smooth functions on M. On the other hand, we get

$$\begin{aligned}g\left((A_N\phi - \phi A_N)X, (A_N\phi + \phi A_N)X\right) &= g(A_N \phi X, A_N \phi X) - g(\phi A_N X, \phi A_N X) \\ &= g(A_N^2 \phi X, \phi X) - g(\phi^2 A_N X, A_N X).\end{aligned}$$

If we consider $\phi^2 X = X - \eta(X)\xi$ in the last equation, then we obtain

$$g\left((A_N\phi - \phi A_N)X, (A_N\phi + \phi A_N)X\right) = g(A_N^2 \phi X, \phi X) - g(A_N X, A_N X) + \left[g(X, A_N \xi\right]^2. \tag{48}$$

Using the fact that $A_N \xi = \alpha\xi + \beta U$, we write

$$(X, A_N\xi) = g(X, \alpha\xi + \beta U) = \alpha g(X, \xi) + \beta g(X, U).$$

If we choose X is perpendicular to ξ, we get

$$g(X, A_N\xi) = \beta g(X, U). \tag{49}$$

Putting (49) in (48), we obtain

$$g\left((A_N\phi - \phi A_N)X, (A_N\phi + \phi A_N)X\right) = g(A_N^2\phi X, \phi X) - g(A_N X, A_N X) + \beta^2\left[g(X, U)\right]^2. \tag{50}$$

Furthermore, considering (26), we write

$$g(\phi A_N^2 X, \phi X) = g(A_N^2 X, X) - \eta(A_N^2 X)\eta(X). \tag{51}$$

Using $g(A_N X, A_N X) = g(A_N^2 X, X)$ and (51) in (50), we obtain

$$g\left((A_N\phi - \phi A_N)X, (A_N\phi + \phi A_N)X\right) = g(A_N^2\phi X, \phi X) - \left(g(\phi A_N^2 X, \phi X) + \eta(A_N^2 X)\eta(X)\right) + \beta^2\left[g(X, U)\right]^2. \tag{52}$$

Using the fact that $\eta(X) = 0$ in the last equation, then we find

$$g\left((A_N\phi - \phi A_N)X, (A_N\phi + \phi A_N)X\right) = g(A_N^2\phi X - \phi A_N^2 X, \phi X) + \beta^2\left[g(X, U)\right]^2. \tag{53}$$

If we put $Y = \phi XY = \phi X$ in (47), we also have

$$g\left((L_\xi A_N)X, \phi X\right) - g\left((L_\xi A_N)\phi X, X\right) = g(\phi A_N^2 X - A_N^2\phi X, \phi X). \tag{54}$$

With the help of (53), we can rewrite (54) as follows:

$$g\left((L_\xi A_N)X, \phi X\right) - g\left((L_\xi A_N)\phi X, X\right) = -g\left((A_N\phi - \phi A_N)X, (A_N\phi + \phi A_N)X\right) + \beta^2\left[g(X, U)\right]^2.$$

Since $g(X, \phi Y) = g(\phi X, Y)$, we get
$g\left((L_\xi A_N)X, \phi X\right) - g\left((L_\xi A_N)\phi X, X\right) = 0$.
Then we obtain

$$g\left((A_N\phi - \phi A_N)X, (A_N\phi + \phi A_N)X\right) = \beta^2\left[g(X, U)\right]^2,$$

which is equivalent to

$$\left\|A_N\phi X\right\|^2 = \left\|\phi A_N X\right\|^2 + \beta^2\left[g(X, U)\right]^2.$$

If $A_N\phi = 0$, we get

$$\left\|\phi A_N X\right\|^2 + \beta^2\left[g(X, U)\right]^2 = 0.$$

Using $g(X,U)\neq 0$, we obtain $\phi A_N=0$ and $\beta=0$. Thus, we obtain either $A_N\xi=\alpha\xi$ or $A_N=0$. By these facts, the proof of proposition is completed.

Proposition 3.15. Let (M,g) be a semi-invariant hypersurface of a locally product Riemann manifold $(\tilde{M},\tilde{g},F)$. If $L_\xi g=0$ then ξ is a principal direction.

Proof: Using the definition of Lie derivative, we have

$$(L_\xi g)(X,Y)=g(\nabla_X\xi,Y)+g(X,\nabla_Y\xi)$$

for any $X,Y\in\Gamma(TM)$. Then we write from (32) that

$$\begin{aligned}(L_\xi g)(X,Y)&=-g(\phi A_N X,Y)-g(X,\phi A_N Y)\\ &=-g(\phi A_N X+A_N\phi X,Y).\end{aligned} \tag{55}$$

Now, we suppose that $L_\xi g=0$. In this case, we get from (55) that $\phi A_N X=-A_N\phi X$ for any $X\in\Gamma(TM)$. Therefore, we obtain

$$\begin{aligned}\phi^2 A_N X&=-\phi A_N\phi X\\ &=A_N\phi^2 X\\ &=A_N X-\eta(X)A_N\xi.\end{aligned}$$

Since $\phi^2 A_N X=A_N X-\eta(A_N X)\xi$, we get

$$A_N\xi=\frac{\eta(A_N X)}{\eta(X)}\xi,$$

which implies that ξ is a principal direction.

4. SOME CHARACTERIZATIONS ON SEMI-INVARIANT HYPERSURFACES IN TERMS OF RIEMANNIAN CURVATURE TENSOR FIELDS

Let (M,g,ϕ,η,ξ) be a semi-invariant hypersurface of an almost product Riemannian manifold such that

$$TM=D\oplus D^\perp,$$

where D is the invariant distribution and $D^\perp=\mathrm{Span}\{\xi\}$ is the anti-invariant distribution. It is clear that ϕX and X are not always orthogonal and ϕ is a product structure on D such that $\phi^2X=X$ for any $X\in\Gamma(D)$. Thus, eigenvalues of symmetric operator ϕ are 1, 0, -1. Hence ϕX and X do not always span a plane which is known as the holomorphic plane section in complex geometry and contact geometry.

When ϕ acts non-trivially on D, that is, ϕX is perpendicular to X for any $X \in \Gamma(D)$, we can state the definition of para-holomorphic sectional curvature for any semi-invariant hypersurface of an almost product Riemannian manifold.

Definition 4.1. Let (M, g) be a semi-invariant hypersurface of an almost product Riemannian manifold. Suppose that ϕ acts non-trivially on D and Π^H is a plane section spanned by orthonormal vector fields X and ϕX. In this case, Π^H is called a para-holomorphic section. The para-holomorphic sectional curvature of Π^H is defined by

$$K(\Pi^H) = g(R(X,\phi X)\phi X, X). \tag{56}$$

Now we shall give the following lemmas for later usage:

Lemma 4.2. Let (M, g) be a semi-invariant hypersurface of an almost constant curvature manifold $\tilde{M}(a,b)$. Then we have

$$\begin{aligned}\tilde{g}(\tilde{R}(X,Y)Z,W) = a\{&g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + g(X,\phi W)g(Y,\phi Z)\\ &-g(X,\phi Z)g(Y,\phi W)\} + b\{g(X,\phi W)g(Y,Z) - g(X,\phi Z)g(Y,W)\\ &+g(X,W)g(Y,\phi Z) - g(X,Z)g(Y,\phi W)\},\end{aligned} \tag{57}$$

$$\tilde{g}(\tilde{R}(X,\xi)\xi,W) = a\{g(X,W) - \eta(X)\eta(W)\} + b\, g(X,\phi W), \tag{58}$$

$$\tilde{g}(\tilde{R}(X,\xi)Z,W) = a\{g(X,W)\eta(Z) - g(X,Z)\eta(W)\} + b\{g(X,\phi W)\eta(Z) - g(X,\phi Z)\eta(W)\}, \tag{59}$$

$$\tilde{R}(X,Y)N = a\{\eta(Y)\phi X - \eta(X)\phi Y\} + b\{\eta(Y)X - \eta(X)Y\} \tag{60}$$

for any $X, Y, Z, W \in \Gamma(TM)$.

Proof: The proof is straightforward by using (19) in (12). □

Lemma 4.3. Let (M, g) be a semi-invariant hypersurface of $\tilde{M}(a,b)$ and suppose that ϕ acts non-trivially on D.

i. For a plane section Π spanned by any orthonormal vector pair $\{X, Y\}$, we have

$$\tilde{K}(\Pi) = a\{1 + g(\phi X, X)g(\phi Y, Y) - g(\phi X, Y)^2\} + b\{g(\phi X, X) + g(\phi Y, Y)\}. \tag{61}$$

ii. For a para-holomorphic plane section Π^H spanned by any orthonormal vector pair $\{X, \phi X\}$, we have

$$\tilde{K}(\Pi^H) = a. \tag{62}$$

Here $\tilde{K}(\Pi)$ and $\tilde{K}(\Pi^H)$ denote the curvatures of plane sections Π and Π^H in terms of $\tilde{R}$, respectively.

Proof: Taking into account the equations (56) and (57), we get the proof immediately. □

Proposition 4.4. Let (M,g) be a semi-invariant hypersurface of $\tilde{M}(a,b)$ admitting a para-Sasakian structure. For a para-holomorphic plane section Π^H spanned by X and ϕX, we have

$$K(\Pi^H)=a+1. \tag{63}$$

Proof: From (17), we get

$$K(\Pi^H)=a+g(A_N\phi X,\phi X)g(A_N X,X)-g(A_N X,\phi X)g(A_N\phi X,X). \tag{64}$$

Since M admits a para-Sasakian structure, we get from (44) that

$$A_N X=-X+\eta(X)\xi$$

and

$$A_N\phi X=-\phi X.$$

are satisfied. Using these facts in (64), we have

$$\begin{aligned}K(\Pi^H)&=a-g(\phi X,\phi X)g(-X+\eta(X)\xi,X)-g(-X+\eta(X)\xi,\phi X)g(-\phi X,X)\\&=a+g(\phi X,\phi X)g(X,X)\\&=a+1.\end{aligned}$$

□

Lemma 4.5. For any semi-invariant hypersurface of $\tilde{M}(a,b)$, we have

$$(\nabla_Y A_N)X-(\nabla_X A_N)Y=a\{\eta(Y)\phi X-\eta(X)\phi Y\}+b\{\eta(Y)X-\eta(X)Y\}. \tag{65}$$

Proof: By comparing the tangent parts of (18) and (60), the proof is straightforward. □

Proposition 4.6. Let (M,g) be a semi-invariant hypersurface of $\tilde{M}(a,b)$. If ξ is a principal direction and D is a totally geodesic distribution, then $a=b=0$, i.e., $\tilde{M}$ is an Euclidean space.

Proof: Putting $X=\xi$ in (65), we write

$$(\nabla_Y A_N)\xi-(\nabla_\xi A_N)Y=-a\phi Y-bY+b\eta(Y)\xi \tag{66}$$

for any $Y\in\Gamma(TM)$. On the other hand, since ξ is principal, there exists a smooth function α such that $A_N\xi=\alpha\xi$. Using the fact that $(\nabla_X A_N)Y=\nabla_X A_N Y-A_N\nabla_X Y$ for $\forall X,Y\in\Gamma(TM)$ and (32), we have

$$
\begin{aligned}
(\nabla_Y A_N)\xi-(\nabla_\xi A_N)Y &= \nabla_Y A_N\xi - A_N\nabla_Y\xi-\left(\nabla_\xi A_N Y - A_N\nabla_\xi Y\right)\\
&= \nabla_Y \alpha\xi + A_N\phi A_N Y-\nabla_\xi A_N Y + A_N\nabla_\xi Y\\
&= Y[\alpha]\nabla_\xi Y+\alpha\nabla_\xi Y + A_N\phi A_N Y-\nabla_\xi A_N Y + A_N\nabla_\xi Y\\
&= -Y[\alpha]\phi A_N Y-\alpha\phi A_N Y + A_N\phi A_N Y-\nabla_\xi A_N Y + A_N\nabla_\xi Y\ .
\end{aligned} \tag{67}
$$

Since D is totally geodesic we can write $A_N Y=0$ for any $Y\in\Gamma(D)$. Considering this fact in the last equation, we get

$$
(\nabla_Y A_N)\xi-(\nabla_\xi A_N)Y = A_N\nabla_\xi Y\ . \tag{68}
$$

In view of (66) and (68), we obtain

$$
A_N\nabla_\xi Y=-a\phi Y-bY\ . \tag{69}
$$

Consider an orthonormal basis $\{e_1,e_2,\dots,e_n,\xi\}$ on $\Gamma(TM)$. Then we can write

$$
\nabla_\xi Y=\sum_{i=1}^{n}\gamma_i e_i+\gamma_2 e_2+\gamma_{n+1}\xi\ ,
$$

where γ_i are smooth functions on M for all $i\in\{1,\dots,n+1\}$. Thus, we have

$$
A_N\nabla_\xi Y=\gamma_{n+1}A_N\xi.
$$

Using that $A_N\xi=\alpha\xi$, we get

$$
A_N\nabla_\xi Y=\gamma_{2n}\alpha\xi\ . \tag{70}
$$

From (69) and (70), we get

$$
-a\phi Y-bY=\gamma_{2n}\alpha\xi\ .
$$

Since Y and ϕY are orthogonal to ξ, we obtain $a=b=0$. □

Proposition 4.7. Let (M,g) be a semi-invariant hypersurface of $\tilde{M}(a,b)$. If ξ is a principal direction and D is a totally umbilical distribution, then A_N is parallel and $a=0$.

Proof: Since D is totally umbilical, there exists a smooth function λ on M such that $A_N Y=\lambda Y$ for any $Y\in\Gamma(D)$. Using this fact in (67) we get

$$
\begin{aligned}
(\nabla_Y A_N)\xi-(\nabla_\xi A_N)Y &= -Y[\alpha]\phi\lambda Y-\alpha\phi\lambda Y + A_N\phi\lambda Y-\nabla_\xi\lambda Y + A_N\nabla_\xi Y\\
&=\left[-\lambda Y[\alpha]-\lambda\alpha+\lambda^2\right]\phi Y-\xi[\lambda]Y\ .
\end{aligned} \tag{71}
$$

In view of (66) and (71), we obtain

$$a = \lambda Y[\alpha] + \lambda\alpha - \lambda^2 \text{ and } b = \xi[\lambda]. \tag{72}$$

Furthermore, we can write

$$\nabla_{\xi} A_N Y = \xi[\lambda]Y + \lambda\nabla_{\xi} Y. \tag{73}$$

If we consider (72), (73) and using the fact that $\nabla_{\xi} A_N Y = (\nabla_{\xi} A_N)Y + A_N \nabla_{\xi} Y$, we obtain

$$(\nabla_{\xi} A_N)Y = \xi[\lambda]Y = bY. \tag{74}$$

From (66) and (74), we derive

$$(\nabla_Y A_N)\xi - bY = -a\phi Y - bY, \tag{75}$$

which implies

$$(\nabla_Y A_N)\xi = -a\phi Y.$$

The last equation shows that $\nabla_X A_N = 0$ for any $X \in \Gamma(TM)$ and $a = 0$. □

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