ORIGINAL PAPER

CHARACTERIZATION OF THE NATURAL LIFT ACCORDING TO THE CURVE

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Abstract. *In this article we have characterized the natural lift curve of a curve in IR^3 according to its torsion and curvature. The natural lift curve of a timelike curve in IR_1^3 is characterized by the torsion and curvature of the natural lift curve, taking into account whether it is a spacelike curve with a timelike binormal or a spacelike curve with a space binormal. The natural lift curve of a spacelike curve with a timelike binormal in IR_1^3 is characterized according to the torsion and curvature of the natural lift curve, considering that the natural lift curve is a spacelike curve with a timelike binormal or a spacelike curve with a space binormal.*

Keywords: *Natural lift curve; Darboux vector; Frenet frame.*

1. INTRODUCTION

The concepts of natural lift curve and geodesic spray were given by Thorpe in [1]. Proved by Thorpe that the natural lift $\overline{\alpha}$ of the curve α is an integral curve of the geodesic spray iff α is an geodesic on M. Çalışkan at al. studied the natural lift curves of the spherical indicatries of tangent, principal normal, binormal vectors and fixed centrode of a curve in [2]. They gave some interesting results about the original curve, depending on the assumption that the natural lift curve should be the integral curve of the geodesic spray on the tangent bundle $T(S^2)$. Ergün and Çalışkan defined the concepts of the natural lift curve and geodesic spray in Minkowski 3-space in [3]. The anologue of the theorem of Thorpe was given in Minkowski 3-space by Ergün and Çalışkan in [3]. Walrave characterized the curve with constant curvature in Minkowski 3-space in [4]. In differential geometry, especially the theory of space curve, the Darboux vector is the areal velocity vector of the Frenet frame of a spacere curve. It is named after Gaston Darboux who discovered it. In term of the Frenet-Serret apparatus, the darboux vector W can be expressed as $W = \tau T + \kappa B$, details are given in Lambert et al. in [5].

Let $\alpha : I \rightarrow IR^3$ be a parametrized curve. We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve α , where T, N and B are the tangent, the principal normal and the binormal vector fields of the curve α , respectively.

Let α be a reguler curve in IR^3. Then

$$T = \frac{\alpha'}{\|\alpha'\|},\ N = B \times T,\ B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}, \kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \tau = \frac{det(\alpha', \alpha'', \alpha''')}{\|\alpha' \times \alpha''\|^2}$$

If α is a unit speed curve, then

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$$T=\alpha',\ N=\frac{\alpha''}{\|\alpha''\|},\ B=T\times N, \kappa=\langle T',N\rangle,\ \tau=\langle N',B\rangle \text{, [6].}$$

Let α be a unit speed space curve with curvature κ and torsion τ. Let Frenet vector fields of α be $\{T,N,B\}$. Then, Frenet formulas are given by

$$T'=\kappa N, N'=-\kappa T+\tau B, B'=-\tau N \text{, [6].}$$

For any unit speed curve $\alpha : I\to IR^3$, we call $W(s)=\tau T(s)+\kappa B(s)$ the Darboux vector field of α , [5].

Theorem 1. Let $\alpha : I\to IR^3$ be a parametrized curve. Then we have
1) $\kappa=0$ if and only if α is a part of a straight line;
2) $\tau=0$ if and only if α is a planar curve;
3) $\tau=0$ and $\kappa=$ constant >0 if and only if α is a circle;
4) $\kappa=$ constant >0 , and $\tau=$ constant $\neq 0$ if and only if α is a circular helix, [4].

Definition 2. Let M be a hypersurface in IR^3 and let $\alpha : I\to M$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{ds}\big(\alpha(s)\big)=X\big(\alpha(s)\big) \text{ (for all } s\in I)$$

where X is a smooth tangent vector field on M. We have

$$TM=\bigcup_{P\in M}T_PM=\chi(M)$$

where T_PM is the tangent space of M at P and $\chi(M)$ is the space of vector fields of M [7].

Definition 3. For any parametrized curve $\alpha : I\to M$ in IR^3 , $\overline{\alpha} : I\to TM$ given by

$$\overline{\alpha}(s)=\big(\alpha(s),\alpha'(s)\big)=\alpha'(s)|_{\alpha(s)}$$

is called the natural lift of α on TM [1].

Theorem 4. We denote by $\{\overline{T}(s),\overline{N}(s),\overline{B}(s)\}$ the moving Frenet frame along the curve $\overline{\alpha}$, where $\overline{T},\overline{N}$ and $\overline{B}$ are the tangent, the principal normal and the binormal vector fields of the curve $\overline{\alpha}$, respectively.

Now we have

$$\overline{T}(s)=N(s)$$

$$\overline{\boldsymbol{N}}(\boldsymbol{s})=-\frac{\boldsymbol{\kappa}}{\|\boldsymbol{W}\|}\boldsymbol{T}(\boldsymbol{s})+\frac{\boldsymbol{\tau}}{\|\boldsymbol{W}\|}\boldsymbol{B}(\boldsymbol{s})$$

$$\overline{\boldsymbol{B}}(\boldsymbol{s})=\frac{\boldsymbol{\tau}}{\|\boldsymbol{W}\|}\boldsymbol{T}(\boldsymbol{s})+\frac{\boldsymbol{\kappa}}{\|\boldsymbol{W}\|}\boldsymbol{B}(\boldsymbol{s})$$

$$\overline{\kappa}=\frac{\|W\|}{\kappa},\overline{\tau}=\frac{\kappa\tau'+\kappa'\tau}{\kappa\|W\|^2} \text{[10].}$$

Let Minkowski 3-space IR_1^3 be the vector space IR^3 equipped with the Lorentzian inner product g given by

$$g(X,X) = -x_1^2 + x_2^2 + x_3^2$$

where $X = (x_1, x_2, x_3) \in IR^3$. A vector $X = (x_1, x_2, x_3) \in IR^3$ is said to be timelike if $g(X,X) < 0$, spacelike if $g(X,X) > 0$ and lightlike (or null) if $g(X,X) = 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in IR_1^3 where t is a pseudo-arclength parameter, can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively timelike, spacelike or null (lightlike), for every $t \in I \subset IR$. A lightlike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ ($resp. x_1 < 0$) and a timelike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ ($resp.$ $x_1 < 0$). The norm of a vector X is defined by $\|X\|_{IL} = \sqrt{|g(X,X)|}$, [8].

The vectors $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in IR_1^3$ are orthogonal if and only if $g(X,X) = 0$, [8]. Now let X and Y be two vectors in IR_1^3 , then the Lorentzian cross product is given by

$$X \times Y = \left(x_3y_2 - x_2y_{3,}\ x_1y_3 - x_3y_{1,}\ x_1y_2 - x_2y_1\right).$$

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve α . Then T, N and B are the tangent, the principal normal and the binormal vector of the curve α , respectively.

Let α be a unit speed timelike space curve with curvature κ and torsion τ . Let Frenet vector fields of α be $\{T, N, B\}$. In this trihedron, T is timelike vector field, N and B are spacelike vector fields. For this vectors, we can write

$$T \times N = B, \quad N \times B = -T, \quad B \times T = N,$$

where $\times$ is the Lorentzian cross product in space IR_1^3 Then, Frenet formulas are given by

$$T' = \kappa N, N' = \kappa T + \tau B, B' = -\tau N, [4].$$

The Frenet instantaneous rotation vector for the timelike curve is given by $W = \tau T + \kappa B$. Let α be a unit speed spacelike space curve with a spacelike binormal. In this trihedron, we assume that T and B are spacelike vector fields and N is a timelike vector field In this situation,

$$T \times N = B, \quad N \times B = T, \quad B \times T = -N.$$

Then, Frenet formulas are given by

$$T' = \kappa N, N' = \kappa T + \tau B, B' = \tau N, [4].$$

The Frenet instantaneous rotation vector for the spacelike space curve with a spacelike binormal is given by $W = \tau T - \kappa B$. Let α be a unit speed spacelike space curve with a timelike binormal. In this trihedron, we assume that T and N are spacelike vector fields and B is a timelike vector field In this situation,

$$T \times N = -B, \quad N \times B = T, \quad B \times T = N,$$

Then, Frenet formulas are given by

$$T' = \kappa N, N' = -\kappa T + \tau B, B' = \tau N, [4].$$

The Frenet instantaneous rotation vector for the spacelike space curve with a spacelike binormal is given by $W = -\tau T + \kappa B$.

Lemma 5. Let X and Y be nonzero Lorentz orthogonal vectors in IR_1^3 . If X is timelike, then Y is spacelike, [9] .

Lemma 6. Let X and Y be pozitive (negative) timelike vectors in IR_1^3 . Then

$$g(X,Y) \leq \|X\|\|Y\|$$

with equality if and only if X and Y are linearly dependent [9] .

Lemma 7.
i) Let X and Y be pozitive (negative) timelike vectors in IR_1^3 . By the Lemma 6, there is unique nonnegative real number $\phi(X,Y)$ such that

$$g(X,Y) = \|X\|\|Y\| \cosh \phi\,(X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\phi(X,Y)$.
ii) Let X and Y be spacelike vectors in IR_1^3 *t*hat span a spacelike vector subspace. Then we have

$$|g(X,Y)| \leq \|X\|\|Y\|.$$

Hence, there is a unique real number $\phi(X,Y)$ between 0 and π such that

$$g(X,Y) = \|X\|\|Y\| \cos \phi\,(X,Y)$$

the Lorentzian spacelike angle between X and Y is defined to be $\phi(X,Y)$.
iii) Let X and Y be spacelike vectors in IR_1^3 that span a timelike vector subspace. Then we have

$$g(X,Y) > \|X\|\|Y\|.$$

Hence, there is a unique pozitive real number $\phi(X,Y)$ between 0 and π such that

$$|g(X,Y)| = \|X\|\|Y\| \cosh \phi\,(X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\phi(X,Y)$.
iv) Let X be a spacelike vector and Y be a pozitive timelike vector in IR_1^3 . Then there is a unique nonnegative reel number $\phi(X,Y)$ such that

$$|g(X,Y)| = \|X\|\|Y\| \sinh \phi\,(X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\phi(X,Y)$ [9].

Definition 8. (Unit Vector C of Direction W for Non-null Curves**)**
1. For the curve α with a timelike tanget, θ being a Lorentzian timelike angle between the spacelike binormal unit $-B$ and the Frenet instantaneous rotation vector W.
(i) If $|\kappa| > |\tau|$, then W is a spacelike vector. In this situation, from Lemma 7 iii) we can write

$$\kappa = \|W\| \cosh \theta$$

$$\tau = \|W\| \sinh \theta$$

$$\|W\|^2 = g(W,W) = \kappa^2 - \tau^2$$

and

$$C = \frac{W}{\|W\|} = \sinh\theta\, T + \cosh\theta\, B\ ,$$

where C is unit vector of direction W .

(ii) If $|\kappa| < |\tau|$, then W is a timelike vector. In this situation, from Lemma 7 iv) we can write

$$\kappa = \|W\| \sinh\theta$$

$$\tau = \|W\| \cosh\theta$$

$$\|W\|^2 = -g(W,W) = -(\kappa^2 - \tau^2)$$

and

$$C = \cosh\theta\, T + \sinh\theta\, B\ .$$

2. For the curve α with a timelike principal normal, θ being an angle between the B and the W , if B and W spacelike vectors that span a spacelike vektor subspace then by the Lemma 7 ii) we can write

$$\kappa = \|W\| \cos\theta$$

$$\tau = \|W\| \sin\theta$$

$$\|W\|^2 = g(W,W) = \kappa^2 + \tau^2$$

and

$$C = \sin\theta\, T - \cos\theta\, B\ .$$

3. For the curve α with a timelike binormal, θ being a Lorentzian timelike angle between the $-B$ and the W.

(i) If $|\kappa| < |\tau|$, then W is a spacelike vector. In this situation, from Lemma 7 iv) we can write

$$\kappa = \|W\| \sinh\theta$$

$$\tau = \|W\| \cosh\theta$$

$$\|W\|^2 = g(W,W) = \tau^2 - \kappa^2$$

and

$$C = -\cosh\theta\, T + \sinh\theta\, B.$$

(ii) If $|\kappa| > |\tau|$, then W is a timelike vector. In this situation, from Lemma 7 i) we have

$$\kappa = \|W\| \cosh\theta$$

$$\tau = \|W\| \sinh\theta$$

$$\|W\|^2 = -g(W,W) = -(\tau^2 - \kappa^2)$$

and

$$C = -\sinh\theta\, T + \cosh\theta\, B\ .$$

Theorem 9. Let α be a unit speed timelike space curve. Then we have

1) $\kappa = 0$ if and only if α is a part of a timelike straight line;

2) $\tau = 0$ if and only if α is a planar timelike curve;
3) $\tau = 0$ and $\kappa =$ constant > 0 if and only if α is a part of a orthogonal hyperbola*;*
4) $\kappa =$ constant > 0 , $\tau =$ constant $\neq 0$ and $|\tau| > \kappa$ *if and only if* α is a part of a timelike circular helix,

$$\alpha(s) = \frac{1}{K}\left(\sqrt{\tau^2 K}s, \kappa\, cos\left(\sqrt{K}s\right), \kappa\, sin\left(\sqrt{K}s\right)\right)$$

with $K = \tau^2 - \kappa^2$ *;*
5) $\kappa =$ constant > 0 , $\tau =$ constant $\neq 0$ and $|\tau| < \kappa$ if and only if α is a timelike hyperbolic helix,

$$\alpha(s) = \frac{1}{K}\left(\kappa\, sinh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa\, cosh\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 - \tau^2$ *;*
6) $\kappa =$ constant > 0 , $\tau =$ constant $\neq 0$ and $|\tau| = \kappa$ *if and only if* α can be parameterized by

$$\alpha(s) = \tfrac{1}{6}(\kappa^2 s^3 + 6s, 3\kappa s^2, \kappa\tau s^3), [4].$$

Theorem 10. Let α be a unit speed spacelike space curve with a spacelike binormal. Then we have:
1) $\tau = 0$ and $\kappa =$ constant > 0 if and only if α is a part of a orthogonal hyperbola;
2) $\kappa =$ constant > 0 , $\tau =$ constant $\neq 0$ if and only if α *is a part of a spacelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(, \kappa\, cosh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa\, sinh\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 + \tau^2$, [4].

Theorem 11. Let α be a unit speed spacelike space curve with a timelike binormal. Then we have
1) $\tau = 0$ and $\kappa =$ constant > 0 *if and only if* α is a part of a circle;
2) $\kappa =$ constant > 0 , $\tau =$ constant $\neq 0$ and $|\tau| > \kappa$ if and only if α is a part of a spacelike hyperbolic helix,

$$\alpha(s) = \frac{1}{K}\left(\kappa\, sinh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa\, cosh\left(\sqrt{K}s\right)\right)$$

with $K = \tau^2 - \kappa^2$ *;*
3) $\kappa =$ constant > 0 , $\tau =$ constant $\neq 0$ and $|\tau| < \kappa$ if and only if α is a part of a spacelike circular helix,

$$\alpha(s) = \frac{1}{K}\left(\sqrt{\tau^2 K}s, \kappa\, cos\left(\sqrt{K}s\right), \kappa\, sin\left(\sqrt{K}s\right)\right)$$

with $K = \kappa^2 - \tau^2$ *;*
4) $\kappa =$ constant > 0 , $\tau =$ constant $\neq 0$ and $|\tau| = \kappa$ if and only if α can be parameterized by

$$\alpha(s) = \tfrac{1}{6}(\kappa\tau s^3, -\kappa^2 s^3 + 6s, 3\kappa s^2), [4].$$

Definition 12. Let M be a hypersurface in R_1^3 and let $\alpha : I \to M$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{ds}\big(\alpha(s)\big) = X\big(\alpha(s)\big) \text{ (for all } t \in I)$$

where X is a smooth tangent vector field on M. We have

$$TM = \bigcup_{P\in M} T_P M = \chi(M)$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of vector fields of M, [8].

Definition 13. For any parametrized curve $\alpha : I \to M$, $\overline{\alpha} : I \to TM$ given by

$$\overline{\alpha}(s) = \left(\alpha(s), \alpha'(s)\right) = \alpha'(s)|_{\alpha(s)}$$

is called the natural lift of α on TM [3] .

Proposition 14. Let α be a unit speed timelike space curve. Then the natural lift $\overline{\alpha}$ of α is a spacelike space curve, [3] .

Proposition 15. Let α be a unit speed spacelike space curve with a spacelike binormal. Then the natural lift $\overline{\alpha}$ of α is a timelike space curve, [3] .

Proposition 16. Let α be a unit speed spacelike space curve with a timelike binormal. Then the natural lift $\overline{\alpha}$ of α is a spacelike space curve [3].

Theorem 17. Let α be a unit speed timelike space curve and $\overline{\alpha}$ be the natural lift of α. Then

$$\overline{T}(s) = N(s)$$

$$\overline{N}(s) = -\frac{\kappa}{\|W\|}T(s) - \frac{\tau}{\|W\|}B(s)$$

$$\overline{B}(s) = -\frac{\tau}{\|W\|}T(s) - \frac{\kappa}{\|W\|}B(s)$$

$$\overline{\kappa} = \frac{\|W\|}{\kappa}, \overline{\tau} = -\frac{\kappa\tau' + \kappa'\tau}{\kappa\|W\|^2}, [10] .$$

Theorem 18. Let α be a unit speed spacelike space curve with a spacelike binormal and $\overline{\alpha}$ be the natural lift of α. Then

$$\overline{T}(s) = N(s)$$

$$\overline{N}(s) = \frac{\kappa}{\|W\|}T(s) + \frac{\tau}{\|W\|}B(s)$$

$$\overline{B}(s) = \frac{\tau}{\|W\|}T(s) - \frac{\kappa}{\|W\|}B(s)$$

$$\overline{\kappa} = \frac{\|W\|}{\kappa}, \overline{\tau} = \frac{-\kappa\tau' + \kappa'\tau}{\kappa\|W\|^2}, [10].$$

Theorem 19. Let α be a unit speed spacelike space curve with a timelike binormal and $\overline{\alpha}$ be the natural lift of α. Then

$$\overline{T}(s) = N(s)$$

$$\overline{N}(s) = -\frac{\kappa}{\|W\|}T(s) - \frac{\tau}{\|W\|}B(s)$$

$$\overline{B}(s) = \frac{\tau}{\|W\|}T(s) + \frac{\kappa}{\|W\|}B(s)$$

$$\overline{\kappa} = \frac{\|W\|}{\kappa}, \overline{\tau} = \frac{-\kappa\tau' - \kappa'\tau}{\kappa\|W\|^2}$$, [10].

2. CHARACTERIZATION OF ITS THE NATURAL LIFT ACCORDING TO THE CURVE

Corollary 20. Let $\alpha : I \to IR^3$ be a parametrized curve. Then we have

1) If α is a part of a straight line; then the natural lift $\overline{\alpha}$ of α is a undefined space curve;
2) If α is a planar curve; then the natural lift $\overline{\alpha}$ of α is a circle;
3) If α is a circle; then the natural lift $\overline{\alpha}$ of α is a circle;
4) If α is a circular helix, then the natural lift $\overline{\alpha}$ of α is a circle.

Corollary 21. Let α be a unit speed timelike space curve. Then the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a spacelike binormal or a spacelike space curve with a timelike binormal. Then we have

1) If α is a part of a timelike straight line; then the natural lift $\overline{\alpha}$ of α is a undefined space curve;
2) Let α be a planar timelike curve;
   a) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a spacelike binormal, then $\overline{\alpha}$ is a part of a orthogonal hyperbola;
   b) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a timelike binormal, then $\overline{\alpha}$ is a part of a circle.
3) Let α be a part of a orthogonal hyperbola;
   a) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a spacelike binormal, then $\overline{\alpha}$ is a part of a orthogonal hyperbola;
   b) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a timelike binormal, then $\overline{\alpha}$ is a part of a circle.
4) Let α be a part of a timelike circular helix;
   a) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a spacelike binormal,then $\overline{\alpha}$ is a part of a orthogonal hyperbola;
   b) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a timelike binormal, then $\overline{\alpha}$ is a part of a circle.
5) Let α be a part of a timelike hyperbolic helix;
   a) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a spacelike binormal, then $\overline{\alpha}$ is a part of a orthogonal hyperbola;
   b) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a timelike binormal, then $\overline{\alpha}$ is a part of a circle.
   c) If α can be parameterized by

$$\alpha(s) = \frac{1}{6}(\kappa^2 s^3 + 6s, 3\kappa s^2, \kappa\tau s^3)$$

then the natural lift $\overline{\alpha}$ of α is a straight line.

Corollary 22. Let α be a unit speed spacelike space curve with a spacelike binormal. Then the natural lift $\overline{\alpha}$ of α is a timelike space curve. Then we have

1) If α is a part of a orthogonal hyperbola, then the natural lift $\overline{\alpha}$ is a part of a orthogonal hyperbola;
2) If α is a part of a spacelike hyperbolic helix, then the natural lift $\overline{\alpha}$ is a part of a orthogonal hyperbola.

Corollary 23. Let α be a unit speed spacelike space curve with a timelike binormal. Then the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a spacelike binormal or a spacelike space curve with a timelike binormal. Then we have

1) Let α be a part of a circle.
   a) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a spacelike binormal, then $\overline{\alpha}$ is a part of a orthogonal hyperbola;
   b) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a timelike binormal, then $\overline{\alpha}$ is a part of a circle.
2) Let α be a part of a spacelike hyperbolic helix;
   a) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a timelike binormal, then $\overline{\alpha}$ is a part of a circle.
   b) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a spacelike binormal, then $\overline{\alpha}$ is a part of a orthogonal hyperbola;
3) Let α be a part of a spacelike circular helix;
   a) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a spacelike binormal, then $\overline{\alpha}$ is a part of a orthogonal hyperbola;
   b) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a timelike binormal, then $\overline{\alpha}$ is a part of a circle.
4) Let α be parameterized by

$$\alpha(s) = \frac{1}{6}(\kappa\tau s^3, -\kappa^2 s^3 + 6s, 3\kappa s^2);$$

   a) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a spacelike binormal, then $\overline{\alpha}$ is a spacelike straight line;
   b) If the natural lift $\overline{\alpha}$ of α is a spacelike space curve with a timelike binormal, then $\overline{\alpha}$ is a spacelike straight line.

Example 24. Let α be a unit speed curve

$$\alpha(s) = \left(\frac{\sqrt{6}}{3}s, \frac{1}{3}cos(\sqrt{3}s), \frac{1}{3}sin(\sqrt{3}s)\right).$$

Then the natural lift $\overline{\alpha}$ of α (see Fig.1):

$$\overline{\alpha}(s) = \left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3}sin(\sqrt{3}s), \frac{\sqrt{3}}{3}cos(\sqrt{3}s)\right).$$

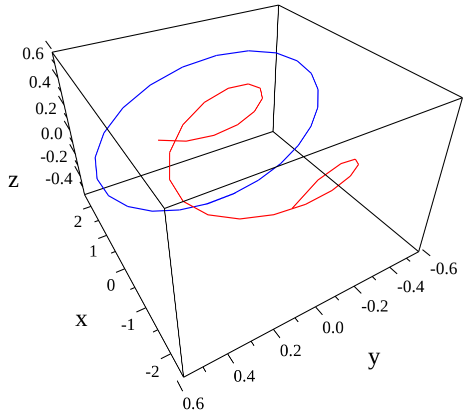

Figure 1. α **and its natural lift** $\overline{\alpha}$

Example 25. Let α be a unit speed curve

$$\alpha(s) = (cos(s), 0, sin(s))$$

Then the natural lift $\overline{\alpha}$ of α (see Fig. 2):

$$\overline{\alpha}(s) = (-sin(s), 0, cos(s))$$

Example 26. Let α be a unit speed timelike circular helix

$$\alpha(s) = \left(\frac{2\sqrt{3}}{3}s, \frac{1}{3}cos(\sqrt{3}s), \frac{1}{3}sin(\sqrt{3}s)\right)$$

Then the natural lift $\overline{\alpha}$ of α (see Fig. 3):

$$\overline{\alpha}(s) = \left(\frac{2\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}sin(\sqrt{3}s), \frac{\sqrt{3}}{3}cos(\sqrt{3}s)\right)$$

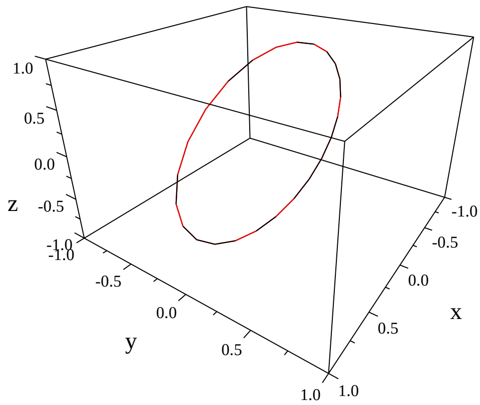

Figure 2. α **and its natural lift** $\overline{\alpha}$

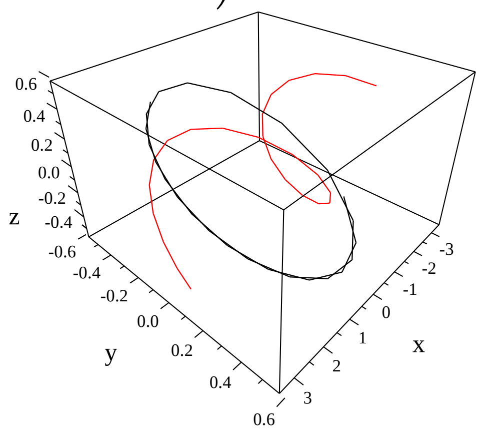

Figure 3. α **and its natural lift** $\overline{\alpha}$

Example 27. Let α be a unit speed spacelike hyperbolic helix

$$\alpha(s) = \left(cosh\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}, sinh\left(\frac{s}{\sqrt{2}}\right)\right).$$

Then the natural lift $\overline{\alpha}$ of α (see Fig. 4):

$$\overline{\alpha}(s) = \left(\frac{1}{\sqrt{2}}sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}cosh\left(\frac{s}{\sqrt{2}}\right)\right).$$

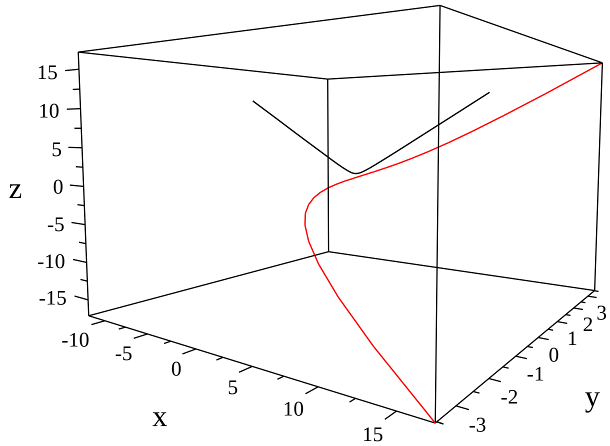

Figure 4. $\boldsymbol{\alpha}$ **and its natural lift** $\overline{\boldsymbol{\alpha}}$

Example 28. Let α be a unit speed spacelike circular helix

$$\alpha(s) = \left(\frac{\sqrt{3}}{3}s, \frac{2}{3}cos(\sqrt{3}s), \frac{2}{3}sin(\sqrt{3}s)\right).$$

Then the natural lift $\overline{\alpha}$ of α (see Fig. 5):

$$\overline{\alpha}(s) = \left(\frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}sin(\sqrt{3}s), \frac{2\sqrt{3}}{3}cos(\sqrt{3}s)\right)$$

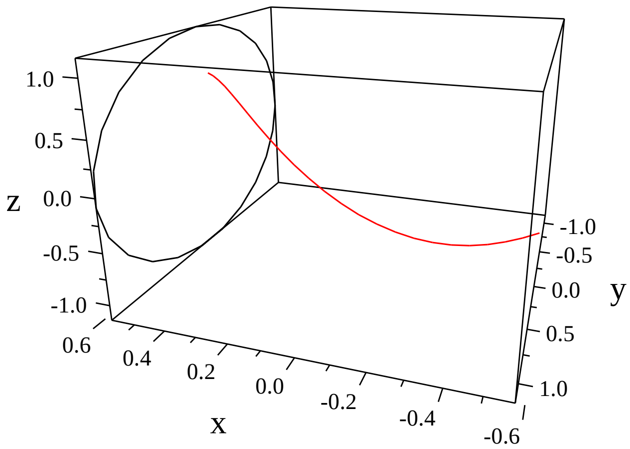

Figure 5. $\boldsymbol{\alpha}$ **and its natural lift** $\overline{\boldsymbol{\alpha}}$

3. CONCLUSION

Using the characterization according to the torsion and curvature of a curve in $\boldsymbol{IR^3}$, considering the torsion and curvature of the natural lift curve, it is obtained that the natural lift curve is a circle curve. Then, using the characterization according to the torsion and curvature of a timelike curve in $\boldsymbol{IR_1^3}$, considering the torsion and curvature of the natural lift curve, it is obtained that the natural lift curve is an orthogonal hyperbola curve or circle curve or straight line. Then, using the characterization based on the torsion and curvature of a spacelike curve with a spacelike binormal in $\boldsymbol{IR_1^3}$, by considering the torsion and curvature of the natural lift curve, it is obtained that the natural lift curve is an orthogonal hyperbola curve. Finally, using the characterization based on the torsion and curvature of a spacelike curve with a timelike binormal in $\boldsymbol{IR_1^3}$, given the torsion and curvature of the natural lift curve, we obtain that the natural lift curve is an orthogonal hyperbola curve or circle curve or straight line. And then examples are given .

REFERENCES

[1] Thorpe, J. A., *Elementary Topics In Differential Geometry*, Springer-Verlag, New York, Heidelberg-Berlin, 1979.
[2] Çalışkan, M., Sivridag, A. I., Hacisalihoglu, H. H., *Communications Faculty of Sciences University of Ankara*, **33**, 235, 1984.
[3] Ergün, E., Çalışkan, M, *International Journal of Contemporary Mathematical Sciences*, **6**(39), 1929, 2011.
[4] Walrave, J., PhD. Thesis *Curves and surfaces in Minkowski space*, K.U. Leuven, 1995.
[5] Lambert, M. S, Mariam T. T., Susan, F. H., *Darboux Vector*, VDM Publishing House, Germany, 2010.
[6] Do Carmo, M. P., *Differential Geometry of Curves and Surfaces* Prentice-Hall Inc. Englewood Cliffs, New Jersey,1976
[7] O'Neill, B, *Elementery Differential Geometry,* Academic Press, New York and London, 1967.
[8] O'Neill, B., *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.
[9] Ratcliffe, J. G*., Foundations of Hyperbolic Manifold*, Springer-Verlag, New York, 1994.
[10] Ergün, E., Çalışkan, M., *Pure Mathematical Sciences*, **1**(2), 85, 2012.