ORIGINAL PAPER

ON THE WEIGHTED PADOVAN AND PERRIN SUMS

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Abstract. *In this paper, we obtain various weighted sum formulas using several sum formulas of Padovan and Perrin numbers.*

Keywords: *Padovan numbers; Perrin numbers; sum formulas.*

1. INTRODUCTION

There are many studies in the literature about special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Tribonacci, Padovan, and Perrin; the most known of these are the Fibonacci and Lucas numbers [1, 2]. Fibonacci and Lucas numbers are defined, respectively, by

$$F_{n+2} = F_{n+1} + F_n \text{ in which } F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n \text{ in which } L_0 = 2,\ L_1 = 1.$$

It is well known that the partial sums of the Fibonacci and Lucas numbers are

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

and

$$\sum_{i=1}^{n} L_i = L_{n+2} - 3$$

Also, the weighted sums of the Fibonacci and Lucas numbers are as follows:

$$\sum_{i=1}^{n} iF_i = nF_{n+2} - F_{n+3} + 2$$

and

$$\sum_{i=1}^{n} iL_i = nL_{n+2} - L_{n+3} + 4.$$

The other important special numbers are the Padovan and Perrin numbers (for more information see [3]). Let's first consider the definitions of the Padovan and Perrin numbers.

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In 1996, Ian Stewart wrote about the Padovan numbers in his Scientific American column Mathematical Recreations. The Perrin numbers $\{R_n\}$ was introduced by Lucas in 1876, and Stewart named it the Perrin sequence in honor of R. Perrin (see [4-6]).

The Padovan and Perrin sequences are defined, respectively, by

$$P_{n+3} = P_{n+1} + P_n \text{ in which } P_0 = P_1 = P_2 = 1$$

and

$$R_{n+3} = R_{n+1} + R_n \text{ in which } R_0 = 3,\ R_1 = 0,\ R_2 = 2\,.$$

It is well known that the sum of the first n terms for the Padovan and Perrin sequences can be given as follows, respectively,

$$\sum_{k=1}^{n} P_k = P_{n+5} - 3 \tag{1}$$

and

$$\sum_{k=1}^{n} R_k = R_{n+5} - 5 \tag{2}$$

Generalizations and some properties of the Padovan sequence can be found in [7-11]. It is the aim of this paper to explore some of the properties of the third-order sequences of thePadovan and Perrin numbers $\{P_n\}$ and $\{R_n\}$, respectively, and their weighted sums.

2. THE WEIGHTED PADOVAN AND PERRIN SUMS

There are a lot of studies related to weighted sums as follows: Gauthier examined Fibonacci sums of the type $\sum r^m F_m$ and derivation of a formula for $\sum r^k x^r$ in [12, 13]. Koshy obtained the weighted Fibonacci and Lucas sums in [14]. Brousseau showed a summation of the infinite Fibonacci series in [15]. Clarke investigated a formula for weighted Fibonacci sums in [16]. Ozeki studied weighted Fibonacci and Lucas sums using the differential operator method in [17]. Cerin et al. explored some formulas for several sums of generalized Fibonacci and generalized Lucas numbers in [18]. In this study we derive new identities in more general form for the Padovan and Perrin sequences.

Let's consider the sums of the Padovan sequence as

$$\alpha_n = \sum_{k=1}^{n} P_k$$

and

$$\beta_n = \sum_{k=1}^{n} kP_k\,.$$

Then, we have

$$\begin{aligned}\beta_n &= P_1 + 2P_2 + 3P_3 + \cdots + nP_n \\ &= \sum_{k=1}^{n} P_k + \sum_{k=2}^{n} P_k + \sum_{k=3}^{n} P_k + \cdots + \sum_{k=n}^{n} P_k\end{aligned}$$

$$
\begin{aligned}
&= \alpha_n + (\alpha_n - \alpha_1) + (\alpha_n - \alpha_2) + \cdots + (\alpha_n - \alpha_{n-1}) \\
&= \underbrace{(\alpha_n + \alpha_n + \cdots + \alpha_n)}_{n} - (\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}) \\
&= n\alpha_n - \sum_{k=1}^{n-1} \alpha_k \\
&= n(P_{n+5} - 3) - \sum_{k=1}^{n-1} (P_{k+5} - 3)
\end{aligned}
$$

Thus, we prove the first identity as

$$\beta_n = nP_{n+5} - P_{n+9} + 9 \tag{3}$$

For example,

$$
\begin{aligned}
\beta_5 = \sum_{k=1}^{5} kP_k &= 1P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5 \\
&= 1.1 + 2.1 + 3.2 + 4.2 + 5.3 = 32 \\
&= 5.12 - 37 + 9 \\
&= 5P_{10} - P_{14} + 9
\end{aligned}
$$

Now let's consider the sums of the Perrin sequence as

$$\gamma_n = \sum_{k=1}^{n} R_k$$

and

$$\delta_n = \sum_{k=1}^{n} kR_k \ .$$

Similarly, we have

$$\delta_n = nR_{n+5} - R_{n+9} + 12 \tag{4}$$

For example,

$$
\begin{aligned}
\delta_5 = \sum_{k=1}^{5} kR_k &= 1R_1 + 2R_2 + 3R_3 + 4R_4 + 5R_5 \\
&= 1.0 + 2.2 + 3.3 + 4.2 + 5.5 = 46 \\
&= 5.17 - 51 + 12 \\
&= 5R_{10} - R_{14} + 12
\end{aligned}
$$

Now let's find the sum

$$\varepsilon_n = \sum_{k=1}^{n} (n - k + 1) P_k \ .$$

It is clearly that

$$\beta_n+\varepsilon_n=\sum_{k=1}^{n}kP_k+\sum_{k=1}^{n}(n-k+1)P_k$$
$$=\sum_{k=1}^{n}(n+1)P_k$$
$$=(n+1)(P_{n+5}-3)$$

So, we reach

$$\varepsilon_n=(n+1)(P_{n+5}-3)-\beta_n$$
$$=(n+1)(P_{n+5}-3)-(nP_{n+5}-P_{n+9}+9)$$
$$=P_{n+9}+P_{n+5}-3n-12$$

For example,

$$\varepsilon_3=\sum_{k=1}^{3}(3-k+1)P_k=3P_1+2P_2+P_3$$
$$=3.1+2.1+2=7$$
$$=21+7-3.3-12$$
$$=P_{12}+P_8-3.3-12$$

Similarly, we find

$$\sum_{k=1}^{n}(n-k+1)R_k=R_{n+9}+R_{n+5}-5n-17$$

For example,

$$\sum_{k=1}^{3}(3-k+1)R_k=3R_1+2R_2+R_3$$
$$=3.0+2.2+3=7$$
$$=29+10-5.3-17$$
$$=R_{12}+R_8-5.3-17$$

We know that

$$\varphi_n=\sum_{k=1}^{n}P_{2k-1}=P_{2n+2}-1$$

and

$$\gamma_n=\sum_{k=1}^{n}P_{2k}=P_{2n+3}-2\,.$$

So, let's find the sums

$$\eta_n=\sum_{k=1}^{n}(2k-1)P_{2k-1}$$

and

$$\kappa_n=\sum_{k=1}^{n}2kP_{2k}\,.$$

Therefore, we get

$$\begin{aligned}
\eta_n &= P_1 + 3P_3 + 5P_5 + \cdots + (2n-1)P_{2n-1} \\
&= \sum_{k=1}^{n} P_{2k-1} + 2\sum_{k=2}^{n} P_{2k-1} + 2\sum_{k=3}^{n} P_{2k-1} + \cdots + 2\sum_{k=n}^{n} P_{2k-1} \\
&= \varphi_n + 2(\varphi_n - \varphi_1) + 2(\varphi_n - \varphi_2) + \cdots + 2(\varphi_n - \varphi_{n-1}) \\
&= \varphi_n + 2(n-1)\varphi_n - 2\sum_{k=1}^{n-1} \varphi_k \\
&= (2n-1)(P_{2n+2} - 1) - 2\sum_{k=1}^{n-1} (P_{2k+2} - 1) \\
&= (2n-1)P_{2n+2} - 2P_{2n+3} + 5
\end{aligned}$$

and

$$\begin{aligned}
\kappa_n &= 2P_2 + 4P_4 + 6P_6 + \cdots + 2nP_{2n} \\
&= 2\sum_{k=1}^{n} P_{2k} + 2\sum_{k=2}^{n} P_{2k} + 2\sum_{k=3}^{n} P_{2k} + \cdots + 2\sum_{k=n}^{n} P_{2k} \\
&= 2\gamma_n + 2(\gamma_n - \gamma_1) + 2(\gamma_n - \gamma_2) + \cdots + 2(\gamma_n - \gamma_{n-1}) \\
&= 2n\gamma_n - 2\sum_{k=1}^{n-1} \gamma_k \\
&= 2n(P_{2n+3} - 2) - 2\sum_{k=1}^{n-1} (P_{2k+3} - 2) \\
&= 2nP_{2n+3} - 2P_{2n+4} + 4
\end{aligned}$$

For example,

$$\begin{aligned}
\eta_3 &= \sum_{k=1}^{3} (2k-1)P_{2k-1} = P_1 + 3P_3 + 5P_5 \\
&= 1 + 3.2 + 5.3 = 22 \\
&= (2.3-1)7 - 2.9 + 5 \\
&= (2.3-1)P_8 - 2P_9 + 5
\end{aligned}$$

and

$$\begin{aligned}
\kappa_3 &= \sum_{k=1}^{3} 2kP_{2k} = 2P_2 + 4P_4 + 6P_6 \\
&= 2.1 + 4.2 + 6.4 = 34 \\
&= 2.3.9 - 2.12 + 4 \\
&= 2.3P_9 - 2P_{10} + 4
\end{aligned}$$

We know that

$$\lambda_n = \sum_{k=1}^{n} R_{2k-1} = R_{2n+2} - 2$$

and

$$\mu_n = \sum_{k=1}^{n} R_{2k} = R_{2n+3} - 3.$$

Similarly, we write

$$\sum_{k=1}^{n}(2k-1)R_{2k-1}=(2n-1)R_{2n+2}-2R_{2n+3}+8$$

and

$$\sum_{k=1}^{n}2kR_{2k}=2nR_{2n+3}-2R_{2n+4}+4\,.$$

For example,

$$\begin{aligned}\sum_{k=1}^{3}(2k-1)R_{2k-1}&=R_1+3R_3+5R_5\\&=0+3.3+5.5=34\\&=(2.3-1)10-2.12+8\\&=(2.3-1)R_{2n+2}-2R_{2n+3}+8\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{3}2kR_{2k}&=2R_2+4R_4+6R_6\\&=2.2+4.2+6.5=42\\&=2.3.12-2.17+4\\&=2.3P_9-2P_{10}+4\end{aligned}$$

Now let's find the sums

$$\pi_n=\sum_{k=1}^{n}(2n-2k+1)P_{2k-1}$$

and

$$\rho_n=\sum_{k=1}^{n}(2n-2k)P_{2k}\,.$$

Then, we have

$$\begin{aligned}\eta_n+\pi_n&=\sum_{k=1}^{n}(2k-1)P_{2k-1}+\sum_{k=1}^{n}(2n-2k+1)P_{2k-1}\\&=\sum_{k=1}^{n}2nP_{2k-1}=2n(P_{2n+2}-1)\end{aligned}$$

So, we obtain

$$\begin{aligned}\pi_n&=2n(P_{2n+2}-1)-\eta_n\\&=2nP_{2n+2}-2n-2nP_{2n+2}+P_{2n+2}+2P_{2n+3}-5\\&=2P_{2n+3}+P_{2n+2}-2n-5\end{aligned}$$

For example,

$$\pi_3 = \sum_{k=1}^{3}(6-2k+1)P_{2k-1} = 5P_1 + 3P_3 + P_5$$
$$= 5.1 + 3.2 + 3 = 14$$
$$= 2.9 + 7 - 2.3 - 5$$
$$= 2P_9 + P_8 - 2.3 - 5$$

Moreover, we write

$$\kappa_n + \rho_n = \sum_{k=1}^{n} 2kP_{2k} + \sum_{k=1}^{n}(2n-2k)P_{2k}$$
$$= \sum_{k=1}^{n} 2nP_{2k} = 2n(P_{2n+3} - 2)$$

So, we reach

$$\rho_n = 2nP_{2n+3} - 4n - \kappa_n$$
$$= 2nP_{2n+3} - 4n - 2nP_{2n+3} + 2P_{2n+4} - 4.$$
$$= 2P_{2n+4} - 4n - 4$$

For example,

$$\rho_3 = \sum_{k=1}^{3}(6-2k)P_{2k} = 4P_2 + 2P_4 + 0P_6$$
$$= 4.1 + 2.2 + 0.4 = 8$$
$$= 2.12 - 4.3 - 4$$
$$= 2P_{10} - 4.3 - 4$$

Similarly, we can find

$$\sum_{k=1}^{n}(2n-2k+1)R_{2k-1} = 2R_{2n+3} + R_{2n+2} - 4n - 8$$

and

$$\sum_{k=1}^{n}(2n-2k)R_{2k} = 2R_{2n+4} - 6n - 4.$$

For example,

$$\sum_{k=1}^{3}(6-2k+1)R_{2k-1} = 5R_1 + 3R_3 + R_5$$
$$= 5.0 + 3.3 + 5 = 14$$
$$= 2.12 + 10 - 4.3 - 8$$
$$= 2R_9 + R_8 - 4.3 - 8$$

and

$$\sum_{k=1}^{3}(6-2k)R_{2k} = 4R_2 + 2R_4 + 0R_6$$
$$= 4.2 + 2.2 + 0.4 = 12$$
$$= 2.17 - 6.3 - 4$$
$$= 2R_{10} - 4.3 - 4$$

The identity whose first term a and common difference d can be obtained as follows:

$$\begin{aligned} S_n &= \sum_{k=1}^{n}\left[a+(k-1)d\right]P_k \\ &= a\sum_{k=1}^{n}P_k + d\sum_{k=1}^{n}kP_k - d\sum_{k=1}^{n}P_k \\ &= a\left(P_{n+5}-3\right)+d\left(nP_{n+5}-P_{n+9}+9\right)-d\left(P_{n+5}-3\right) \\ &= \left(a+nd-d\right)P_{n+5}-dP_{n+9}+12d-3a \end{aligned}$$

Similarly, we have

$$\begin{aligned} T_n &= \sum_{k=1}^{n}\left[a+(k-1)d\right]R_k \\ &= \left(a+nd-d\right)R_{n+5}-dR_{n+9}+7d-5a \end{aligned}$$

Using formulas (1) and (3), we can derive a formula for $\sum_{k=1}^{n}k^2P_k$. It can be shown that

$$\begin{aligned} \sum_{k=1}^{n}k^2P_k &= P_1+4P_2+9P_3+\cdots+n^2P_n \\ &= \sum_{k=1}^{n}P_k+3\sum_{k=2}^{n}P_k+5\sum_{k=3}^{n}P_k+\cdots+\left(2n-1\right)\sum_{k=n}^{n}P_k \\ &= \alpha_n+3\left(\alpha_n-\alpha_1\right)+5\left(\alpha_n-\alpha_2\right)+\cdots+\left(2n-1\right)\left(\alpha_n-\alpha_{n-1}\right) \\ &= n^2\alpha_n-\sum_{k=1}^{n-1}\left(2k+1\right)\alpha_k \\ &= n^2\left(P_{n+5}-3\right)-\sum_{k=1}^{n-1}\left(2k+1\right)\left(P_{k+5}-3\right) \\ &= n^2\left(P_{n+5}-3\right)-\sum_{k=6}^{n+4}\left(2k-9\right)\left(P_k-3\right) \\ &= n^2P_{n+5}-3n^2-2\sum_{k=6}^{n+4}kP_k+9\sum_{k=6}^{n+4}P_k+6\sum_{k=6}^{n+4}k-\sum_{k=6}^{n+4}27 \\ &= n^2P_{n+5}-3n^2-2nP_{n+9}-2P_{n+6}-4P_{n+4}+46+9P_{n+9} \\ &\quad -108+3n^2+27n-30-27(n-1) \\ &= n^2P_{n+5}+\left(9-2n\right)P_{n+9}-2P_{n+6}-4P_{n+4}-65 \end{aligned}$$

Similarly, using formulas (2) and (4), we obtain

$$\sum_{k=1}^{n}k^2R_k = n^2R_{n+5}+\left(9-2n\right)R_{n+9}-2R_{n+6}-4R_{n+4}-90 .$$

Now, we reach the general form of the weighted Padovan and Perrin sum formulas as follows:

Theorem 1. The formula for Padovan sums weighted by k^m is

$$\sum_{k=1}^{n} k^m P_k = n^m P_{n+3} + (n+1)^m P_{n+2} - 1 - 2\sum_{i=0}^{\left[\frac{m-1}{2}\right]} \binom{m}{2i+1} \sum_{k=1}^{n} k^{m-2i-1} P_{k+2}$$

where $m = 1, 2, 3, \dots$.

Proof: Using the recursive of the Padovan sequence, we have

$$\begin{aligned}
\sum_{k=1}^{n} k^m P_k &= \sum_{k=1}^{n} k^m \left(P_{k+3} - P_{k+1}\right) \\
&= \sum_{k=2}^{n+1} (k-1)^m P_{k+2} - \sum_{k=0}^{n-1} (k+1)^m P_{k+2} \\
&= n^m P_{n+3} + (n+1)^m P_{n+2} - 1 + \sum_{k=1}^{n} \left((k-1)^m - (k+1)^m\right) P_{k+2} \\
&= n^m P_{n+3} + (n+1)^m P_{n+2} - 1 + \sum_{k=1}^{n} \left(\sum_{j=0}^{m} \binom{m}{j} \left((-1)^j - 1\right) k^{m-j} \right) P_{k+2} \\
&= n^m P_{n+3} + (n+1)^m P_{n+2} - 1 + \sum_{k=1}^{n} \left(\sum_{i=0}^{\left[\frac{m-1}{2}\right]} \binom{m}{2i+1} (-2) k^{m-2i-1} \right) P_{k+2} \\
&= n^m P_{n+3} + (n+1)^m P_{n+2} - 1 - 2\sum_{i=0}^{\left[\frac{m-1}{2}\right]} \binom{m}{2i+1} \sum_{k=1}^{n} k^{m-2i-1} P_{k+2}
\end{aligned}$$

Similarly, we can find

$$T_m(n) = \sum_{k=1}^{n} k^m R_k = n^m R_{n+3} + (n+1)^m R_{n+2} - 2 - 2\sum_{i=0}^{\left[\frac{m-1}{2}\right]} \binom{m}{2i+1} \sum_{k=1}^{n} k^{m-2i-1} R_{k+2}.$$

3. CONCLUSIONS

In this paper, we obtain weighted sum formulas $\sum_{k=1}^{n} kP_k$ and $\sum_{k=1}^{n} kR_k$ using sum formulas $\sum_{k=1}^{n} P_k$ and $\sum_{k=1}^{n} R_k$ of Padovan and Perrin numbers. Moreover, we show that the identities involve $\sum_{k=1}^{n} kP_k$ and $\sum_{k=1}^{n} kR_k$ can be extended to any weighted Padovan and Perrin sum formulas, where weights form an arbitrary arithmetic sequence with first term a and common difference d: $\sum_{k=1}^{n}\left[a+(k-1)d\right]P_k$ and $\sum_{k=1}^{n}\left[a+(k-1)d\right]R_k$. Then, we prove that it is possible to develop formulas for Padovan and Perrin sums which are weighted by k^m .

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