ORIGINAL PAPER

BINARY SEMI-LOCALLY CLOSED SETS IN BINARY TOPOLOGICAL SPACES

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Abstract. *In this paper we introduce new weaker forms of locally closed sets called binary semi-locally closed, binary locally semi-closed sets and binary generalized locally semi-closed sets and study their relations with the existing ones. We also try to establish the conditions under which these weaker forms coincide with one another. Further it is established that in a particular space, all the weaker forms of binary locally closed sets coincide with* $\mathbb{P}(X, Y)$, *the power set of* (X, Y).

Keywords: *bsl-closed; bsgl-closed; binary semi-dense and bsg-submaximal.*

1. INTRODUCTION AND PRELIMINARIES

In 2011, S. Nithyanantha Jothi and P. Thangavelu [1-3] were introduced topology between two sets and also studied some of their properties. Topology between two sets is the binary structure from X to Y which is defined to be the ordered pairs (A, B) where $A \subseteq X$ and $B \subseteq Y$. In this paper we introduce new weaker forms of locally closed sets called binary semi-locally closed, binary locally semi-closed sets and binary generalized locally semi-closed sets and study their relations with the existing ones.

We also try to establish the conditions under which these weaker forms coincide with one another. Further it is established that in a particular space, all the weaker forms of binary locally closed sets coincide with $\mathbb{P}(X, Y)$, the power set of (X, Y).

Let X and Y be any two nonempty sets. A binary topology [2] from X to Y is a binary structure $\mathcal{M} \subseteq \mathbb{P}(X) \times \mathbb{P}(Y)$ that satisfies the axioms namely

1. (ϕ, ϕ) and $(X, Y) \in \mathcal{M}$,
2. $(A_1 \cap A_2, B_1 \cap B_2) \in \mathcal{M}$ whenever $(A_1, B_1) \in \mathcal{M}$ and $(A_2, B_2) \in \mathcal{M}$, and
3. If $\{(A_\alpha, B_\alpha): \alpha \in \delta\}$ is a family of members of $\mathcal{M}$, then $(\bigcup_{\alpha\in\delta} A_\alpha, \bigcup_{\alpha\in\delta} B_\alpha) \in \mathcal{M}$.

If $\mathcal{M}$ is a binary topology from X to Y then the triplet $(X, Y, \mathcal{M})$ is called a binary topological space and the members of $\mathcal{M}$ are called the binary open subsets of the binary topological space $(X, Y, \mathcal{M})$. The elements of $X \times Y$ are called the binary points of the binary topological space $(X, Y, \mathcal{M})$. If $Y = X$ then $\mathcal{M}$ is called a binary topology on X in which case we write $(X, \mathcal{M})$ as a binary topological space.

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Definition 1.1. [2] Let X and Y be any two nonempty sets and let (A, B) and $(C, D) \in \mathbb{P}(X) \times \mathbb{P}(Y)$. We say that $(A, B) \subseteq (C, D)$ if $A \subseteq C$ and $B \subseteq D$.

Definition 1.2. [2] Let $(X, Y, \mathcal{M})$ be a binary topological space and $A \subseteq X$, $B \subseteq Y$. Then (A, B) is called binary closed in $(X, Y, \mathcal{M})$ if $(X \backslash A, Y \backslash B) \in \mathcal{M}$.

Definition 1.3. [3] Let $(X, Y, \mathcal{M})$ be a binary topological space. Let $(A, B) \subseteq \mathbb{P}(X) \times \mathbb{P}(Y)$. Then (A, B) is called binary g-closed if $\text{b-cl}(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary open in $(X, Y, \mathcal{M})$.

Definition 1.4. A subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is called

1. a binary semi open set [2] if $(A, B) \subseteq \text{b-cl}(\text{b-int}(A, B))$.
2. a binary pre open set [4] if $(A, B) \subseteq \text{b-int}(\text{b-cl}(A, B))$.

Definition 1.5. [5] A subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is called a binary α-open if $(A, B) \subseteq \text{b-int}(\text{b-cl}(\text{b-int}(A, B)))$.

Definition 1.6. [6] A subset (A, B) of a binary toplogical space (X, Y, M) is binary β-open in (X, Y) if $(A, B) \subseteq \text{b-cl}(\text{b-int}(\text{b-cl}(A, B)))$. The set of all binary β-open sets of (X, Y) is denoted by $b\beta O(X, Y, M)$.

2. BINARY SEMI-LOCALLY CLOSED SETS AND BINARY SEMI-GENERALIZED LOCALLY CLOSED SETS

Definition 2.1. A subset (A, B) of $(X, Y, \mathcal{M})$ is called binary semi-locally closed (briefly bslc) if

$$(A, B) = (P, Q) \cap (M, N)$$

where (P, Q) is binary semi-open and (M, N) is binary semi-closed in $(X, Y, \mathcal{M})$.

The class of all binary semi locally closed sets is denoted by $BSLC(X, Y)$.

Definition 2.2. A subset (A, B) of $(X, Y, \mathcal{M})$ is called binary semi-generalized locally closed (briefly bsglc) if

$$(A, B) = (P, Q) \cap (M, N)$$

where (P, Q) is bsg-open and (M, N) is bsg-closed in (X, Y).

The class of all binary semi-generalized locally closed sets of $(X, Y, \mathcal{M})$ is denoted by $BSGLC(X, Y)$.

Definition 2.3. A subset (A, B) of $(X, Y, \mathcal{M})$ is called binary locally semi-closed (briefly blsc) if

$$(A, B) = (P, Q) \cap (M, N)$$

where (P, Q) is binary open and (M, N) is binary semi-closed in $(X, Y, \mathcal{M})$.

Every binary semi-closed (resp. open) set is binary locally semi-closed. The collection of all binary locally semi-closed sets of $(\mathrm{X}, \mathrm{Y}, \mathcal{M})$ is denoted by $\mathrm{BLSC}(\mathrm{X}, \mathrm{Y})$.

Definition 2.4. A subset (A, B) of $(\mathrm{X}, \mathrm{Y}, \mathcal{M})$ is called binary generalized locally semi-closed (briefly bglsc) if

$$(\mathrm{A}, \mathrm{B}) = (\mathrm{P}, \mathrm{Q}) \cap (\mathrm{M}, \mathrm{N})$$

where (P, Q) is bg-open and (M, N) is binary semi-closed in (X, Y).

Every bg-open (resp. binary semi-closed) set is bglsc. The class of all binary generalized locally semi-closed sets is denoted by $\mathrm{BGLSC}(\mathrm{X}, \mathrm{Y})$.

Proposition 2.5. *For a binary topological space $(X, Y, \mathcal{M})$, the following implications hold.*

1. $\mathrm{BLSC}(\mathrm{X}, \mathrm{Y}) \subseteq \mathrm{BSLC}(\mathrm{X}, \mathrm{Y}) \subseteq \mathrm{BSGLC}(\mathrm{X}, \mathrm{Y})$,
2. $\mathrm{BLSC}(\mathrm{X}, \mathrm{Y}) \subseteq \mathrm{BGLSC}(\mathrm{X}, \mathrm{Y})$,
3. $\mathrm{BGLC}^*(\mathrm{X}, \mathrm{Y}) \subseteq \mathrm{BGLSC}(\mathrm{X}, \mathrm{Y})$.

Proof: The proofs follow easily from definitions.

Proposition 2.6.

1.If $(A, B) \in BLSC(X, Y)$ and $(C, D) \in BLSC(X, Y)$, then

$$(A, B) \cap (C, D) \in BLSC(X, Y).$$

2.If $(A, B) \in BLSC(X, Y)$ and $(C, D) \in BGLC^*(X, Y)$, then

$$(A, B) \cap (C, D) \in BGLSC(X, Y).$$

3.If $(A, B) \in BGLSC(X, Y)$ and $(C, D) \in BGLSC(X, Y)$, then

$$(A, B) \cap (C, D) \in BGLSC(X, Y).$$

4.If $(A, B) \in BLSC(X, Y)$ and $(C, D) \in BGLSC(X, Y)$, then

$$(A, B) \cap (C, D) \in BGLSC(X, Y).$$

5.If $(A, B) \in BGLSC(X, Y)$ and $(C, D) \in BGLC^*(X, Y)$, then

$$(A, B) \cap (C, D) \in BGLSC(X, Y).$$

Proof: The proofs are immediate.

Proposition 2.7. Let (A, B) be a subset of $(\mathrm{X}, \mathrm{Y}, \mathcal{M})$. Then

1. $(\mathrm{A}, \mathrm{B}) \in \mathrm{BGLSC}(\mathrm{X}, \mathrm{Y})$ if and only if $(\mathrm{A}, \mathrm{B}) = (\mathrm{U}, \mathrm{V}) \cap \mathrm{bscl}(\mathrm{A}, \mathrm{B})$ for some bg-open set (U, V).

2. $(\mathrm{A}, \mathrm{B}) \in \mathrm{BLSC}(\mathrm{X}, \mathrm{Y})$ if and only if $(\mathrm{A}, \mathrm{B}) - (\mathrm{G}, \mathrm{H}) \cap \mathrm{bscl}(\mathrm{A}, \mathrm{B})$ for some binary open set (G, H).

Proof:
1. (Necessity) Let (A,B) be $bglsc$. Then $(A,B)=(U,V)\cap(P,Q)$ where (U,V) is bg-open and (P,Q) is binary semi-closed. $(A,B)\subseteq(P,Q)$ implies $bscl(A,B)\subseteq(P,Q)$. Now $(A,B)=(A,B)\cap bscl(A,B)=(U,V)\cap(P,Q)\cap bscl(A,B)=(U,V)\cap bscl(A,B)$. Hence (1) is proved.

(Sufficiency) Let $(A,B)=(U,V)\cap bscl(A,B)$ where (U,V) is bg-open. Then (A,B) is $bglsc$ as $bscl(A,B)$ is binary semi-closed.

2. (Necessity) Let $(A,B)=(G,H)\cap(E,F)$ where (G,H) is binary open and (E,F) is binary semi-closed. Since $(A,B)\subseteq(E,F)$, $bscl(A,B)\subseteq(E,F)$ and hence $(G,H)\cap bscl(A,B)\subseteq(G,H)\cap(E,F)=(A,B)$. Conversely, $(A,B)\subseteq(G,H)$ and $(A,B)\subseteq bscl(A,B)$ implies $(A,B)\subseteq(G,H)\cap bscl(A,B)$.

(Sufficiency) It is obvious.

Definition 2.8. A binary topological space $(X,Y,\mathcal{M})$ is called a BSG-space if the intersection of a binary semi-closed set with a bg-closed set is bg-closed.

Theorem 2.9. For a subset (A,B) of a BSG-space $(X,Y,\mathcal{M})$, the following are equivalent.
1. (A,B) is bglsc.
2. $\mathrm{bscl}(A,B)-(A,B)$ is bg-closed.
3. $(A,B)\cup((X,Y)-\mathrm{bscl}(A,B))$ is bg-open.

Proof:
(1) → (2)

By Proposition 2.7, $(A,B)=bscl(A,B)\cap(P,Q)$ where (P,Q) is bg-open.
Now

$$bscl(A,B)-(A,B)=bscl(A,B)-(P,Q)=bscl(A,B)\cap((X,Y)-(P,Q))$$

where $bscl(A,B)$ is binary semi-closed and $(X,Y)-(P,Q)$ is bg-closed.
Since the space is a BSG-space, $bscl(A,B)\cap((X,Y)-(P,Q))$ is bg-closed.
That is, $bscl(A,B)-(A,B)$ is bg-closed.

(2) → (3)

$$(X,Y)-(bscl(A,B)-(A,B))=((X,Y)-bscl(A,B))\cup(A,B).$$

Since $bscl(A,B)-(A,B)$ is bg-closed, $(X,Y)-(bscl(A,B)-(A,B))$ that is $(A,B)\cup((X,Y)-bscl(A,B))$ is bg-open.

(3) → (1)

$$(A,B)=((X,Y)-(bscl(A,B)-(A,B)))\cap bscl(A,B).$$

By (3), $(X,Y)-(bscl(A,B)-(A,B))$ is bg-open. It follows from Proposition 2.7 that (A,B) is $bglsc$.

Definition 2.10. A subset (A,B) of $(\mathrm{X},\mathrm{Y},\mathcal{M})$ is called binary semi-dense in (X,Y) if $\mathrm{bscl}(\mathrm{A},\mathrm{B}) = (\mathrm{X},\mathrm{Y})$.

Definition 2.11. A space $(\mathrm{X},\mathrm{Y},\mathcal{M})$ is called bsg-submaximal if every binary semi-dense subset is bg-open in (X,Y).

Theorem 2.12. A BSG-space $(\mathrm{X},\mathrm{Y},\mathcal{M})$ is bsg-submaximal if and only if $\mathbb{P}(\mathrm{X},\mathrm{Y}) = \mathrm{BGLSC}(\mathrm{X},\mathrm{Y})$ holds.

Proof:
(Necessity) Let (X,Y) be bsg-submaximal.

Let $(\mathrm{A},\mathrm{B}) \in \mathbb{P}(\mathrm{X},\mathrm{Y})$. Let $(\mathrm{U},\mathrm{V}) = (\mathrm{A},\mathrm{B}) \cup ((\mathrm{X},\mathrm{Y}) - \mathrm{bscl}(\mathrm{A},\mathrm{B}))$.
Then $\mathrm{bscl}(\mathrm{U},\mathrm{V}) = (\mathrm{X},\mathrm{Y})$. That is, (U,V) is binary semi-dense in (X,Y).
By hypothesis, (U,V) is bg-open.
By Theorem 2.9, (A,B) is bglsc. This implies $\mathbb{P}(\mathrm{X},\mathrm{Y}) = \mathrm{BGLSC}(\mathrm{X},\mathrm{Y})$.

(Sufficiency)

Let (A,B) be binary semi-dense in (X,Y) and $\mathbb{P}(\mathrm{X},\mathrm{Y}) = \mathrm{BGLSC}(\mathrm{X},\mathrm{Y})$.
Now $(\mathrm{A},\mathrm{B}) \cup ((\mathrm{X},\mathrm{Y}) - \mathrm{bscl}(\mathrm{A},\mathrm{B})) = (\mathrm{A},\mathrm{B})$. $(\mathrm{A},\mathrm{B}) \in \mathrm{BGLSC}(\mathrm{X},\mathrm{Y})$ implies that (A,B) is bg-open by Theorem 2.9.
Hence (X,Y) is bsg-submaximal.

Proposition 2.13. If (X,Y) is binary submaximal, then it is bsg-submaximal.

Proof: Let (X,Y) be binary submaximal and $(\mathrm{A},\mathrm{B}) \subseteq (\mathrm{X},\mathrm{Y})$ be binary semi-dense in (X,Y). Then (A,B) is binary dense in (X,Y) and by assumption (A,B) is binary open in (X,Y). Since every binary open set is bg-open, (A,B) is bg-open in (X,Y). That is, (X,Y) is bsg-submaximal.

3. CONCLUSION

We discussed new weaker forms of locally closed sets called binary semi-locally closed, binary locally semi-closed sets and binary generalized locally semi-closed sets and study their relations with the existing ones. We also try to establish the conditions under which these weaker forms coincide with one another. Further it is established that in a particular space, all the weaker forms of binary locally closed sets coincide with $\mathbb{P}(\boldsymbol{X},\boldsymbol{Y})$, the power set of $(\boldsymbol{X},\boldsymbol{Y})$. In the future, we can study the concepts of new separation axioms. Let's try to see real life applications using this paper.

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