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NONPARAMETRIC FUNCTIONAL HAZARD WITH k NEAREST NEIGHBORS ESTIMATION WHERE THE OBSERVATIONS ARE M.A.R. AND RELATED *VIA* A FUNCTIONAL SINGLE-INDEX STRUCTURE

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Abstract. *The nonparametric local linear hazard functional estimation analyzed by the k.N.N method of the scalar response variable B, not totally observed provided the functional variable A. The purpose of this paper is to illustrate, under some general conditions, the almost complete convergence (with rates) of the constructed estimator.*

Keywords: *hazard function; almost complete convergence (a.co.); local linear method; missing data; k-Nearest Neighbors method (kNN).*

1. INTRODUCTION

Recent years have seen significant developments in computing power an exponentially vast volume of data can be obtained and analyzed. Parametric models have become multivariate computational techniques may be extended to usable data and a good guide on this topic can be contained in Bosq [1] or Ramsay and Silverman [2]. New studies have been carried out recently to indicate nonparametric methods which take functional data into account. We are referring to Ferraty, and Vieu [3] for a more detailed analysis on this subject.

The estimation of hazard function has been a significant topic in statistics. Even then, it is very well known that a local polynomial smoothing method has several advantages over the kernel method (for more information, see Fan and Gijbels [4]). In fact, the former approach has good characteristics, in terms of bias estimation.

In various fields, including surveys, clinical trials and longitudinal research, missing data frequently occurs. Responses may be lacking, and methods for the retrieval of missing data also depends on the generating mechanism. See Efromovich [5] for the missing values. In many practical occupations, such as sample surveys, pharmacy or pharmacy tracing, dependability, data and some responses are also not thoroughly observed and some responses are note completed (M.A.R.) which means Missing At Random.

A weighted estimator is k Nearest Neighbor, or k.N.N. Neighborhood nearest of response variables of a. The current bibliography for estimating the k.N.N. process dates back to Stone [6] and to Royall [7]. Mack [8] derived the convergence rates for both bias and variation, as well as asymptotic normality in multivariate case, various forms of convergence

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(probability, almost.sure (a.s), almost.completely (a.co)) has been studied by Collomb [8] for regression function estimator.

For functional data studies, in the monograph of Ferraty and Vieu [3], the k.N.N kernel estimate was first implemented, Burba et al. [9] obtained the rate of almost complete convergence. Using the k.N.N. approach for independent regression functional data. Attouch et al. [10, 11] achieved almost complete convergence with rates in the functional data case (i.i.d.) and dependent data respectively for the k.N.N. conditional data case in the function mode estimation.

During this article, with the k.N.N. methods of incomplete data at random structure, of univariate response variable B, given the functional variable A, we examine nonparametric estimation by the local linear method of conditional density. The purpose of this analysis is to demonstrate the convergence under some general conditions of our estimator.

The results obtained are established under general conditions that allow the incomplete case of functional data to be covered, mainly when the response variables contain some MAR observations. In practice, the feasibility and efficacy of the L.L.E.-k.N.N. estimator was tested by real data.

The remainder of the article is structured like this. In Section 2 we'll present our model. In Section 3, we indicate the nearly complete pointwise convergence of the constructed estimator, by specifying the convergence rate, Section 4 is then dedicated to a few comments highlighting the effect of our study to the N.F.D.A contribution. Some simulation results are discussed in Section 5. Lastly, proofs of the various outcomes are assigned to the last segment.

2. MODEL DEFINITION, NOTATION, AND PROSPECTS

2.1. MODEL AND ESTIMATOR

Let's consider a $(A_i, B_i)_{i\geq 1}$ sequence of independent and identically random pairs according to the (A, B) pair distribution, which is valued in the $\mathcal{F} \times \mathbb{R}$ space, where $(\mathcal{F}, d)$ is a semi-metrical space.

We will refer to a fixed curve in $\mathcal{F}$ in the remainder of this article, $\mathcal{N}_a$ is the neighborhood of a.

In addition, the B conditional cumulative distribution function (c.d.f) is denoted by $F(a,.)$ given $(A = a)$, and we suppose that $F(a,;)$ is totally continuous with respect to the Lebesgue measure with radon-Nikodym derivative $f(a.)$, Lebesgue measure, which is a the B conditional probability density function (p.d.f) given $(A = a)$.

Accordingly, the conditional hazard function (c.h.f) of B, given $A = a$, is

$$h(a,b) = \frac{f(a,b)}{1-F(a,b)}, b \in \mathbb{R} \text{and} F(a,b) < 1. \tag{1}$$

Our main goal,To estimate the conditional hazard function $\breve{h}(a,.)$, for a fixed a, so we have

$$\breve{h}(a,b) = \frac{\breve{f}(a,b)}{1-\breve{F}(a,b)}, b \in \mathbb{R} \text{ and } \breve{F}(a,b) < 1. \tag{2}$$

There are several ways that allow the local linear approach to be described. For example, we may discuss the quick version proposed in Barrientos et al. [13] of banachique or the version introduced in Baìllo and Grané [12] of Hilbertian. All in all, Local Linear Estimation shortly L.L.E modeling is based on the hypothesis that the $F(a,.)$ is sufficiently smooth to be approximated by a loval linear operator.

Formally,$\forall a_0$ in the neighborhood of a in $\mathcal{F}$, we assume that $f(a,.)$.

For any $a_0 \in \mathcal{N}_a$:

$$f(a_0,b) = f(a,b) + rf_{a,b}(a_0 - a) + Rf_{a,b}(a_0 - a, a_0 - a) + o(\| (a_0, a) \|^2). \quad (3)$$

and we assume that $F(a,.)$ for any $a_0 \in \mathcal{N}_a$:

$$F(a_0,b) = \zeta_{a,b} + \eta_{a,b}(a_0 - a) + \rho_{a,b}(a_0 - a, a0 - a) + o(\| a_0 - a \|^2). \quad (4)$$

where from $\mathcal{F} \times \mathcal{F}$ to $\mathbb{R}$, the continuous linear operator are $rF_{a,b}$ and $\rho_{a,b}$, and a bilinear continuous operator is $RF_{a,b}$.

It is possible to interpret the cumulative conditional function (cdf) with the response variable $(e_\ell G(e_\ell^{-1}(. - B_i)))$, where G is a given Rosenblatt kernel(see, for example, Demongeot et al. [15]; Fan and Gijbels [4]). Next, the, local linear approximation elements, $f(a,b)$ and $rf_{a,b}$see (3), are calculated in local linear estimation, by the k.N.N method, as the minimizers of

$$f(a,b) = (\zeta_{a,b}, \eta_{a,b}) = \arg\min_{(\zeta,\eta)\in\mathbb{R}\times\mathcal{F}} \sum_{i=1}^{n} (e_\ell^{-1} G(e_\ell^{-1}(. - B_i)) - \zeta - < \eta, A_i - a >)^2 K(\frac{\|a - A_i\|}{h_k}). \quad (5)$$

Then the local linear approximation elements, $F(a,b)$ and $\rho_{a,b}$see (4), are calculated in local linear estimation, by the k.N.N method, as the minimizers of

$$F(a,b) = \arg\min_{(\zeta,\eta)\in\mathbb{R}\times\mathcal{F}} \sum_{i=1}^{n} (B_i - \zeta - < \eta, A_i - a >)^2 K(\frac{\|a - A_i\|}{h_k}). \quad (6)$$

K is a kernel function, h_k is a sequence of positive real numbers tending to zero as n tends to infinity, and $e_\ell = \min\{e \in \mathbb{R}^+ such that \sum_{i=1}^{n} \mathrm{I}_{b-e,b+e}(B_i) = \ell\}$ where I_A denotes the indicator function on the set A, this form of estimate has been studied by Baíllo and Grané [12]. We can deduced from (5) the conditional density function as

$$f(a,b) = \arg\min_{(\zeta,\eta)\in\mathbb{R}\times\mathcal{F}} \sum_{i=1}^{n} (e_\ell^{-1} G(e_\ell^{-1}(. - B_i)) - \zeta - < \eta, A_i - a >)^2 K(\frac{\|a - A_i\|}{H_k}). \quad (7)$$

The replacement of the arbitrary scalar sequence h_k by the random sequence expressed by $H_k(.)$ in (5,6) is however given by the following expression where H_k is a positive random variable which relies on $(A_1, \cdots, A_n)$ so:

$$H_k = \min\{h_k \in \mathrm{IR}^+ / \sum_{i=1}^{n} \mathrm{I}_{\mathcal{L}(a,h_k)}(A_i) = k\}.$$

where the topologically closed ball in $\mathcal{F}$ is denoting by $\mathcal{L}(a, h_k) = d \in \mathcal{F} such that \| a - d \| \leq h_k$, centered at a and of radius h_k.

The second aspect of our contribution is the fact that we are working with incomplete data. This is a more general case than complete data, which therefore includes special care. We concentrate in particular on the case of missed answer observations. In this case, we observe a set of three variables: $(A_i, B_i, \omega_i)_{i=1,\dots,n}$, where ω_i is a random variable of Bernoulli, which is such that $\omega_i = 1$ is observed if B_i is observed and $\omega_i = 0$ instead. It is assumed that the Bernoulli random variable ωi is such that

$$\mathbb{P}(\omega_i = 1|A_i, B_i) = \mathbb{P}(\omega_i = 1|A_i) = \varrho(A_i).$$

where $\varrho(.)$ is a functional operator that calculates the conditional probability of B being observed given A, this considerations means that knowing A, the variables B and ω are independent.

The criteria is modified by combining the two properties k.N.N method with M.A.R (5) using the sample $(A_i, B_i, \omega_i)_{i=1,\ldots,n}$ as follows

$$f(a,b) = \arg\min_{(\zeta,\eta)\in\mathbb{R}\times\mathcal{F}} \sum_{i=1}^{n} (e_\ell^{-1}(G(e_\ell^{-1}(b - B_i)) - \zeta - <\eta, A_i - a>)^2 \omega_i K(\frac{\|a - A_i\|}{H_k}), \tag{8}$$

and

$$F(a,b) = (\zeta_{a,b}, \eta_{a,b}) = \arg\min_{(\zeta,\eta)\in\mathbb{R}\times\mathcal{F}} \sum_{i=1}^{n} (B_i - \zeta - <\eta, A_i - a>)^2 \omega_i K(\frac{\|a - A_i\|}{H_k}). \tag{9}$$

$\mathcal{F}$ is a separable Hilbert space with an orthonormal base $(\nu_j)_{j\geq 1}$ then we make utilize of the fact that to write

$$\eta_{a,b)} = \sum_j <\eta, \nu_j> \nu_j and A_i - a = \sum_j <A_i - a, \nu_j> \nu_j.$$

Thus, we put for all $j \geq 1, \eta_j =<\eta, \nu_j>$ and $m_{ij}(x) =<A_i - a, \nu_j>$ and we estimate the parameters $\zeta_{a,b}$ and $\eta_{a,b}$ firstly of (8) by estimating their components in the orthonormal basis $(\nu_j)_{j\geq 1}$ with respect to the following rule

$$\min_{(\zeta,\eta_1,\ldots)\in\mathbb{R}\times\mathbb{R}\times\ldots} \sum_{i=1}^{n} \left(e_\ell^{-1}\ \ G(e_\ell^{-1}(b - B_i)) - \zeta - \sum_{j\geq 1} m_{ij}\eta_j\right)^2 \omega_i K(\frac{\|a - A_i\|}{H_k}).$$

and secondly of (9)

$$\min_{(\zeta,\eta_1,\ldots)\in\mathbb{R}\times\mathbb{R}\times\ldots} \sum_{i=1}^{n} \left(B_i - \zeta - \sum_{j\geq 1} m_{ij}\eta_j\right)^2 \omega_i K(\frac{\|a - A_i\|}{H_k}).$$

while this criterion is optimized, it can not be used in practice on an infinite set of components. Therefore, we recommend that its truncated form be used at the threshold J. Parseval 's theorem justifies this concern, which guarantees that the development of Fourier converges

$$\sum_{j=1}^{J} m_j(a)\nu_j \to (A - a), as\ J \to \infty.$$

where $(m_j(a))_j$ are the coefficients of $(A - a)$ in the base $(\nu_j)_{j\geq 1}$. Then, the estimator of $F(a,b)$ is obtained by estimating their components by:

$$\begin{pmatrix}\hat{\zeta}_{a,b}\\ \hat{\eta}_1\\ .\\ .\\ .\\ \hat{\eta}_J\end{pmatrix} .= \text{argmin}_{(\zeta,\eta_1,\ldots,\eta_J)\in\mathbb{R}^{J+1}} \sum_{i=1}^{n} \left(B_i - \zeta - \sum_{j=1}^{J} m_{ij}(a)\eta_j\right)^2 \omega_i K\left(\frac{\|a-A_i\|}{H_k}\right).$$

Explicitly this estimator is given by:

$$(P'_Y K P_Y)\begin{pmatrix}\hat{\zeta}_{a,b}\\ \hat{\eta}_1\\ .\\ .\\ .\\ \hat{\eta}_J\end{pmatrix} .= (P'_Y K B) \text{where } P_Y = \begin{pmatrix}1 & m_{11}(a) & . & . & m_{1j}(a)\\ . & . & . & . & .\\ . & . & . & . & .\\ . & . & . & . & .\\ 1 & m_{n1}(a) & . & . & m_{nj}(a)\end{pmatrix}.$$

with

$$K = diag(\omega_1 K(H_k^{-1} \parallel a - A_1 \parallel), \ldots, \omega_n K(H_k^{-1} \parallel a - A_n \parallel),$$

and

$$B' = (B_1, \cdots, B_n)\ .$$

Assuming that $(P'_Y K P_Y)$ is a non singular matrix then

$$\begin{pmatrix}\hat{\zeta}_{a,b}\\ \hat{\eta}_1\\ .\\ .\\ .\\ \hat{\eta}_J\end{pmatrix} = (P'_Y K P_Y)^{-1}(P'_Y K B).$$

Therefore, the local linear estimator of the conditional density $F(a,b)$ in MAR data, when the matrix $(P'_Y K P_Y)$ is non-singular, is defined under the decomposition 4.

$$\breve{F}(a,b) = (\hat{\zeta}_{a,b}, \hat{\eta}_{a,b}) = E'_1 (P'_Y K P_Y)^{-1}(P'_Y K B). \qquad (10)$$

where E'_1 denotes the transpose vector of the first vector of the canonical basis of $\mathbb{R}^{J+1}$.

In the same manner we can deduce from the previous step that

$$\breve{f}(a,b) = E'_1 (P'_Y K P_Y)^{-1}(P'_Y K G). \qquad (11)$$

with

$$G = (e_\ell^{-1} G(e_\ell^{-1}(b - B_1)), \cdots, e_\ell^{-1} G(e_\ell^{-1}(b - B_n)))$$

2.2. ASSUMPTIONS

Let us introduce a set of hypotheses which will be needed to state our main result.
(W1) For any $d > 0, \mathbb{P}(A \in L(a,d)) := \varphi_a(d) > 0$ is an invertible function and there exists $0 < c < 1 < c^* < \infty$, such that

$$\lim_{d\to 0} \frac{\varphi_a(dc)}{\varphi_a(d)} < 1 < \lim_{d\to 0} \frac{\varphi_a(dc^*)}{\varphi_a(d)}.$$

(W2) The operators $Rf_{a,b}$ and $\rho_{a,b}$ are continuous on $\mathcal{N}_a$ and the coefficients $(m_j(a))_j \geq 1$ are such that for some $\upsilon > 0$ we have:

$$\sum_{j=J+1} m_j^2(a) = O_{a.co}(J^{-\upsilon}).$$

(W3) The function $\varrho(.)$ is continuous on $\mathcal{N}_a$ with $\varrho(\kappa) > 0$, for all $\kappa \in \mathcal{N}_a$.
(W4) Kernel K is a differentiating function that is provided in the $(0,1)$ range. Furthermore, there is its first derivative, K', and it is such that there is two constants C^* and C^* satisfying

$$-\infty < C^* *< K'(\tau) < C^* < 0 \text{ for } \tau \in (0,1)$$

(W5) The kernel G is a 2-times continuously differentiable function and satisfies

$$\int G(\tau)d\tau = 1.$$

(W6) The neighbors k is such that

$$\frac{n\log n}{k\ell} \to 0 \text{ as } n \to \infty.$$

(W7)$\exists\Theta < \infty, f^x(y) \leq \Theta, \forall(a,b) \in \mathcal{F} \times \mathbb{R}$ and

$$\exists 0 < \delta < 1,\ F(a,b) \leq 1 - \delta, \forall(a,b) \in \mathcal{F} \times \mathbb{R}.$$

2.3. SOME COMMENTARIES

Baìllo and Grané [12] specifically used hypotheses (W1), (W2)-(W3) and (W4) Ferraty et al. [16] employed the hypothesis (W2)-(W3) and (W5), Finally, we underline that Burba et al. [13] also uses the last theory (W6), the hypothesis (W7) is a technical assumption and is also similar to that considered in Burba et al. [13].

3. RESULTS AND DISCUSSION

3.1. RESULTS

The following Theorem then gives the complete convergence with estimator rate.

Theorem 1. Under assumptions (W1)-(W6), we obtain:

$$\breve{h}(a,b) - h(a,b)| = O(J^{-v}) + O(\frac{\ell}{n})^2 + \varphi_x^{-1}(\frac{K}{n})^2 + O_{a.co}(\sqrt{\frac{\log n}{\ell k}}) \quad \text{as } n \to \infty.$$

Proof: By using the following decomposition

$$\breve{h}(a,b) - h(a,b) = \frac{1}{1-\breve{F}(a,b)}\left[(\breve{f}(a,b) - f(a,b)) + \frac{f(a,b)}{1-F(a,b)}(\breve{F}(a,b) - F(a,b))\right]$$
$$\leqslant \frac{1}{1-\breve{F}(a,b)}\left[(\breve{f}(a,b) - f(a,b)) + \frac{\tau}{\delta}(\breve{F}(a,b) - F(a,b))\right]$$
$$\leqslant \left[(\breve{f}(a,b) - f(a,b)) + \frac{\Theta}{\delta}(\breve{F}(a,b) - (a,b)))\right].$$

The proof of Theorem 1 can be deduced from Theorem 2 and Theorem 3 and the following result

$$\exists \epsilon > 0 \text{ such that} \sum_{n\in\mathbb{N}} \mathbb{P}\big(1 - \breve{F}(a,b) < \epsilon\big) < \infty. \tag{12}$$

Under the same assumption of Theorem 1 we obtain the following Theorems:

Theorem 2. Under the same assumption of Theorem 1, we have

$$\breve{f}(a,b) - f(a,b) = O(J^{-v}) + O(\frac{\ell}{n})^2 + \varphi_x^{-1}(\frac{k}{n})^2 + O_{a.co}(\sqrt{\frac{\log n}{\ell k}}) \quad as\ n \to \infty. \tag{13}$$

Proof:In the first place, we put

$$K_i(a,H_k) = K(H_k^{-1} \parallel a - A_i \parallel, G_i(b,e_\ell) = e_\ell G(e_\ell^{-1}(b - B_1)),$$

and we add the notations below for $j, j' = 1, \ldots, J$

$$S_{n,j',j}(a) = \frac{1}{nH_k^2\varphi_a(H_k)} \sum_{i=1}^{n} m_{ij'}(a) m_{ij}(a) \omega_i K_i(a,H_k).$$

$$S_{n,j',0}(a) = \frac{1}{nH_k\varphi_a(H_k)} \sum_{i=1}^{n} m_{ij'}(a) \omega_i K_i(a,H_k).$$

$$T_{n,j}(a) = \frac{1}{nH_k\varphi_a(H_k)} \sum_{i=1}^{n} m_{ij}(a) \omega_i K_i(a,H_k) G_i(b,e_\ell).$$

$$T_{n,0}(a) = \frac{1}{n\varphi_a(H_k)} \sum_{i=1}^{n} \omega_i K_i(a,H_k) G_i(b,e_\ell).$$

Then, we can write $\begin{pmatrix} \hat{\zeta}_{a,b} \\ H_k\hat{\eta}_1 \\ . \\ . \\ . \\ H_k\hat{\eta}_J \end{pmatrix}$ $=(S_n(a))^{-1}(T_n(a))$

where $S_n(a) = (S_{n,j',j}(a))_{j',j=0,\dots,J}$ and $T_n(a) = (T_{n,j}(a))_{j=0,\dots,J}$.

On the other hand, we introduce the vectors

$$T_n^\bullet(a) = (T_{n,j}^\bullet(a))_{j=0,\dots,J}, E_n = (E_{n,j})_{j=0,\dots,J}$$

and

$$E_n^\bullet(a) = (E_{n,j}^\bullet(a))_{j=0,\dots,J},$$

where

$$T_{n,j}^\bullet(a) = \frac{1}{nH_k\varphi_a(h_k)}\sum_{i=1}^n m_{ij}(a)\omega_i K_i(a,h_k)(G_i(b,e_\ell) - f(A_i,b),$$
$$T_{n,0}^\bullet(a) = \frac{1}{n\varphi_a(H_k)}\sum_{i=1}^n \omega_i K_i(a,H_k)(G_i(b,e_\ell) - f(A_i,b),$$

$$E_{n,j}(a) = \frac{1}{nH_k\varphi_a(H_k)}\sum_{i=1}^n m_{ij}(x)\omega_i K_i(x,h_k)Rf_{a,b}(A_i - a, A_i - a),$$

$$E_{n,0}(a) = \frac{1}{n\varphi_a(H_k)}\sum_{i=1}^n \omega_i K_i(a,H_k)Rf_{a,b}(A_i - a, A_i - a) \quad and$$
$$E_{n,j}^\bullet(a) = S_{n,j,0}(a).$$

We obtain by using regularity see (3) and hypothesis (W2)

$$f(A_i,b) = f(A_i,b) + \sum_j^J m_{ij} r f_{a,b}(\nu_j) + Rf_{a,b}(A_i - a, A_i - a) + O_{a.co.}(J^{-v}).$$

Therefore:

$$T_n^\bullet(a) = T_n(a) - (T_n(a) - T_n^\bullet(a))$$

$$T_n^\bullet(a) = S_n(a)\begin{pmatrix} \hat{\zeta}_{a,b} \\ H_k\hat{\eta}_1 \\ . \\ . \\ . \\ H_k\hat{\eta}_J \end{pmatrix} - S_n(a)\begin{pmatrix} \zeta_{a,b} \\ H_k\eta_1 \\ . \\ . \\ . \\ H_k\eta_J \end{pmatrix} + E_n(a) + O_{a.co.}(J^{-v})E_n^\bullet(a).$$

It ensues that

$$\check{f}(a,b) - f(a,b) = \hat{\zeta}_{a,b} - \zeta_{a,b}$$
$$= E'_1(S_n^{-1}(a)T_n^\bullet(a) - S_n^{-1}(a)E_n(a) - O_{a.co.}(J^{-v})S_n^{-1}(a)E_n^\bullet(a)).$$

The result of the Theorem 2 will therefore be the consequence of the following Lemmas.

Lemma 1. Under assumptions (W1),(W2) (W3), (W4) and (W6),we have

$$|S_{n,j',j}(a)| = O_{a.co.}(1)$$

*Proof.*We only prove the case where j and$j'$are distinct from 0. Other cases may be handled in the same way. Indeed, we demonstrate from the second part of the assumption (W1) that, there exists$(\lambda,\mu) \in (0,1)^2$, such that:

$$\varphi_a^{-1}(\frac{k}{\lambda_n}) \leq C_1\varphi_a^{-1}(\frac{\mu k}{n}). \tag{14}$$

Then we consider

$$D^+ = \varphi_a^{-1}(k/\lambda_n); D^- = \varphi_a^{-1}(\mu k/n).$$

We get this by utilizing the same arguments as in Burba et al.[13]

$$\sum_n \mathbb{P}(D_n \not\subset (D_n^+, D_n^-)).$$

Also, we can write

$$\tilde{S}_{n,j',j}(a) \leq S_{n,j',j}(a) \leq \bar{S}_{n,j',j}(a).$$

where

$$\bar{S}_{n,j',j}(a) = \frac{1}{n(D_n^-)^2\varphi_a(D_n^-)}\sum_{i=1}^n m_{ij'}(a)m_{ij}(a)\omega_i K_i(a,D_n^-).$$

$$\tilde{S}_{n,j',j}(a) = \frac{1}{n(D_n^+)^2\varphi_a(D_n^+)}\sum_{i=1}^n m_{ij'}(a)m_{ij}(a)\omega_i K_i(a,D_n^+).$$

Then, it is clear that

$$\bar{S}_{n,j',j}(a) = \frac{n(D_n^+)^2\varphi_a(D_n^+)}{n(D_n^-)^2\varphi_a(D_n^-)}\sum_{i=1}^n m_{ij'}(a)m_{ij}(a)\omega_i K_i(a,D_n^-),$$

and

$$\tilde{S}_{n,j',j}(a) = \frac{n(D_n^-)^2\varphi_a(D_n^+)}{n(D_n^-)^2\varphi_a(D_n^+)}\sum_{i=1}^n m_{ij'}(a)m_{ij}(a)\omega_i K_i(a,D_n^+).$$

It follows that equation (14) indicates that

$$\bar{S}_{n,j',j}(a) \geq C_2\frac{1}{n(D_n+)^2\varphi_a(D_n^-)}\sum_{i=1}^n m_{ij'}(a)m_{ij}(a)\omega_i K_i(a,D_n^-);$$

and

$$\tilde{S}_{n,j',j}(a) \leq C_1\frac{1}{n(D_n^-)^2\varphi_a(D_n^+)}\sum_{i=1}^n m_{ij'}(a)m_{ij}(a)\omega_i K_i(a,D_n^+).$$

Thus, all it remains to do is to give the limit of $\dot{S}_{n,j',j}(a)$, where

$$\dot{S}_{n,j',j}(a) = \frac{1}{nD_n^2\varphi_a(D_n)}\sum_{i=1}^n m_{ij'}(a)m_{ij}(a)\omega_i K_i(a,D_n).$$

To do that, we have to show that

$$\lim_{h\to 0}\mathbb{E}[\dot{S}_{n,j',j}(a)] = O(1).$$

and since the basis $(\nu_j)_j$ is orthonormal, for all $j \leq J$, we get:

$$|m_{1j}(x)| \leq (\nu_j) \parallel a - A_1 \parallel \leq (\nu_j) \parallel a - A_1 \parallel \quad (15)$$

For the expectation term we have

$$\mathbb{E}[\dot{S}_{n,j',j}(a)] = \frac{1}{nD_n^2\varphi_a(D_n)}\mathbb{E}[\sum_{i=1}^{n} m_{1k}(a)m_{ij}(a)\omega_1 K_1(a, D_n)].$$

Once again, we use the fact that the basis $(\nu_j)_j$ is orthonormal to get:

$$\mathbb{E}[m_{1k}(a)m_{ij}(a)\omega_1 K_1(a, D_n)] \leq \mathbb{E}[\parallel x - X_1 \parallel K_1(a, D_n)] \leq C_1 D_n^2 \varphi_a(D_n).$$

Lemma 2. Under the same conditions of Theorem 1, we have:

$$\mathbb{E}[T^{\bullet}_{n,j}(a)] = O(\frac{\ell}{n})^2 \quad \text{and} \quad \mathbb{E}[E_{n,j}(a)] = O(\varphi_a^{-1}(\frac{k}{n})^2)$$

Proof: Similarly to the proof of Lemma 1, we give only the proof for$j \neq 0$. The case$j = 0$can be obtained by a similar proof. For this purpose, we define

$$e^+(b) = (\ell/\lambda_n); \quad e^-(b) = (\mu \ \ell/n), e(b) = e^{\pm}(b).$$

Utilizing the similar ideas as in Kudraszow and Vieu [18], we can prove that

$$\textstyle\sum_n \mathbb{P}(D_n \not\subset (D_n^+, D_n^-)) < \infty,$$

and

$$\textstyle\sum_n \mathbb{P}(e \not\subset (e^+, e^-)) < \infty \quad (16)$$

Again we only give evidence of the asymptotic behavior of the term $T^{\bullet}_{n,j}(a)$ for $j \neq 0$. In fact, proof of the other terms , i.e., $T^{\bullet}_{n,0}(a)$ and $E_{n,j}(a)$, can be carried out in the same way. With respect to $T^{\bullet}_{n,j}(a)$, the result follows from the following statement $\mathbb{E}[T^{\bullet}_{n,j}(a)] = O(\frac{\ell}{n})^2$, and $\mathbb{E}[E_{n,j}(a)] = O(\varphi_a^{-1}(\frac{k}{n})^2)$.

$$T^{\bullet}_{n,j}(a) = \frac{1}{nD_n\varphi_a(D_n)}\textstyle\sum_{i=1}^{n} m_{ij}(a)\omega_i K_i(a, D_n)(G_i(b, e) - f(A_i, b).$$

$$E_{n,j}(a) = \frac{1}{nD_n\varphi_a(D_n)}\textstyle\sum_{i=1}^{n} m_{ij}(a)\omega_i K_i(x, D_n)Rf_{a,b}(A_i - a, A_i - a).$$

We start by $\mathbb{E}[E_{n,j}(a)]$. For this aim, we use the continuity property of the operator $Rf_{a,b}$ to write that

$$\parallel Rf_{a,b}(A_i - a, A_i - a) \parallel \leq C_1 \parallel A_1 - a \parallel^2. \quad (17)$$

Combining (15) and (21), we deduce that

$$\mathbb{E}[|m_{1j}(a)|\omega_1 K_1(x, D_n)Rf_{a,b}(A_i - a, A_i - a)] \leq \mathbb{E}[\parallel a - A_1 \parallel K_1(a, D_n)] \leq CD_n^3\varphi_a(D_n).$$

Accordingly

$$\mathbb{E}[E_{nj}(a)] \leq C_1 D_n^2.$$

Now we are back to

$$\mathbb{E}[T^{\bullet}_{n,j}(a)] = \frac{1}{nD_n\varphi_a(D_n)}\mathbb{E}[\sum_{i=1}^{n}|m_{1j}(a)|\omega_1 K_1(a,D_n)(G_1(b,e(b)) - f(A_1,b)].$$

Since the variables ω_i and B_i are independent given A_i, then under (W3), we obtain

$$\mathbb{E}(\omega_i G_1|A_1) = (\varrho(A_1) + o(1))\mathbb{E}(G_1|A_1) \leq \tfrac{C_1}{e}. \quad (18)$$

So,

$$\mathbb{E}[T^{\bullet}_{n,j}(a)] = \frac{1}{nD_n\varphi_a(D_n)}\mathbb{E}[\sum_{i=1}^{n} m_{1j}(a)\varrho(A_1)K_1(a,D_n)\mathbb{E}[(G_1(b,e)|/A_1 - f(A_1,b)]].$$

As for all $a_0 \in \mathcal{N}_a$, the conditional density $f(a_0,.)$ is of 2-times continuously differentiable with respect to b, we get

$$\mathbb{E}[(G_1(b,e)/A_1] - f(A_1,b) = O(e^2).$$

We combine this last with 21 and the fact that $E[K_1(a,D_n)] \leq C_1\varphi_a(D_n)$, we can write that

$$\mathbb{E}[T^{\bullet}_{n,j}(a)] = O(e^2).$$

We attain the proof by noting that a similar proof can be obtained for the case $j = 0$.

Lemma 3. Under the same conditions of Theorem 1, we have:

$$T^{\bullet}_{n,j}(a) - \mathbb{E}[T^{\bullet}_{n,j}(a)] = O_{a.co.}\left(\sqrt{\frac{\log n}{\ell k}}\right).$$

$$E_{n,j}(a) - \mathbb{E}[E_{n,j}(a)] = O(\varphi_a^{-1}(\tfrac{k}{n})^2) + O_{a.co.}\left(\sqrt{\frac{\log n}{k}}\right).$$

Proof: Again we only give evidence of the asymptotic behavior of the term$Tn,j^*(x)$for$j \neq 0$. In fact, proof of the other terms, so$Tn,0^{\bullet}(a)$and$En,j(a)$, can be carried out in the same way. With respect to$Tn,j^{\bullet}(a)$, we use (16) to get, almost completely rate, so the result follows from the following statement:

$$\ddot{T}^{*}_{n,j}(a) - \mathbb{E}[\ddot{T}^{\bullet}_{n,j}(a)] = O_{a.co.}(\sqrt{\frac{\log n}{n\varphi_a(D_n)}}). \quad (19)$$

with

$$\ddot{T}^{\bullet}_{n,j}(a) = \tfrac{1}{nD_n\varphi_a(D_n)}\textstyle\sum_{i=1}^{n} m_{ij}(a)\omega_i K_i(a,D_n)(G_i(b,e(b))) - f(A_i,b). \quad (20)$$

So ,

$$\hat{T}^{*}_{n,j}(a) - \mathbb{E}[\tilde{T}^{\bullet}_{n,j}(a)] \leq T^{*}_{n,j}(a) - \mathbb{E}[T^{\bullet}_{n,j}(a)] \leq \tilde{T}^{*}_{n,j}(a) - \mathbb{E}[\hat{T}^{\bullet}_{n,j}(a)].$$

$$\hat{T}^{\bullet}_{n,j}(a) = \frac{1}{nD_n^-\varphi_a(D_n^-)}\sum_{i=1}^{n} m_{ij}(a)\omega_i K_i(a, D_n^-)(G_i(b, e^-(b))) - f(A_i, b).$$

$$\tilde{T}^{*}_{n,j}(a) = \frac{1}{nD_n^+\varphi_a(D_n^+)}\sum_{i=1}^{n} m_{ij}(a)\omega_i K_i(a, D_n^+)(G_i(b, e^+(b))) - f(A_i, b).$$

We note that

$$\hat{T}^{\bullet}_{n,j}(a) = \frac{nD_n^+\varphi_a(D_n^+)}{nD_n^-\varphi_a(D_n^-)}\frac{1}{nD_n^+\varphi_a(D_n^+)}\sum_{i=1}^{n} m_{ij}(a)\omega_i K_i(a, D_n^+)(G_i(b, e^+(b))) - f(A_i, b).$$

$$\tilde{T}^{*}_{n,j}(a) = \frac{nD_n^-\varphi_a(D_n^-)}{nD_n^+\varphi_a(D_n^+)}\frac{1}{nD_n^-\varphi_a(D_n^-)}\sum_{i=1}^{n} m_{ij}(a)\omega_i K_i(a, D_n^-)(G_i(b, e^-(b))) - f(A_i, b).$$

and

$$\hat{T}^{\bullet}_{n,j}(a) - \mathbb{E}[\tilde{T}^{\bullet}_{n,j}(a)] = O_{a.co.}\left(\frac{\ell}{n}\right)^2.$$

This last equality is deduced from the result of Lemma 2. By using (14), we get:

$$\hat{T}^{\bullet}_{n,j}(a) - \mathbb{E}[\hat{T}^{\bullet}_{n,j}(a)] \leq C_1 \left|\frac{1}{nD_n^+\varphi_a(D_n^+)}(\sum_{i=1}^{n} m_{ij}(a)\omega_i K_i(a, D_n^+)(G_i(b, e^+(b))) - f(A_i, b))\right| - \left|\mathbb{E}\left[\sum_{i=1}^{n} m_{ij}(a)\omega_i K_i(a, D_n^+)(G_i(b, e^+(b))) - f(A_i, b)\right]\right|.$$

and

$$\tilde{T}^{\bullet}_{n,j}(a) - \mathbb{E}[\tilde{T}^{\bullet}_{n,j}(a)] \leq C_2 | \frac{1}{nD_n^-\varphi_a(D_n^-)}(\sum_{i=1}^{n} m_{ij}(a)\omega_i K_i(a, D_n^-)(G_i(b, e^-(b))) - f(A_i, b)))] - \left|\mathbb{E}\left[\sum_{i=1}^{n} m_{ij}(a)\omega_i K_i(a, D_n^-)(G_i(b, e^-(b))) - f(A_i, b)\right]\right|.$$

As define in (19) and (20), we apply the unlimited version of the exponential inequality of the Bernstein (see pages 234 specially the Corollary A8 in Ferraty and Vieu [3]). We state that this is based on an asymptotic assessment of the α^{th} order moments of the following random variables,

$$\Upsilon_{ij} = \frac{|m_{ij}(a)|\omega_i K_i(a, D_n)(G_i(b, e(b))) - f(A_i, b)}{D_n\varphi_a(D_n)} - \mathbb{E}[[\frac{|m_{ij}(a)|\omega_i K_i(a, D_n)(G_i(b, e(b))) - f(A_i, b)}{D_n\varphi_a(D_n)}]$$

Note that we obtain by the binomial expansion of the Newton

$$\mathbb{E}|(|m_{ij}(a)|\omega_i K_i(a,D_n)(G_i(b,e(b))) - f(A_i,b) - \mathbb{E}[|m_{ij}(a)|\omega_i K_i(a,D_n)(G_i(b,e(b))) - f(A_i,b)])^{\alpha}|$$
$$= \mathbb{E}|\sum_{\beta=0}^{\alpha} C_{\alpha}^{\beta}(|m_{ij}(a)|\omega_i K_i(a,D_n)(G_i(b,e(b))) - f(A_i,b))^{\beta} \times (\mathbb{E}[|m_{ij}(a)|\omega_i K_i(a,D_n)(G_i(b,e(b))) - f(A_i,b)])^{\alpha-\beta}(-1)^{\alpha-\beta})$$

$$\leq \sum_{\beta=0}^{\alpha} C_{\alpha}^{\beta}(\mathbb{E}||m_{ij}(a)|\omega_i K_i(a,D_n)(G_i(b,e(b))) - f(A_i,b)|^{\beta})|\mathbb{E}|[m_{ij}(a)|\omega_i K_i(a,D_n)(G_i(b,e(b))) - f(A_i,b)]^{\beta}|^{\alpha-\beta}.$$
$$\leq \sum_{\beta=0}^{\alpha} C_{\alpha}^{\beta}\mathbb{E}||m_{ij}(a)|\omega_i K_i(a,D_n)(G_i(b,e(b))) - f(A_i,b)^{\beta}||\mathbb{E}|[m_{ij}(a)|\omega_i K_i(a,D_n)(G_i(b,e(b))) - f(A_i,b)]^{\beta}|^{\alpha-\beta}.$$

where $C_{k,\alpha} = \alpha!/(k!\,(\alpha-k)!)$.

Since the variables ω and B are independent given A then, under assumption (W3), we utilize (18), and we obtain for all $\beta \leq \alpha$:

$$D_n^{-\alpha}\varphi_a^{-\alpha}(D_n)\sum_{\beta=0}^{\alpha} C_{\alpha}^{\beta}\mathbb{E}||m_{ij}(a)|\omega_i K_i(a,D_n)(G_i(b,e(b))) - f(A_i,b)^{\beta}| \times ||\mathbb{E}|[m_{ij}(a)|\omega_i K_i(a,D_n)(G_i(b,e(b))) - f(A_i,b)]^{\beta}|^{\alpha-\beta} \leq C_1(\varphi_a(D_n),e)^{-\alpha+1}.$$

So we obtain

$$\mathbb{E}|\Upsilon_{i,j}|^{\alpha} = O(\varphi_a(D_n),e)^{-\alpha+1}.$$

Consequently, it suffices to apply the Corollary A8 in Ferraty and Vieu [3], for $\zeta_n = (\varphi_a((D_n),e))^{-1/2}$, to conclude that

$$\sum_n \mathbb{P}\left\{|\ddot{T}^{\bullet}_{n,j}(a) - \mathbb{E}[\ddot{T}^{\bullet}_{n,j}(a)]| > \vartheta\sqrt{\frac{\log n}{n\varphi_a D_n e}}\right\} < \infty.$$

The following corollary is then deduced.

Corollary 1. Under the conditions of Theorem 1, we have

$$E'S_n^{-1}(a)T^{\bullet}_{n,j}(a) = O_{a.co.}(\sqrt{\frac{\log n}{k}})$$

and

$$E'S_n^{-1}(a)E_n(a) = O(\varphi_a^{-1}(\frac{k}{n})^2) + O_{a.co.}(\sqrt{\frac{\log n}{k}})$$

Theorem 3.

$$\breve{F}(a,b)-F(a,b)=O_{a.co}(J)^{-v}+\varphi_x^{-1}(\frac{k}{n})^2+\sqrt{(\frac{logn}{k})}\ as\ n\to\infty.$$

Proof:

$$S^1_{n,j',j}(a)=\frac{1}{nH_k^2\varphi_a(H_k)}\sum_{i=1}^{n}m_{ij'}(a)m_{ij}(a)\omega_iK_i(a,H_k).$$

$$S^1_{n,j',0}(a)=\frac{1}{nH_k\varphi_a(H_k)}\sum_{i=1}^{n}m_{ij'}(a)\omega_iK_i(a,H_k).$$

$$T^1_{n,j}(a)=\frac{1}{nH_k\varphi_a(H_k)}\sum_{i=1}^{n}m_{ij}(a)\omega_iK_i(a,H_k)B_i.$$

$$T^1_{n,0}(a)=\frac{1}{n\varphi_a(H_k)}\sum_{i=1}^{n}\omega_iK_i(a,H_k)B_i.$$

We should write, then $\begin{pmatrix}\hat{\zeta}_{a,b}\\ H_k\hat{\eta}_1\\ .\\ .\\ .\\ H_k\hat{\eta}_J\end{pmatrix}=(S^1_n(a))^{-1}(T^1_n(a)).$

where $S^1_n(a)=(S^1_{n,j',j}(a))_{j',j=0,\dots,J}$ and $T^1_n(a)=(T^1_{n,j}(a))_{j=0,\dots,J}$. On the other hand, we introduce the vectors $T^\star_n(a)=(T^\star_{n,j}(a))_{j=0,\dots,J}, E^1_n=(E^1_{n,j})_{j=0,\dots,J}$ and

$$E^\star_n(a)=(E^\star_{n,j}(a))_{j=0,\dots,J},$$

where

$$T^\star_{n,j}(a)=\frac{1}{nH_k\varphi_a(h_k)}\sum_{i=1}^{n}m_{ij}(a)\omega_iK_i(a,h_k)(B_i-F(A_i,b),$$

$$T^\star_{n,0}(a)=\frac{1}{n\varphi_a(H_k)}\sum_{i=1}^{n}\omega_iK_i(a,H_k)(B_i-F(A_i,b),$$

$$E^1_{n,j}(a)=\frac{1}{nH_k\varphi_a(H_k)}\sum_{i=1}^{n}m_{ij}(x)\omega_iK_i(x,h_k)\rho_{a,b}(A_i-a,A_i-a),$$

$$E^1_{n,0}(a)=\frac{1}{n\varphi_a(H_k)}\sum_{i=1}^{n}\omega_iK_i(a,H_k)\rho_{a,b}(A_i-a,A_i-a)andE^\star_{n,j}(a)=S_{n,j,0}(a).$$

We obtain by using regularity (see 4) and assumption (W2)

$$F(A_i,b)=\zeta_{a,b}+\sum_{j}^{J} m_{ij}\eta_{a,b}(\nu_j)+\varrho_{a,b}(A_i-a,A_i-a)+O_{a.co.}(J^{-\nu}).$$

Therefore:

$$T_n^{\star}(a)=T_n^1(a)-(T_n^1(a)-T_n^{\star}(a))$$

$$=S_n^1(a)\begin{pmatrix}\hat{\zeta}_{a,b}\\ H_k\hat{\eta}_1\\ .\\ .\\ .\\ H_k\hat{\eta}_J\end{pmatrix}-S_n^1(a)\begin{pmatrix}\zeta_{a,b}\\ H_k\eta_1\\ .\\ .\\ .\\ H_k\eta_J\end{pmatrix}+E_n^1(a)+O_{a.co.}(J^{-\nu})E_n^{\star}(a).$$

It implies that

$$\breve{F}(a,b)-F(a,b)$$
$$=(E^1)'_1((S_n^1)^{-1}(a)T_n^{\star}(a)-(S_n^1)^{-1}(a)E_n^1(a)-O_{a.co.}(J^{-\nu})(S_n^1)^{-1}(a)E_n^{\star}(a)).$$

Consequently, the outcome of the Theorem 3 will be the result of the corresponding Lemmas for which their proof is given in the concluding part.

Lemma 4. Under assumptions (W1),(W2) (W3), (W4) and (W6),we have

$$|S_{n,j',j}^1(a)|=O_{a.co.}(1)$$

Proof: The demonstration of this Lemma is the same as Lemma 1.

Lemma 5. Under the same conditions of Theorem 3, we have:

$$\mathbb{E}[T_{n,j}^{\star}(a)]=O(\frac{\ell}{n})^2 \quad \text{and} \quad \mathbb{E}[E_{n,j}(a)]=O(\varphi_a^{-1}(\frac{k}{n})^2)$$

Proof: The proof would be based on the following quantities of behaviour, for all $j\geq 0$

$$T_{n,j}^{\bullet}(a)=\frac{1}{nD_n\varphi_a(D_n)}\sum_{i=1}^{n} m_{ij}(a)\omega_i K_i(a,D_n)(B_i-F(A_i,b).$$

$$E_{n,j}^1(a)=\frac{1}{nD_n\varphi_a(D_n)}\sum_{i=1}^{n} m_{ij}(a)\omega_i K_i(x,D_n)\rho_{a,b}(A_i-a,A_i-a).$$

We start by $\mathbb{E}[E_{n,j}^1(a)]$. For this aim, we use the continuity property of the operator $\rho_{a,b}$ to write that

$$\parallel \rho_{a,b}(A_i-a,A_i-a)\parallel\leq C_1\parallel A_1-a\parallel^2. \tag{21}$$

Combining (15) and (21), we deduce that

$$\mathbb{E}\big[\big|m_{1j}(a)\big|\omega_1K_1(x,D_n)\rho_{a,b}(A_i-a,A_i-a)\big]\leq$$

$$\mathbb{E}[\parallel a - A_1 \parallel K_1(a, D_n)] \leq C D_n^3 \varphi_a(D_n).$$

Accordingly

$$\mathbb{E}[E_{nj}^1(a)] \leq C_1 D_n^2.$$

Now we are back to

$$\mathbb{E}[T_{n,j}^{\bullet}(a)] = \frac{1}{nD_n\varphi_a(D_n)} \mathbb{E}[\sum_{i=1}^{n} |m_{1j}(a)| \omega_1 K_1(a, D_n)(B_1) - F(A_1, b)].$$

Since the variables ω_i and B_i are independent given A_i, then under (W3), so by conditioning by A_1

$$\mathbb{E}[T_{n,j}^{\star}(a)] = \frac{1}{D_n\varphi_a(D_n)} \mathbb{E}[|m_{1j}(a)| \varrho(A_1) K_1(a, D_n)(\mathbb{E}[B_1/A_1] - F(A_1, b))],$$

So we can write that

$$\mathbb{E}[T_{n,j}^{\bullet}(a)] = O.$$

We attain the proof by noting that a similar proof can be obtained for the case $j = 0$.

Lemma 6. Under the same conditions of Theorem 3, we have:

$$T_{n,j}^{\star}(a) - \mathbb{E}[T_{n,j}^{\star}(a)] = O_{a.co.}\left(\sqrt{\frac{\log n}{k}}\right).$$

$$E_{n,j}^1(a) - \mathbb{E}[E_{n,j}^1(a)] = O(\varphi_a^{-1}(\frac{k}{n})^2) + O_{a.co.}\left(\sqrt{\frac{\log n}{k}}\right).$$

Proof: The same manner as Lemma 3, just finally it suffices to apply the Corollary A8 in Ferraty and Vieu [3], for$\zeta_n = (\varphi_a((D_n)))^{-1/2}$, to conclude that

$$\sum_{\boldsymbol{n}} \mathbb{P}\left\{|\ddot{\boldsymbol{T}}_{\boldsymbol{n,j}}^{\bullet}(\boldsymbol{a}) - \mathbb{E}[\ddot{\boldsymbol{T}}_{\boldsymbol{n,j}}^{\bullet}(\boldsymbol{a})]| > \boldsymbol{\vartheta}\sqrt{\frac{\mathbf{log}\boldsymbol{n}}{\boldsymbol{n\varphi_a D_n}}}\right\} < \infty$$

We deduce then the following corollary.

Corollary 2. Under the conditions of Theorem1, we have

$$(E^1)'(S_n^1)^{-1}(a)T_{n,j}^{\star}(a) = O_{a.co.}(\sqrt{\frac{\log n}{k}})$$

and

$$(E^1)'(S_n^1)^{-1}(a)E_n^1(a) = O(\varphi_a^{-1}(\frac{k}{n})^2) + O_{a.co.}(\sqrt{\frac{\log n}{k}})$$

3.2. DISCUSSION AND COMMENTS

In practice, on the characteristics of the constructed estimator This paper's primary objective is to create a non-parametric hazard estimator by mixing the concepts of local linear estimation when the data is not completed. The new plan enables to prosper from advantages offered both approaches. There are no results on local linear estimation data with MAR of the hazard function in N.F.D.A, as far as we know. The current work can be viewed as a first research study in that direction.

The results obtained are established under general conditions that allow the incomplete case of functional data to be covered, mainly when the response variables contain some M.A.R observations. In practice, the feasibility and efficacy of local linear estimation with M.A.R estimator was tested on simulated data.

On the effect of the calculation of conditional hazard function in incomplete function analyzing results. Grow within the context of other statistical approaches throughout reality, an analysis of missing usable data is critical. It is inspired by the impossibility of analyzing all the usable data absolutely. Furthermore, the high volume of data observed and the continuous exploration of the instruments used the problem of missing findings makes the data collection very popular in Nonparametric Functional Data Analysis (N.F.D.A). In this section we go back to one of the most important functional applications regression when usable data is M.A.R, the mean estimate of failed observation. Ferraty et al.[16] has looked at this critical application using regression kernel estimator. In addition, we are debating the use of the alternative estimator L.L.E hazard with M.A.R data. Clearly, for specific fields such as medicine, biology, imaging, economics, social sciences, ..., factor B for some subjects is difficult to examine, but at the beginning, we can observe the causal variables A with function space $\mathcal{F}$ values. Of course using the L.L.E hazard with M.A.R data estimator increase the estimator efficiency for conditional hazard. The quantification of this benefit is still an important perspective of this work. A comparative analysis study between the two estimators the first one is the classical hazard and the second one is our estimator hazard with LLE-MAR data, it will be the subject of section of application.

4. SIMULATION STUDY

We test the performance of our asymptotic normality results over finite sample data in this section. More specifically, our main objective is to demonstrate the simple implementation of the hazard conditional function and to analyze the effect of this asymptotic property with incomplete data. For this reason we produce functional observations by considering the following nonparametric functional model:

$$B_i = r(A_i) + \epsilon_i fori = 1, \ldots, n \tag{13}$$

In which the ϵi's are distributed in accordance with normal $N(0, .5)$ distribution. Then to prove the effectiveness of our study, we aim to show the performance of our approach regarding the percentage of the missing observations. It is precisely the behavior of L.L.E with M.A.R estimator defined in (2) comparing with the classical estimator defined in Ferraty et al.[3] defined in (1).

We generate the curves, $\mathcal{A}_i(t)$ for $t \in [0,1]$ and $i = 1, \ldots, n$ according to the following expression

$$\mathcal{A}_i(t) = 3z_i \sin(2\pi t) + \theta_i t \text{ where} z_i \sim \mathcal{N}(0{,}0.5) \text{ and } \theta_i \sim \mathcal{N}(0{,}1).$$

On the same grid that is created from $m = 100$ equi-spaced points in $[0,1]$ all the curves are discretized. Fig. 1 shows an example of such curves.

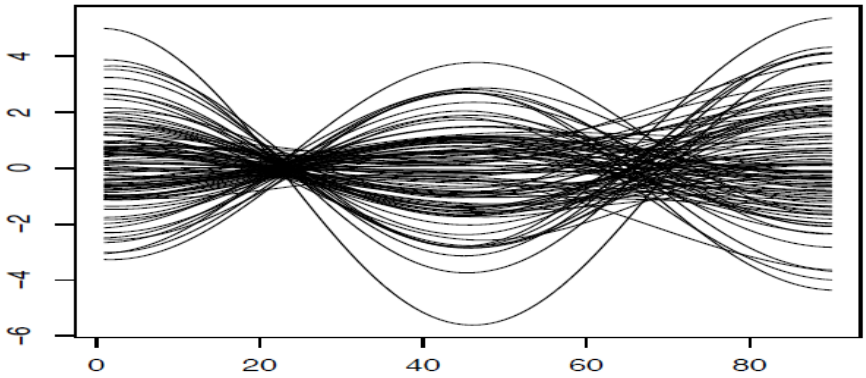

Figure 1. A sample of explanatory curves.

In order to introduce different levels of missing observations we generate a Bernoulli random variable ω with respect to the following different examples of $\mathcal{R}(a)$.

$$\begin{cases} \mathcal{R}(a) = 1 - expt(\int_0^1 a^2(t)dt), \ \ Case\ of\ weak\ missing \\ \mathcal{R}(a) = (\sin(\pi * \int_0^1 a^2(t)dt))^2, \ \ Case\ of\ mean\ missing \\ \mathcal{R}(a) = |\sin(\pi * \int_0^1 a^2(t)dt)|, Case\ of\ strong\ missing \end{cases}$$

where $expt(v) = e^v/(1 + ev))$ is given. We point out that in the poor case the proportion of missing data is close to 11%, and in the mean missing case is close to 33%, while this is 59% for the strong case. That percentage is the mean of $n^{-1} \sum i = 1^n \omega_i$ after 100 repeats.

We generate $n = 100$ observations, and take a quadratic kernel which is supported within (0,1), we use the L^2 metric and the fourier basis with J. Finally, the computational results are plotted in Figs. 2-4.

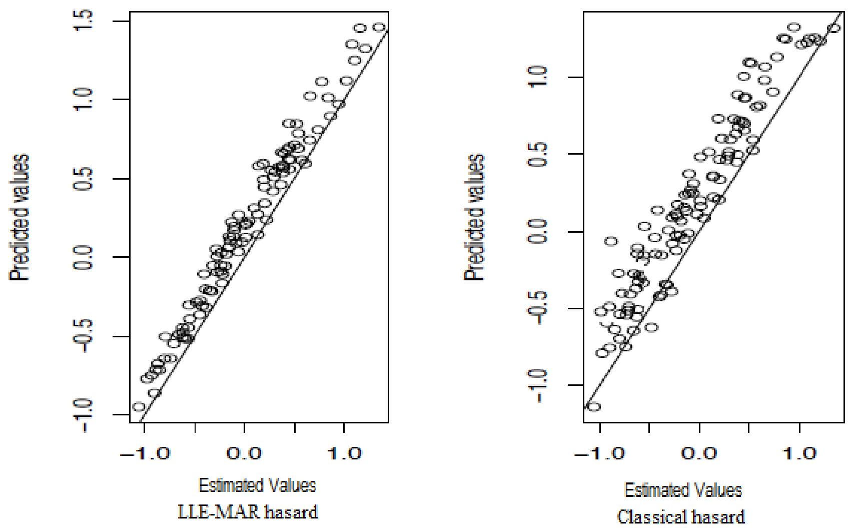

Figure 2.The illustration in the case of weak missing.

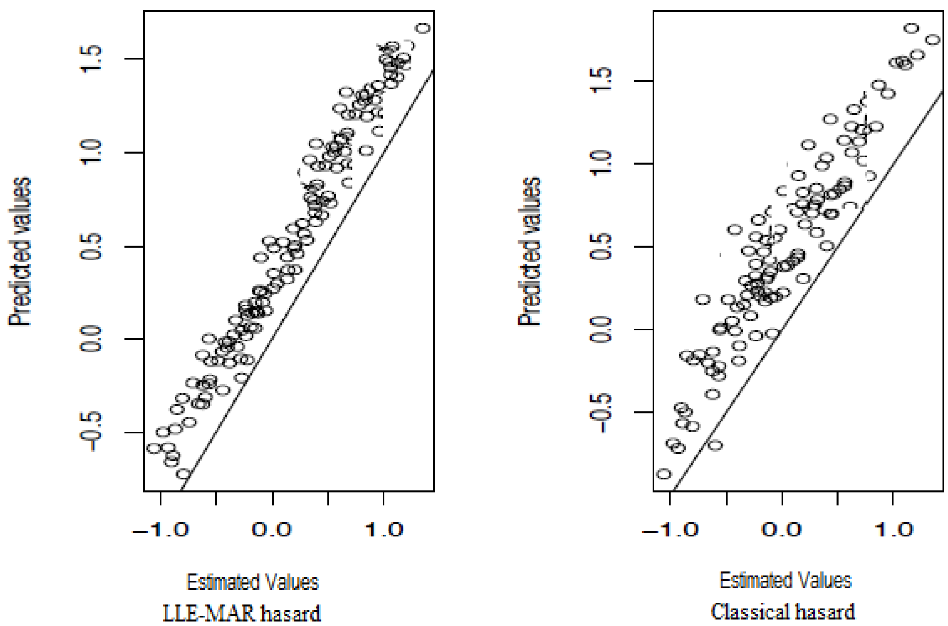

Figure 3. The illustration in the case of mean missing.

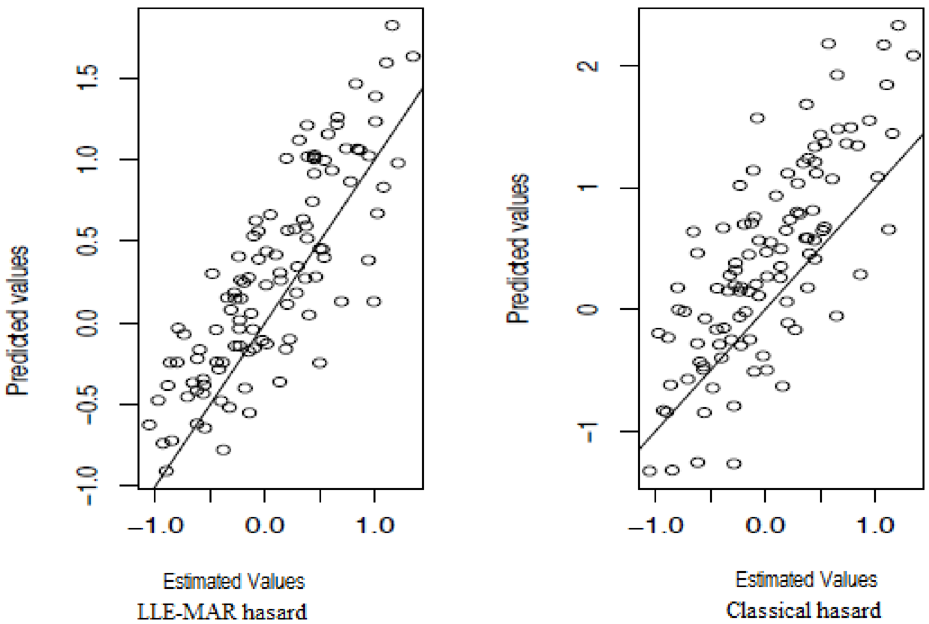

Figure 4. The illustration in the case of strong missing.

So, the theoretical conditional hazard $\breve{h}(a,b)$ can be easily determined in Table 1, we give the following Mean.Squared.Errors (M.S.E.)

$$MSE(Classic) = \frac{1}{n}\sum_{i=1}^{n}(h(\mathcal{A}_i) - \hat{h}^{-i}(\mathcal{A}_i))^2$$

and

$$MSE(LLE - MAR) = \frac{1}{n}\sum_{i=1}^{n}(\breve{h}(\mathcal{A}_i) - \breve{\hat{h}}^{-i}(\mathcal{A}_i))^2$$

Cases of the three missing rates for the two estimators.

Table 1. Mean.squared.errors for the three missing rates of(classic and L.L.E-M.A.R)estimators.

MissingRates	$\mathbf{M.S.E.\,(L.L.E.-M.A.R.)}$	$\mathbf{M.S.E.\,(Classic)}$
Weak	0.20	0.29
Mean	0.66	0.87
Strong	1.79	1.96

5. CONCLUSIONS

The uniform k.N.N. reliability approach is a smoothing alternative that allows for the development of an adaptive estimator for a variety of statistical problems, including bandwidth choice.In our situation, the assumptionthat the bandwidth parameter in the k.N.N. method is a random variable adds to the complexity ofthis problem.

The key innovation of this approach is to estimate the hazard function by mixing two essential statistical techniques: the k.N.N. procedures and local lineaire estimation with MAR. This strategy allowed for the development of a new estimator that combinesthe benefits of both methods. To summarize, the behavior of the developed estimator is unaffected by the number of outliers inthe data collection. In comparison to the classical kernel method, the mixture of the k.N.N. algorithmand the robust method allows for a reduction in the impact of outliers in results.

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