ORIGINAL PAPER

NUMERICAL ACCURACY OF FREDHOLM LINEAR INTEGRO-DIFFERENTIAL EQUATIONS BY USING ADOMIAN DECOMPOSITION METHOD, MODIFIED ADOMIAN DECOMPOSITION METHOD AND VARIATIONAL ITERATION METHOD

ASIYA ANSARI[1], NAJMUDDIN AHMAD[1]

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Abstract. *In this article, we present as a comparative result of Adomian Decomposition Method (ADM), Modified Adomian Decomposition Method (MADM) and Variational Iteration Method (VIM). These methods used for developed to find the analytical approximate solution of linear Fredholm integro-differential equations. The main purpose of this paper was to show a better method for Numerical equations which does not give easily analytical solution. So, in this paper, we find approximate solutions of linear Fredholm integro-differential equations. We explain the convergence of ADM, MADM and VIM by using examples of a deterministic model by graphs and tables. All the calculations performed by the help of MATLAB (2018) Version 9.4.*

Keywords: *Fredholm integro-differential equations; Adomian Decomposition Method; Modified Adomian Decomposition Method and Variational Iteration Method.*

1. INTRODUCTION

In 1980, George Adomian introduced a new method to solve linear and nonlinear integro-differential equations. This method named as Adomian Decomposition Method (ADM) and this term arose from many investigations [1-3]. The Integro-differential equation has gained considerable popularity and importance due to their numerous applications in many fields of science and engineering including physics, biotechnology, electrodynamics, chemical technology, population dynamics and so on [1].

Recently, the authors used several methods to solve both linear and nonlinear Volterra and Fredholm integro-differential equations numerically or analytically [4, 5]. In this section, we provide a Adomian Decomposition Method, Modified Adomian Decomposition Method and Variational Iteration Method for linear Fredholm integro-differential equations [6, 7] of the form

$$u^n(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, u(0) = u_0 \tag{1.1}$$

where u_0 is real, constant and u^n indicate the n^{th} derivative of unknown function $u(x)$ that will be determined, occurs inside and outside the integral sign. The kernel $K(x,t)$ of the integral function and the function $f(x)$ are given real-valued functions, λ is a parameter and $u(t)$ is a linear function of t. A variety of powerful methods for solving linear and nonlinear integro-differential equations has been presented, such as the Adomian Decomposition Method [8],

[1] Integral University, Mathematics and Statistics, 226026 Lucknow, India.
E-mail: asiyaansari.du.786@gmail.com; najmuddinahmad33@gmail.com.

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Modified Adomian Decomposition Method, Variational Iteration Method, Homotopy Analysis Method [9-11], Homotopy Perturbation Method [12] and the Exp-function Method 13]. An intriguing feature of these methods was the ability to obtain an analytical solution in many cases. The Adomian Decomposition Method, Modified Adomian Decomposition Method, and Variational Iteration Method [14-20] have all been suggested by numerous researches as effective ways to solve linear and nonlinear integro-differential equations such as Fredholm and Volterra. All of these methods were used to approximate the solutions of the linear and nonlinear Fredholm and Volterra integro-differential equations. We obtain analytical solutions for the linear Fredholm integro-differential equations [21] using the ADM, MADM, and VIM. The methods were also evaluated in relation to the exact solution, if exists. The Adomian Decomposition Method, Modified Adomian Decomposition Method, and Variational Iteration Method are briefly discussed in Section 2, as well as exact solutions for selected examples.

In Section 3, the Adomian Decomposition Method (ADM), Modified Adomian Decomposition Method (MADM), and Variational Iteration Method (VIM) are used to solve the Fredholm linear integro-differential equations numerically. The results show how efficient and quick these approaches are to compute. Furthermore, some authors concluded that these methods could be used to find an exact solution in some cases. We examine comparison graphs in Section 4 before comparing the methodologies in Section 5. The results and conclusion are briefly discussed in Section 6, which maintaining confidentiality this paper.

2. TECHNIQUES AND METHODOLOGY

In this section, we describe some powerful methods have focused on developing more advanced and effective methods for both linear and nonlinear equations Fredholm integro-differential and also Volterra integro-differential such as the Adomian Decomposition Method, Modified Adomian Decomposition Method and Variational Iteration Method. We will describe these methods in this section.

2.1. ADOMIAN DECOMPOSITION METHOD (ADM)

To illustrate, the following general differential equation is considered:

$$Lu + Ru + Nu = f(x), \tag{2.1}$$

where $u(x)$ is the unknown function, linear terms are decomposed into $L + R$, and nonlinear terms are denoted by Nu. Because L is easily invertible, R is the nonlinear operator's remainder, and $f(x)$ is a non-homogeneous term. From Eq. (2.1)

$$Lu = f(x) - Ru - Nu, \tag{2.2}$$

Using the provided conditions and multiplying L^{-1} on both sides of Eq. (2.2), we obtain

$$u(x) = g(x) - L^{-1}(Ru) - L^{-1}(Nu) \tag{2.3}$$

where $L^{-1}\big(f(x)\big) = g(x)$.

The Adomian decomposition method is to decompose the unknown function $u(x)$ of any equation in a sum of an infinite number of components defined by the decomposition series. The Adomian approach is used to define the series solution $u(x)$ as follows:

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{2.4}$$

or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \ldots \ldots \tag{2.5}$$

where the components $u_n(x), n \geq 0$ are to be determined in a recursive manner. The method of decomposition concerns itself with finding the components. We substitute the linear Fredholm integro-differential equation to get

$$\sum_{n=0}^{\infty} u_n(x) = \mathrm{f(x)} + \lambda \int_a^b K(x,t) \sum_{n=0}^{\infty} u_n(t)\, dt \tag{2.6}$$

or equivalently (from Eq. (2.5))

$$\begin{aligned} &u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots \\ &\qquad = f(x) + \lambda \int_a^b K(x,t).[u_0(t) + u_1(t) + u_2(t) + \cdots]dt \end{aligned} \tag{2.7}$$

The $zeroth$ component $u_0(x)$ is identified by all terms which are not included with the integral sign. Therefore, the components $u_i(x), i \geq 0$ of the unknown function $u(x)$ are entirely determined by defining the recurrence relation:

Let

$$u_0(x) = f(x)$$

$$u_{n+1}(x) = \lambda \int_a^b K(x,t).u_n(t)dt, n \geq 0. \tag{2.8}$$

That is,

$$\begin{aligned} u_1^{(n_1)}(x) &= f_1(x) + \int_{c_1}^{d_1} k_1\left(x, t, u_1(t), u_2(t), \ldots, u_p(t)\right) dt, \\ u_2^{(n_2)}(x) &= f_2(x) + \int_{c_2}^{d_2} k_2\left(x, t, u_1(t), u_2(t), \ldots, u_p(t)\right) dt, \\ &\vdots \\ u_p^{(n_p)}(x) &= f_p(x) + \int_{c_p}^{d_p} k_p\left(x, t, u_1(t), u_2(t), \ldots, u_p(t)\right) dt. \end{aligned}$$

With initial conditions:

$$u_i^{(j)}(x_0) = u_{ij}, i = 1, \ldots, p, j = 0,1, \ldots, n_i - 1. \tag{2.9}$$

Let us write Adomian series of $u_i(x)$ as the following

$$\begin{cases} u_1(x) = \sum_{j=0}^{\infty} u_{1j}(x), \\ u_2(x) = \sum_{j=0}^{\infty} u_{2j}(x), \\ \vdots \\ u_p(x) = \sum_{j=0}^{\infty} u_{pj}(x). \end{cases} \tag{2.10}$$

Or generally we have

$$u_i(x) = \sum_{j=0}^{\infty} u_{ij}(x),\ i = 1,2, \dots\ and\ j = 0,1, \dots$$

2.2. MODIFIED ADOMIAN DECOMPOSITION METHOD (MADM)

We present the Modified Adomian Decomposition Method [22], which can be used to solve both linear and nonlinear Fredholm integro-differential equations (Eq. 1.1). Wazwaz [23, 24] created a modified version of the Adomian Decomposition Method to analyse the system of Fredholm integro-differential equations. By computing only two terms from the decomposition series, the Modified Adomian Decomposition Method [23, 24] provides the exact solution. This has the positive attraction of being computationally efficient.

Now, series solution $u(x)$ (Eq. 2.4) from the Adomian Decomposition Method,

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{2.11}$$

the following relations make it simple to iteratively find the components

$$\begin{gathered} u_0, u_1, u_2, \dots u_0(x) = g(x) \\ u_{n+1}(x) = -L^{-1}(Ru_n) - L^{-1}(Nu_n), n \geq 0. \end{gathered} \tag{2.12}$$

The decomposition method usually identifies the $zeroth$ components $u_0(x)$ as the function $g(x)$. But according to the modified decomposition approach, the function $g(x)$ stated in Eq. (2.3) can be divided into two components,

$$g_0(x) \text{ and } g_1(x), \text{ i.e., } g(x) = g_0(x) + g_1(x). \tag{2.13}$$

The proper selection of $g_0(x)$ and $g_1(x)$ in the above equation is critical and is primarily determined by the trail basis. As a result, the recursive relations for the modified decomposition method are written as follows:

$$\begin{gathered} u_0(x) = g_0(x), \\ u_1(x) = g_1(x) - L^{-1}(Ru_0) - L^{-1}(Nu_0), \\ u_{n+1}(x) = -L^{-1}(Ru_n) - L^{-1}(Nu_n), n \geq 1. \end{gathered} \tag{2.14}$$

The above equation (2.14) demonstrates dependability in that it accelerates solution convergence and reduces the number of computations when compared to the Adomian Decomposition Method.

2.3. VARIATIONAL ITERATION METHOD (VIM):

This method is used for solving a large class of both linear and nonlinear equations Fredholm integro-differential as well as linear and nonlinear Volterra integro-differential equations with rapidly converging approximations to exact solutions. In this paper, we consider solving only the Linear Fredholm integro-differential equations. For a complete review of the Variational Iteration method, see [25, 26]. The initial approximation may be freely chosen with possible unknowns that may be determined by the initial conditions. To illustrate, the following general differential equation is considered:

$$LG(x) + NG(x) = f(x) \tag{2.15}$$

where L is a linear operator, N is a nonlinear operator and $f(x)$ is a non-homogeneous term. The G_n terms are calculated by a correction functional in the following manner:

$$G_{n+1}(x) = Z_n(x) + \int_0^x \lambda(\tau)\big(LG_n(\tau) + N\tilde{y}(\tau) - f(\tau)\big)d\tau \tag{2.16}$$

where λ is a general Lagrange multiplier, noticing that within this method λ can be a constant or a function and $\widetilde{G_n}$ is a restricted value means that it behaves as a constant, hence $\delta\widetilde{G_n} = 0$, δ is the variational derivative. The successive approximation $G_n(x), n \geq 0$ of the solution $G_n(x)$ will be easily obtained by using the Lagrange multiplier and using any selective function G_0. The $zeroth$ approximation G_0 may be selected using any function that just satisfies at least the initial and boundary conditions, with λ determined, several approximations $G_n(x), n \geq 0$ follow immediately.

For the IVB approximation solution (1.1), according to the VIM, the iteration formula (2.15) can be written as follows:

$$G_{n+1}(x) = G_n(x) + L^{-1}\left[\lambda(x)\left[\sum_{j=0}^{k} \mathcal{E}_j(x)G_n^j(x) - f(x) - \gamma\int_a^b K(x,t)U\big(G_n(t)\big)dt\right]\right],$$

where L^{-1} is the multiple integration operator, and it is defined as follows:

$$L^{-1}(.) = \int_a^b \int_a^b \cdots \int_a^b (.)dxdx \dots\ dx\ (k - times).$$

We continue as follows to find the optimal $\lambda(x)$:

$$\begin{aligned}\delta G_{n+1}(x) &= \delta G_n(x) \\ &\quad + \delta L^{-1}\left[\lambda(x)\left[\sum_{j=0}^{k} \mathcal{E}_j(x)G_n^j(x) - f(x) - \gamma\int_a^b K(x,t)U\big(G_n(t)\big)dt\right]\right] \\ &= \delta G_n(x) + \lambda(x)\delta G_n(x) - L^{-1}[\delta G_n(x)\lambda'(x)].\end{aligned} \tag{2.17}$$

The stationary conditions can be calculated using equation (2.17) as follows:

$$\lambda'(x) = 0, and\ 1 + \lambda(x)_{x=t} = 0.$$

Consequently, the Lagrange multipliers can be identified as $\lambda(x) = -1$ and by putting this value into the equation (2.17), the iteration formula shown below is obtained:

$$G_0(x) = L^{-1}\left[\frac{f(x)}{\mathcal{E}_k(x)}\right] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r,$$

$$G_{n+1}(x) = G_n(x) + L^{-1}\left[\sum_{j=0}^{k} \mathcal{E}_j(x) G_n^j(x) - f(x) - \gamma \int_a^b K(x,t) U\big(G_n(t)\big) dt\right], n \geq 0.$$

The expression $\sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r$ is obtained from the initial conditions, $\mathcal{E}_k(x) \neq 0$. We can use the relation (2.18) to recursively figure out the components of $G_n(x)$ for $n \geq 0$. Consequently, the approximate solution can be obtained by using

$$G(x) = \lim_{n\to\infty} G_n(x). \tag{2.18}$$

3. NUMERICAL RESULTS

In this section, we demonstrated the semi-analytical techniques based on Adomian Decomposition Method (ADM), Modified Adomian Decomposition Method (MADM) and Variational Iteration Method to solve some numerical examples of linear Fredholm integro-differential equations. We consider the comparison of these methods using tables and graphs in three examples for testing the accuracy of linear Fredholm integro-differential equations with the exact solution. The errors have also been considered as well.

Example 3.1. Consider the following linear Fredholm integro-differential equation: $G'(x) = e^x - x + xe^x + \int_0^1 xG(t)dt$, with the initial condition $G(0) = 0$, and the exact solution is $G(x) = xe^x$.

Using inverse operation, we get

$$G(x) = xe^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 G(t)dt \tag{3.1}$$

The Adomian Decomposition Method: From eq. (2.4) $G(x) = \sum_{n\to 0}^{\infty} G_n(x)$

$$\sum_{n\to 0}^{\infty} G_n(x) = xe^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 \sum_{n\to 0}^{\infty} G_n(t)\, dt$$

$$f(x) = xe^x - \frac{x^2}{2}$$

we know that $G_0(x) = f(x)$.

So $G_0(x) = xe^x - \frac{x^2}{2}$

$$G_{n+1}(x) = \frac{x^2}{2}\int_0^1 \sum_{n\to 0}^{\infty} G_n(t)\, dt$$

$$G_1(x) = \frac{5}{12}x^2$$

$$G_2(x) = \frac{5}{72}x^2$$

$$G_3(x) = \frac{5}{432}x^2$$
$$\vdots$$
$$and\ so\ on.$$

Or equivalently $G(x) = \sum_{n\to 0}^{\infty} G_n(x) = G_0(x) + G_1(x) + G_2(x) + G_3(x) + \cdots$

$$G(x) \approx xe^x + \frac{1}{2}x^2 + \frac{5}{12}x^2 + \frac{5}{72}x^2 + \frac{5}{432}x^2$$
$$\approx xe^x + \frac{431}{432}x^2$$

The Modified Adomian Decomposition Method:

From Eq. (3.1) where $g(x) = xe^x - \frac{1}{2}x^2$, splitting $g(x)$ into two parts i.e., $g_0(x) = xe^x$, $g_1(x) = -\frac{1}{2}x^2$ and use recursive relations to obtain

$$G_0(x) = g_0(x) = xe^x$$
$$G_1(x) = -\frac{x^2}{2} + \frac{x^2}{2}\int_0^1 te^t dt = 0$$
$$G_2(x) = 0$$
$$\vdots$$
$$and\ so\ on.$$

Additionally, we take $G_{n+1}(x) = 0, n \geq 1$. Thus, using these into series form (2.10), we have the solution with the closed forms

$$G(x) = xe^x.$$

which corresponds to the exact solution.

The Variational Iteration Method:

The correction functional for equation (3.1) is given by (from eq. 2.16) $G_{n+1}(x) = Z_n(x) + \int_0^x \lambda(\tau)\big(LG_n(\tau) + N\tilde{y}(\tau) - f(\tau)\big)d\tau$.

Let $G_0(x) = G(0) = 0$.

$$G_1(x) = G_0(x) - \int_0^x [G_0'(t) - x + e^x + xe^x - \int_0^1 [G_0(r)dr]dt] = \frac{x^2}{2} - xe^x$$
$$G_2(x) = x^2 - 2xe^x$$
$$G_3(x) = \frac{3}{2}x^2 - 3xe^x$$
$$G_4(x) = 2x^2 - 4xe^x$$
$$G_5(x) = \frac{5}{2}x^2 - 5xe^x$$
$$G_6(x) = 3x^2 - 3xe^x$$
$$\vdots$$
$$and\ so\ on.$$

Consequently, the approximate solution can be obtained by using (2.18)

$$G(x) = \lim_{n\to\infty} G_n(x).$$

So,

$$G(x) = 1 + \frac{x^2}{2} - xe^x + x^2 - 2xe^x + \frac{3}{2}x^2 - 3xe^x + 2x^2 - 4xe^x + \frac{5}{2}x^2 - 5xe^x + 3x^2 - 6xe^x + \cdots$$

Therefore, $G(x) = 1 - 21xe^x + \frac{21}{2}x^2 - \cdots$. This is the approximate solution of (3.1) using Variational Iteration Method.

Table 1. Exact and Approximate solutions by ADM, MADM and VIM for Example 3.1.

x	Exact	ADM	MADM	VIM	Er-ADM	Er-MADM	Er-VIM
0.1	0.1105170918	0.1103782	0.1105170800	0.1105170888	1.39×10^{-4}	1.2×10^{-8}	3.00×10^{-9}
0.2	0.2442805516	0.2437249	0.2442805300	0.2442805397	5.56×10^{-4}	2.2×10^{-8}	1.19×10^{-8}
0.3	0.4049576424	0.4037076	0.4049576100	0.4049576156	1.25×10^{-3}	2.68×10^{-8}	2.68×10^{-8}
0.4	0.5967298792	0.5945076	0.5967298295	0.5967298316	2.22×10^{-3}	3.2×10^{-8}	4.76×10^{-8}
0.5	0.8243606355	0.8208884	0.8243605050	0.8243605611	3.48×10^{-3}	1.31×10^{-7}	7.44×10^{-8}
0.6	1.0932712800	1.0882712	1.0932711503	1.0932711730	5.00×10^{-3}	1.30×10^{-7}	1.07×10^{-7}
0.7	1.4096268950	1.4028213	1.4096266490	1.4096267490	6.81×10^{-3}	2.46×10^{-7}	1.46×10^{-7}
0.8	1.7804327420	1.7715438	1.7804325500	1.7804325510	8.89×10^{-3}	1.92×10^{-7}	1.91×10^{-7}

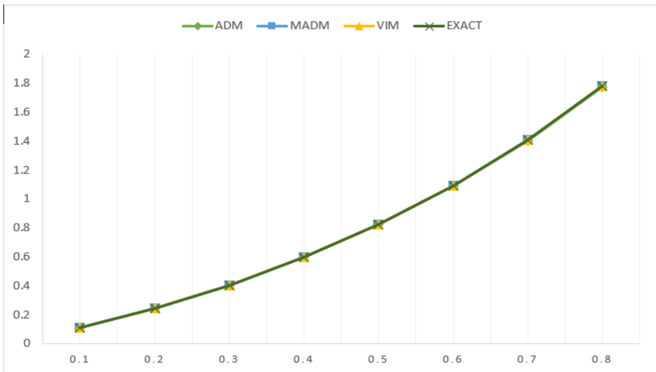

Figure 1. Exact and Approximate solutions by ADM, MADM and VIM for Example 3.1.

Example 3.2. Consider the following linear Fredholm integro-differential equation:

$$\begin{cases} G''(x) = \frac{3x}{10} + 6 - \int_0^1 2xt\big(G(t) - 3H(t)\big)dt, \\ H''(x) = 15x + \frac{4}{5} - \int_0^1 3(2x + t^2)\big(G(t) - 2H(t)\big)dt, \end{cases}$$

with the initial conditions $G(0) = 1, H(0) = -1, G'(0) = 0$ *and* $H'(0) = 2$ and the exact solutions are $G(x) = 3x^2 + 1$ and $H(x) = x^3 + 2x - 1$.

Using inverse operation, we get

$$\begin{cases} G(x) = 1 + \frac{1}{20}x^3 + 3x^2 - \frac{1}{3}x^3 \int_0^1 t\big(G(t) - 3H(t)\big)dt, \\ H(x) = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2 - x^3 \int_0^1 \big(G(t) - 2H(t)\big)dt - \frac{3}{2}x^2 \int_0^1 t^2\big(G(t) - 2H(t)\big)dt. \end{cases} \quad (3.2)$$

where $g(x) = 1 + 3x^2 + \frac{1}{20}x^3, h(x) = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2$.

The Adomian Decomposition Method: From eq. (2.4) $G(x) = \sum_{n \to 0}^{\infty} G_n(x)$

$$\begin{cases} \sum_{n\to 0}^{\infty} G_n(x) = 1 + 3x^2 + \frac{1}{20}x^3 - \frac{1}{3}x^3 \int_0^1 \sum_{n\to 0}^{\infty} t\big(G_n(t) - 3H_n(t)\big)\,dt \\ \sum_{n\to 0}^{\infty} H_n(x) = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2 - x^3 \int_0^1 \sum_{n\to 0}^{\infty} (G_n(t) - 2H_n(t))dt \; - \frac{3}{2}x^2 \int_0^1 \sum_{n\to 0}^{\infty} t^2\big(G(t) - 2H(t)\big)dt\,. \end{cases}$$

We know that $G_0(x) = g(x)$ and $H_0(x) = h(x)$. So

$$\begin{cases} G_0(x) = 1 + 3x^2 + \frac{1}{20}x^3 \\ H_0(x) = -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2 \end{cases}$$

$$\begin{cases} G_1(x) = -\frac{1}{3}x^3 \int_0^1 t\big(G_0(t) - 3H_0(t)\big)dt \\ H_1(x) = -x^3 \int_0^1 \big(G_0(t) - 2H_0(t)\big)dt - \frac{3}{2}x^2 \int_0^1 t^2\big(G_0(t) - 2H_0(t)\big)dt. \end{cases}$$

$$\begin{cases} G_1(x) = 0.34666666667x^3 \\ H_1(x) = 0.6375x^2 - 0.49583333333x^3 \end{cases}$$

$$\vdots$$

$$\textit{and so on.}$$

Or equivalently

$$\begin{cases} G(x) = \sum_{n\to 0}^{\infty} G_n(x) = G_0(x) + G_1(x) + G_2(x) + G_3(x) + \cdots \\ H(x) = \sum_{n\to 0}^{\infty} H_n(x) = H_0(x) + H_1(x) + H_2(x) + H_3(x) + \cdots \end{cases}$$

$$\begin{cases} G(x) \approx 1 + 3x^2 + \frac{1}{20}x^3 + 0.34666666667x^3 \\ H(x) \approx -1 + 2x + \frac{5}{2}x^3 + \frac{2}{5}x^2 + 0.6375x^2 - 0.49583333333x^3 \end{cases}$$

$$\begin{cases} G(x) \approx 1 + 3x^2 + 0.39666666667x^3 \\ H(x) \approx -1 + 2x + 1.0375x^2 + 2.00416666667x^3 \end{cases}$$

Modified Adomian Decomposition Method:

From Eq. (3.2), splitting $g(x)$ into two parts i.e., $g_0(x) = 3x^2 + 1, g_1(x) = \frac{1}{20}x^3$. Also, splitting $h(x)$ into two parts i.e., $h_0(x) = -1 + 2x + x^3, h_1(x) = \frac{2}{5}x^2 + \frac{3}{2}x^3$ and use recursive relations to obtain

$$\begin{cases} G_0(x) = 3x^2 + 1, \\ H_0(x) = x^3 + 2x - 1. \end{cases}$$

and,

$$\begin{cases} G_1(x) = \frac{1}{20}x^3 - \frac{1}{3}x^3 \int_0^1 t\big(G_0(t) - 3H_0(t)\big)dt = \frac{1}{20}x^3 - \frac{1}{3}x^3\left(\frac{3}{20}\right) = 0, \\ H_1(x) = \frac{3}{2}x^3 + \frac{2}{5}x^2 - x^3 \int_0^1 \big(G_0(t) - 2H_0(t)\big)dt - \frac{3}{2}x^2 \int_0^1 t^2\big(G_0(t) - 2H_0(t)\big)dt \\ \quad = \frac{3}{2}x^3 + \frac{2}{5}x^2 - \frac{3}{2}x^3 - \frac{3}{2}x^2\left(\frac{4}{15}\right) = 0. \end{cases}$$

Additionally, we take

$$\begin{cases} G_{n+1}(x) = 0, \\ H_{n+1}(x) = 0, \end{cases} n \geq 1.$$

Thus, using these into series form (2.10), we have the solution with the closed forms $G(x) = 3x^2 + 1$ and $H(x) = x^3 + 2x - 1$. Which corresponds to the exact solution.

The Variational Iteration Method:

The correction functional for this equation is given by

$$\begin{cases} G_{n+1}(x) = G_n(x) - \int_0^x \left(G_n'(t) - \frac{3x}{10} + 6 + \int_0^1 t(r\, G_n(r)dr - rH_n(r)dr) \right) dt \\ H_{n+1}(x) = H_n(x) - \int_0^x (H_n'(t) - 15x + \frac{4}{5} - \int_0^1 3(2x + t^2)\,(rG_n(r)dr - 2H_n(r)dr)dt \end{cases}$$

Consequently, the approximate solution can be obtained by using (2.18)

$$\begin{cases} G(x) = \lim_{n \to \infty} G_n(x). \\ H(x) = \lim_{n \to \infty} H_n(x). \end{cases}$$

So, $G(x) = 3x^2 + 1$ and $H(x) = x^3 + 2x - 1$ which corresponds to the exact solution.

Table 2. Exact and Approximate solutions by ADM for Example 3.2.

x	$Exact - G(x)$	$ADM - G(x)$	$Exact - H(x)$	$ADM - H(x)$
0.1	1.03	1.03039666667	-0.799	-0.7876208333
0.2	1.12	1.12317333333	-0.592	-0.5424666667
0.3	1.27	1.28071000000	-0.373	-0.2525125000
0.4	1.48	1.50538666667	-0.136	0.09426666667
0.5	1.75	1.79958333333	0.125	0.50989583333
0.6	2.08	2.16568000000	0.416	1.00640000000
0.7	2.47	2.60605666667	0.743	1.59580416667

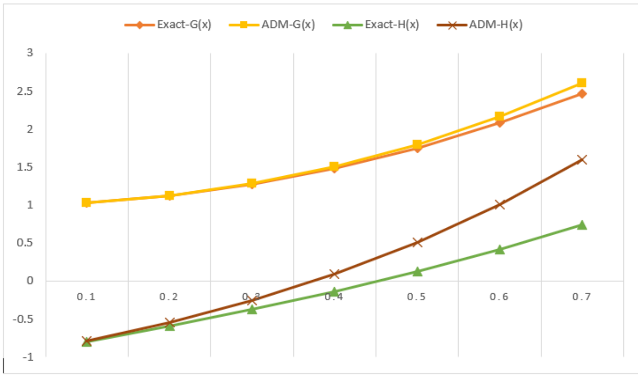

Figure 2. Exact and Approximate solutions by ADM for Example 3.2.

Example 3.3. Consider the following linear Fredholm integro-differential equation: $G'(x) = 2 - \sin x + \int_0^{\pi} tG(t)dt$, with the initial condition $G(0) = 1$, and the exact solution is $G(x) = \cos x$.

Using inverse operation, we get

$$G(x) = \cos x + 2x + \int_0^{\pi} xtG(t)dt. \qquad (3.3)$$

The Adomian Decomposition Method:

From eq. (2.4) $G(x) = \sum_{n\to 0}^{\infty} G_n(x)$

$$\sum_{n\to 0}^{\infty} G_n(x) = \cos x + 2x + \int_0^{\pi} xt \sum_{n\to 0}^{\infty} G_n(t)\, dt.$$

To determine the components of $G(x)$, we use the recurrence relation

$$G_0(x) = \cos x + 2x,$$
$$G_{n+1}(x) = \int_0^{\pi} xtG_n(t)dt, n \geq 0.$$

This in turn gives

$$G_0(x) = \cos x + 2x,$$
$$G_1(x) = \int_0^{\pi} xtG_0(t)dt = \frac{2x(\pi^3 - 3)}{3},$$
$$G_2(x) = \int_0^{\pi} xtG_1(t)dt = 6.223617040x\pi^3,$$
$$G_3(x) = \int_0^{\pi} xtG_2(t)dt = 64.32373063x\pi^3,$$
$$G_4(x) = \int_0^{\pi} xtG_3(t)dt = 664.8131297x\pi^3,$$
$$G_5(x) = \int_0^{\pi} xtG_4(t)dt = 6871.126613x\pi^3,$$
$$G_6(x) = \int_0^{\pi} xtG_5(t)dt = 71016.01763x\pi^3,$$
$$\vdots$$
$$and\ so\ on.$$

Using Eq. (2.4) gives the series solution

$$G(x) = 2x + \cos x + \frac{2x(\pi^3 - 3)}{3} + 6.223617040x\pi^3 + 64.32373063x\pi^3 + 664.8131297x\pi^3 + 6871.126613x\pi^3 + 71016.01763x\pi^3 + \cdots$$
$$\therefore\ G(x) = \cos x + (2.9714632 \times 10^{12})x + \cdots$$

The Modified Adomian Decomposition Method:

From Eq. (3.3), we get $g(x) = \cos x + 2x$. Now splitting $g(x)$ into two parts i.e., $g_0(x) = \cos x, g_1(x) = 2x$. And use recursive relations to obtain

$$G_0(x) = g_0(x) = \cos x$$
$$G_1(x) = 2x + \int_0^{\pi} x(\cos t)dt = 0$$
$$G_2(x) = 0$$
$$\vdots$$
$$and\ so\ on.$$

Additionally, we take

$$G_{n+1}(x) = 0, n \geq 1.$$

Thus, using these into series form (2.10), we have the solution with the closed forms

$$G(x) = \cos x$$

which corresponds to the exact solution.

The Variation Iteration Method: The correction functional for this equation is given by

$$G_{n+1}(x) = G_n(x) - \int_0^x \left(G_n'(t) - 2 + \sin t - \int_0^\pi r\, G_n(r)dr\right)dt.$$

Let $G_0(x) = G(0) = 1$.

$$G_0(x) = 1$$

$$G_1(x) = G_0(x) - \int_0^x (G_0'(t) - 2 + \sin t - \int_0^\pi r\, G_0(r)dr)dt$$

$$= \cos x + 2x + \frac{\pi^2}{2}x$$

$$G_2(x) = G_1(x) - \int_0^x (G_1'(t) - 2 + \sin t - \int_0^\pi r\, G_1(r)dr)dt$$

$$= \cos x + \frac{2\pi^3}{3}x + \frac{\pi^5}{6}x$$

$$G_3(x) = G_2(x) - \int_0^x (G_2'(t) - 2 + \sin t - \int_0^\pi r\, G_2(r)dr)dt$$

$$= \cos x + \frac{2\pi^6}{9}x + \frac{\pi^8}{18}x$$

$$\vdots$$

and so on.

Consequently, the approximate solution can be obtained by using (2.18)

$$G(x) = \lim_{n\to\infty} G_n(x).$$

It gives exact solution., i.e.

$$G(x) = \cos x.$$

Table 3. Exact and Approximate solutions by ADM for Example 3.3.

x	Exact	ADM	MADM	VIM
0.1	0.99999847691	$2.971463199 \times 10^{11}$	0.99999847691	0.99999847691
0.2	0.99999390766	$5.942926399 \times 10^{11}$	0.99999390766	0.99999390766
0.3	0.99998629225	$8.914389598 \times 10^{11}$	0.99998629225	0.99998629225
0.4	0.99997563071	$1.18858528 \times 10^{12}$	0.99997563071	0.99997563071
0.5	0.99996192306	1.4857316×10^{12}	0.99996192306	0.99996192306
0.6	0.99994516937	$1.78287792 \times 10^{12}$	0.99994516937	0.99994516937
0.7	0.99992536966	$2.08002424 \times 10^{12}$	0.99992536966	0.99992536966
0.8	0.99990252401	$2.377170559 \times 10^{12}$	0.99990252401	0.99990252401

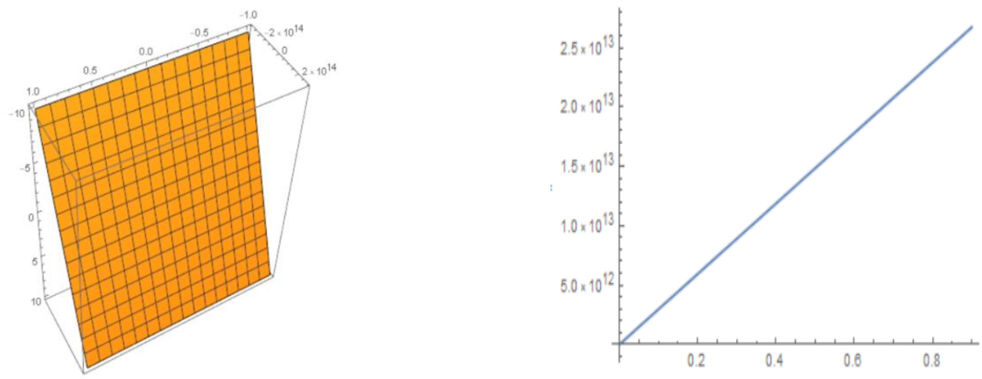

Figure 3. Exact and Approximate solutions by ADM for Example 3.3.

4. COMPARISON AMONG THE METHODS

The comparison among of the methods, it can be seen from the results of the above examples. The techniques are strong, effective, and provide closer approximations with greater precision. They can also generate closed-form solutions, if any exist. However, when applied to linear Fredholm integro-differential equations, the results given by all approaches are approximately the same. Compared to the ADM, VIM and MADM are perceived as being considerably simpler and more practical. However, VIM is more practical than MADM and ADM.

VIM has the benefit of allowing the first solution to be freely selected with some unknowable parameters. This method has a unique feature in that it can generate a very accurate approximation in some situations with a small number of iterations, or even just one.

The VIM converges more quickly than the ADM and MADM. VIM does less calculations than ADM and MADM, as well. The comparison of ADM, MADM, and VIM with the exact solutions was shown in Tables 1-3. The findings from the table's errors demonstrate that VIM provides a better result than ADM and MADM. The statistics also showed that the VIM reached the precise, more quickly than the ADM and MADM.

5. CONCLUSIONS

The Adomian Decomposition Method (ADM), Modified Adomian Decomposition Method (MADM), and Variational Iteration Method (VIM) [27-30] are efficient and effective methods for solving a wide range of problems. The main advantage of these methods is that they do not require the variables to be discretized. Furthermore, these are unaffected by computation round off errors.

In examples 2 and 3, the exact solution arising in many physical and biological models is calculated using a modified decomposition technique and the Variational Iteration Method. We show that the Variational Iteration Method (VIM) and the Modified Decomposition Method (MADM) are both very effective at determining the solution in closed form.

Additionally, while ADM requires the evaluation of an Adomian polynomial, which primarily required time-consuming algebraic calculations, VIM requires the evaluation of a Lagrangian multiplier λ . Also, VIM facilitates the computational work and gives the solution rapidly if compared with ADM and MADM.

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